Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 393095, 15 pages
doi:10.1155/2010/393095

Research Article

Hypothesis Designs for Three-Hypothesis Test Problems

Yan Li and Xiaolong Pu

School of Finance and Statistics, East China Normal University, No. 500 Dongchuan Road, Shanghai 200241, China

Correspondence should be addressed to Yan Li, yli@stat.ecnu.edu.cn

Received 25 January 2010; Accepted 18 March 2010

Academic Editor: Ming Li

As a helpful guide for applications, the alternative hypotheses of the three-hypothesis test problems are designed under the required error probabilities and average sample number in this paper. The asymptotic formulas and the proposed numerical quadrature formulas are adopted, respectively, to obtain the hypothesis designs and the corresponding sequential test schemes under the Koopman-Darmois distributions. The example of the normal mean test shows that our methods are quite efficient and satisfactory for practical uses.

1. Introduction

In practice, the multihypothesis test problems are of considerable interest in the areas of engineering, agriculture, clinical medicine, psychology, and so on. For instance, the multihypothesis tests are involved in pattern recognition [1–4], multiple-resolution radar detection [5–7], products' comparisons [8, 9], and information detection [10]. Before the inspections, the hypotheses must be determined according to such practical needs as the balance of risks and costs. As Wetherill and Glazebrook [11] pointed out, combinations of hypotheses, risks, and costs may need to be tried iteratively until an acceptable design is attained. This bothers and burdens the practitioners.

To avoid too many troublesome trials and to produce the hypotheses directly, we discuss the hypothesis designs under the controlled risks and expected costs in this paper. As an initial exploration, only the three-hypothesis test problems are considered here. Indeed, our methods may extend to the multihypothesis cases.

In practice, test costs are mainly determined by sample sizes. Therefore, the sample size becomes an issue relating to the statistical analysis of problems in many aspects; see for example, Chen et al. [12], Oliveira et al. [13], Li and Zhao [14], Li et al. [15], Bakhoum and Toma [16], Cattani [17], as well as Cattani and Kudreyko [18]. Accordingly, we consider the

Average Sample Number (ASN), which is one of the most important values in evaluating the expected costs of sequential test schemes.

In the three-hypothesis test problem, the null hypothesis is always set as a standard and medium status. For example, Anderson [8] discussed the three-hypothesis test problem to decide whether the difference of two yarns' strength is zero (the null hypothesis), positive or negative. Realistically, the standard and medium status (denoted as θ_0) is definite, while the two alternatives beside it need to be designed to balance the risks and costs. Thus, in this paper, we try to design the alternatives θ_{-1} and θ_1 ($\theta_{-1} < \theta_0 < \theta_1$) under the required error probabilities and ASN for testing the parameter θ of the Koopman-Darmois distribution

$$f_\theta(x) = \exp\{l(x) + \theta x - b(\theta)\}, \quad \text{where } b(\theta) \text{ is a convex function, } \theta \in \Theta. \tag{1.1}$$

To simplify the discussion, we only consider the designs of the two alternative hypotheses symmetric with the null hypothesis, that is, $\theta_1 - \theta_0 = \theta_0 - \theta_{-1} = k(> 0)$. Actually, the asymmetric designs may be obtained by extending our methods slightly.

Then, the test problem here is

$$H_{-1} : \theta = \theta_{-1} = \theta_0 - k \quad \text{vs.} \quad H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1 = \theta_0 + k. \tag{1.2}$$

For the multihypothesis test problems, Armitage [19] provided a classical test scheme by simultaneously applying the method of Sequential Probability Ratio Test (SPRT) on each pair of the hypotheses. This test scheme pattern is simple and easy to implement. When testing the three hypotheses for the Koopman-Darmois distribution (1.1), Armitage's scheme may be illustrated as in Figure 1, where AL//CM are boundaries for "$\theta = \theta_1$ versus $\theta = \theta_0$" and CP//DQ are for "$\theta = \theta_0$ versus $\theta = \theta_{-1}$" when the boundaries for "$\theta = \theta_1$ versus $\theta = \theta_{-1}$" are encircled by AL and DQ and thus are neglected. According to Figure 1, the decision rule should be

$$\begin{aligned}
&\text{Accept } H_1 \text{ if } T_n \geq a + n\tan\psi, \\
&\text{Accept } H_0 \text{ if } c + (n - n_0)\tan\varphi \leq T_n \leq c + (n - n_0)\tan\psi, \\
&\text{Accept } H_{-1} \text{ if } T_n \leq d + n\tan\varphi, \\
&\text{Continue sampling without any decision, otherwise,}
\end{aligned} \tag{1.3}$$

where $T_n = \sum_{i=1}^{n} X_i$ and $X_1, X_2, \ldots$ are independent sequential observations from a Koopman-Darmois distribution.

For the given θ_{-1}, θ_0, and θ_1, the test scheme in Figure 1 is decided by 6 parameters $(n_0, a, c, d, \psi, \varphi)$. ψ and φ may be determined according to Armitage [19], that is, $\tan\psi = [b(\theta_1) - b(\theta_0)]/(\theta_1 - \theta_0)$, $\tan\varphi = [b(\theta_0) - b(\theta_{-1})]/(\theta_0 - \theta_{-1})$ under the Koopman-Darmois distribution (1.1), then the remaining 4 parameters (n_0, a, c, d) form the scheme. Altogether with the hypothesis design value k in the test problem (1.2), the 5 underdetermined values are (k, n_0, a, c, d).

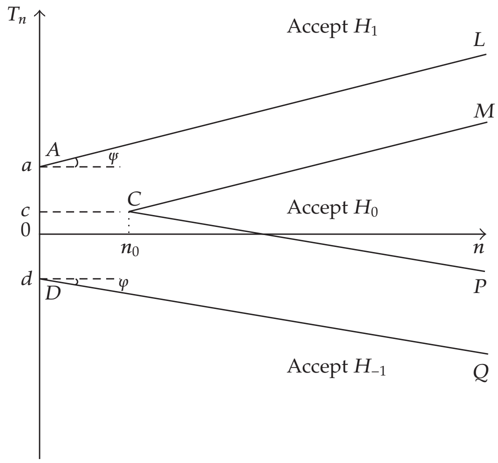

Figure 1

In the three-hypothesis test problems, the error probabilities α and β should be assigned to the error probabilities $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ in correspondence with the requirements

$$\begin{aligned} P(\text{Accept } H_0 \mid H_1) &\leq \gamma_1, \\ P(\text{Accept } H_0 \mid H_{-1}) &\leq \gamma_2, \\ P(\text{Accept } H_1 \mid H_0) &\leq \gamma_3, \\ P(\text{Accept } H_{-1} \mid H_0) &\leq \gamma_4. \end{aligned} \tag{1.4}$$

Commonly, we set $\gamma_1 = \gamma_2 = \beta$, $\gamma_3 = \gamma_4 = \alpha/2$, as Payton and Young [20, 21] indicated.

And the request on the ASN should be

$$\text{ASN}(\theta_{\text{ASN}}) \leq N, \tag{1.5}$$

where $N(> 0)$ is a provided integer and $\theta_{\text{ASN}}(\in \Theta)$ is the point at which the ASN needs to be controlled. θ_{ASN} may take values of θ_{-1}, θ_0, θ_1, and so on according to practical needs.

Then, under the constraints (1.4) and (1.5), we may find the proper (k, n_0, a, c, d) by virtue of their relationships with the error probabilities and ASN.

Unfortunately, however, to the best knowledge of the authors, the accurate formulas for the performances of the three-hypothesis test scheme are still unavailable possibly because of its sequential feature and anomalistic continuing sampling area. In the following, the hypothesis designs and the test scheme parameters are determined under the required error probabilities and ASN in terms of some approximate expressions, that is, the asymptotic formulas and the proposed numerical quadrature formulas.

2. Designs under Asymptotic Formulas

In this section, we try to find the hypothesis designs and test schemes under the required error probabilities and ASN by virtue of the asymptotic formulas of the multihypothesis sequential test scheme by Dragalin et al. in [22, 23].

Firstly, we discuss how to control the error probabilities. Let C_i be the critical value of the logarithmic likelihood ratio function for accepting θ_i, and let R_i be the probability limit of incorrectly accepting θ_i, $i = -1, 0, 1$. According to Dragalin et al. [22], under the condition of equal prior probabilities for the three hypotheses, the probability of wrongly accepting θ_i for the Armitage [19] scheme may be controlled by R_i if the critical value C_i is set as

$$C_i = \ln\left\{\frac{2}{3R_i}\right\}, \quad i = -1, 0, 1. \tag{2.1}$$

Thus, the error probabilities $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ are in control if we follow the critical values in (2.1), where $R_{-1} = \gamma_4$, $R_0 = \min\{\gamma_1, \gamma_2\}$, and $R_1 = \gamma_3$. Setting the critical values (C_{-1}, C_0, C_1) equal to the corresponding logarithmic likelihood ratio functions, we have the following expressions for the test scheme parameters under the Koopman-Darmois distribution (1.1):

$$\begin{gathered}
n_0 = \frac{C_0(1/(\theta_0-\theta_{-1}) - 1/(\theta_0-\theta_1))}{(b(\theta_0)-b(\theta_1))/(\theta_0-\theta_1) - (b(\theta_0)-b(\theta_{-1}))/(\theta_0-\theta_{-1})} = \frac{2C_0}{b(\theta_0+k)+b(\theta_0-k)-2b(\theta_0)}, \\
a = \frac{C_1}{\theta_1-\theta_0} = \frac{C_1}{k}, \\
c = \frac{C_0}{\theta_0-\theta_1} + n_0\tan\psi = -\frac{C_0}{k} + n_0\frac{b(\theta_0+k)-b(\theta_0)}{k}, \\
d = \frac{C_{-1}}{\theta_{-1}-\theta_0} = -\frac{C_{-1}}{k}.
\end{gathered} \tag{2.2}$$

Note that the expressions in (2.2) define the relations between the hypothesis design parameter k and the test scheme parameters (n_0, a, c, d), while k has not been determined so far.

In the following, the hypothesis design parameter k is found with the help of Dragalin et al.'s asymptotic ASN formulas [23].

Based on the nonlinear renewal theory, Dragalin et al. [23] summarized and developed the asymptotic ASN formulas under $\max\{\alpha, \beta\} \to 0$. Specifically, when $\theta_1 - \theta_0 = \theta_0 - \theta_{-1}$, the asymptotic ASN formulas under the two alternatives θ_{-1} and θ_1 are

$$\mathrm{ASN}(\theta_i) \approx \frac{C_i + O_{\theta_i}}{D_{\theta_i}}, \quad i = -1, 1, \tag{2.3}$$

where $D_{\theta_i} = \min_{j\neq i} E_{\theta_i}(\ln\{f_{\theta_i}(x)/f_{\theta_j}(x)\})$ and O_{θ_i} is the expected limiting overshoot under θ_i, $i = -1, 0, 1$.

And for the null hypothesis θ_0 under $\theta_1 - \theta_0 = \theta_0 - \theta_{-1}$, the asymptotic ASN formula is

$$\mathrm{ASN}(\theta_0) \approx \frac{F_2(C_0, D_{\theta_0}, v) + O_{\theta_0}}{D_{\theta_0}}, \tag{2.4}$$

where $F_2(x, q, u) = x + uh_2^*\sqrt{x/q + u^2(h_2^*)^2/(4q^2)} + u^2(h_2^*)^2/(2q)$ ($h_2^* = 0.5641895835$ here), and v is the value related to the covariance of the logarithmic likelihood ratio functions.

Notice that the approximate ASN formulas (2.3) and (2.4) only depend on the hypothesis design parameter k when θ_0 is given. Therefore, to find the proper hypothesis design under the desired number N, we set up an equation about k to meet the ASN requirement on one of the three hypothesis values, that is,

$$\mathrm{ASN}(\theta_{\mathrm{ASN}}) = N, \tag{2.5}$$

where θ_{ASN} may be θ_{-1}, θ_0, or θ_1.

Then, the hypothesis design parameter k is the solution to (2.5) and the test scheme with (n_0, a, c, d) may be obtained correspondingly according to (2.2). Illustrations are provided in Example 1 for testing the normal mean with the variance known.

Example 1. Suppose that the sequential observations $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) with $N(\mu, 1)$. Let $\mu_0 = 0$, $\gamma_1 = \gamma_2 = \beta$, and $\gamma_3 = \gamma_4 = \alpha/2$. Small values ($\le 30$) are set on N as practical sequential inspections always require.

Accordingly, we have $C_0 = \ln\{2/(3\gamma_1)\}$, $C_1 = C_{-1} = \ln\{2/(3\gamma_3)\}$. In this example, the test scheme parameters should be

$$n_0 = \frac{2C_0}{k^2}, \qquad a = \frac{C_1}{k}, \qquad c = 0, \qquad d = -a, \qquad \tan\psi = \frac{k}{2}, \qquad \tan\varphi = -\tan\psi. \tag{2.6}$$

And for the normal distribution $N(\mu, 1)$, there are

$$\begin{gathered} D_{\mu_i} = \min_{j \ne i} \frac{\left(\mu_i - \mu_j\right)^2}{2} = \frac{k^2}{2}, \quad i = -1, 0, 1, \\ O_{\mu_i} = 1 + \frac{k^2}{4} - k \sum_{l=1}^{\infty} \frac{1}{\sqrt{l}} \left[\phi\left(\frac{k}{2}\sqrt{l}\right) - \frac{k}{2}\sqrt{l}\,\Phi\left(-\frac{k}{2}\sqrt{l}\right) \right], \quad i = -1, 0, 1, \\ v = \sqrt{2}k, \end{gathered} \tag{2.7}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of the standard normal distribution, respectively.

Consider the following 4 cases, respectively:

$$\begin{aligned} \mu_{\mathrm{ASN}} &= \mu_0, \qquad \alpha = \beta = 0.002, \\ \mu_{\mathrm{ASN}} &= \mu_0, \qquad \alpha = \beta = 0.05, \\ \mu_{\mathrm{ASN}} &= \mu_1, \qquad \alpha = \beta = 0.002, \\ \mu_{\mathrm{ASN}} &= \mu_1, \qquad \alpha = \beta = 0.05. \end{aligned} \tag{2.8}$$

Then, solving (2.5), we obtain the hypothesis designs k as shown in Column 2 of Tables 1, 2, 3, and 4. The corresponding test scheme parameters $(n_0, a, \tan\psi)$ from (2.6) are listed in Columns 3–5 of Tables 1–4. To evaluate the method's efficiency, we record the Monte Carlo simulation study results with 1,000,000 replicates in Tables 5, 6, 7, and 8, where $\mathrm{ASN}'(\mu_{\mathrm{ASN}})$

Table 1: Hypothesis designs and test schemes for $\mu_{ASN} = \mu_0$, $\alpha = \beta = 0.002$.

N	Under Asymptotic formulas				Under Gaussian quadrature formulas				ε_k (%)
	k	n_0	a	$\tan\psi$	k	n_0	a	$\tan\psi$	
5	2.1090	2.6121	3.0831	1.0545	1.8953	2.6876	3.0725	0.9477	11.28
10	1.4400	5.6030	4.5155	0.7200	1.3193	5.8833	4.6576	0.6597	9.15
15	1.1604	8.6279	5.6034	0.5802	1.0674	9.0266	5.8910	0.5337	8.71
20	0.9977	11.6709	6.5170	0.4989	0.9228	12.3354	6.9039	0.4614	8.12
25	0.8882	14.7257	7.3204	0.4441	0.8246	15.6592	7.7944	0.4123	7.71
30	0.8082	17.7888	8.0458	0.4041	0.7523	18.9949	8.5984	0.3762	7.43

Table 2: Hypothesis designs and test schemes for $\mu_{ASN} = \mu_0$, $\alpha = \beta = 0.05$.

N	Under Asymptotic formulas				Under Gaussian quadrature formulas				ε_k (%)
	k	n_0	a	$\tan\psi$	k	n_0	a	$\tan\psi$	
5	1.6066	2.0072	2.0438	0.8033	1.3025	2.1953	2.2175	0.6513	23.35
10	1.0908	4.3542	3.0102	0.5454	0.9087	5.0028	3.4233	0.4544	20.04
15	0.8767	6.7395	3.7450	0.4384	0.7401	8.0001	4.2242	0.3701	18.46
20	0.7527	9.1448	4.3624	0.3764	0.6392	11.0000	5.2525	0.3196	17.76
25	0.6693	11.5630	4.9054	0.3347	0.5708	14.0000	5.9086	0.2854	17.26
30	0.6085	13.9903	5.3957	0.3043	0.5204	17.0000	6.5244	0.2602	16.93

Table 3: Hypothesis designs and test schemes for $\mu_{ASN} = \mu_1$, $\alpha = \beta = 0.002$.

N	Under Asymptotic formulas				Under Gaussian quadrature formulas				ε_k (%)
	k	n_0	a	$\tan\psi$	k	n_0	a	$\tan\psi$	
5	1.7893	3.6290	3.6340	0.8947	1.7105	3.2864	3.4644	0.8553	4.61
10	1.2179	7.8334	5.3391	0.6090	1.1900	7.1875	5.2255	0.5950	2.34
15	0.9800	12.0973	6.6350	0.4900	0.9667	11.1560	6.5639	0.4834	1.38
20	0.8419	16.3926	7.7236	0.4210	0.8352	15.1594	7.6882	0.4176	0.80
25	0.7490	20.7080	8.6809	0.3745	0.7460	19.1851	8.6770	0.3730	0.40
30	0.6812	25.0379	9.5454	0.3406	0.6803	23.2266	9.5698	0.3402	0.13

Table 4: Hypothesis designs and test schemes for $\mu_{ASN} = \mu_1$, $\alpha = \beta = 0.05$.

N	Under Asymptotic formulas				Under Gaussian quadrature formulas				ε_k (%)
	k	n_0	a	$\tan\psi$	k	n_0	a	$\tan\psi$	
5	1.3089	3.0237	2.5085	0.6545	1.1875	2.9124	2.4858	0.5938	10.22
10	0.8835	6.6369	3.7164	0.4418	0.8264	6.2545	3.8213	0.4132	6.91
15	0.7083	10.3268	4.6358	0.3542	0.6715	10.0000	4.8351	0.3358	5.48
20	0.6071	14.0566	5.4085	0.3036	0.5803	13.4993	5.6872	0.2902	4.62
25	0.5393	17.8118	6.0883	0.2697	0.5183	17.0061	6.4368	0.2592	4.05
30	0.4899	21.5853	6.7022	0.2450	0.4727	20.8842	7.1131	0.2364	3.64

is the simulated value of $\text{ASN}(\mu_{\text{ASN}})$ and ε is the relative difference between $\text{ASN}'(\mu_{\text{ASN}})$ and N. Note that the simulated probabilities under μ_{-1} are neglected here since they are nearly equivalent to their counterparts under μ_1 in terms of the schemes' symmetry.

Obviously, the accuracy of the solution k to (2.5) is decided by the efficiency of the ASN formulas (2.3) and (2.4). On one hand, from Dragalin et al. [23] and the ε's in Tables 5–8, we conclude that the formulas in (2.3) for $\text{ASN}(\theta_{-1})$ and $\text{ASN}(\theta_1)$ are more efficient than

Table 5: Simulated performances for the schemes under asymptotic formulas in Table 1.

N	When μ_1 is true Probability of accepting			When μ_0 is true Probability of accepting			$\text{ASN}'(\mu_0)$	ε (%)
	μ_{-1}	μ_0	μ_1	μ_{-1}	μ_0	μ_1		
5	0	0.0007	0.9993	0.0005	0.9990	0.0005	4.7053	6.26
10	0	0.0009	0.9991	0.0007	0.9986	0.0007	9.3601	6.84
15	0	0.0011	0.9989	0.0007	0.9985	0.0008	13.9845	7.26
20	0	0.0011	0.9989	0.0008	0.9984	0.0008	18.5910	7.58
25	0	0.0012	0.9988	0.0009	0.9982	0.0009	23.1885	7.81
30	0	0.0013	0.9987	0.0010	0.9980	0.0010	27.8046	7.90

Table 6: Simulated performances for the schemes under asymptotic formulas in Table 2.

N	When μ_1 is true Probability of accepting			When μ_0 is true Probability of accepting			$\text{ASN}'(\mu_0)$	ε (%)
	μ_{-1}	μ_0	μ_1	μ_{-1}	μ_0	μ_1		
5	0	0.0186	0.9814	0.0148	0.9703	0.0149	4.4409	12.59
10	0	0.0273	0.9727	0.0195	0.9610	0.0195	8.4229	18.72
15	0	0.0317	0.9682	0.0219	0.9562	0.0219	12.3687	21.27
20	0	0.0323	0.9677	0.0236	0.9529	0.0235	16.4767	21.38
25	0	0.0344	0.9656	0.0246	0.9508	0.0245	20.3535	18.59
30	0	0.0361	0.9638	0.0252	0.9494	0.0255	24.2233	19.26

the one in (2.4) for $\text{ASN}(\theta_0)$ when testing the normal mean. On the other hand, the asymptotic ASN formulas perform better under smaller error probabilities since the asymptotic limit is taken as $\max\{\alpha,\beta\} \to 0$. For applications, with such a simple computation, the efficiency of the design is quite satisfactory for small error probabilities conditions.

However, this method may only serve to control the ASN on the three hypothesis values since the asymptotic ASN formulas out of these points are absent so far. And the quantities $D_{\theta_i}, O_{\theta_i}$ $(i = -1, 0, 1)$, and v should be deduced according to specific distributions (see [23]). Besides, the discrepancies between the real performances and the required ones show the method's conservativeness. In the next section, an improved method is proposed and more efficient formulas are developed through the numerical quadrature.

3. Designs under Numerical Quadrature Formulas

This section proposes a method to obtain more efficient hypothesis designs and test schemes through a system of equations based on the numerical quadrature formulas of the error probabilities and ASN.

In studies by Payton and Young in [20, 21], for the provided hypotheses, the error probabilities $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ are approximately attained by solving a system of equations about the 4 scheme parameters (n_0, a, c, d). This method is hoped to fully make use of the required error probabilities and to obtain efficient designs. Enlightened by Payton and Young, we

Table 7: Simulated performances for the schemes under asymptotic formulas in Table 3.

N	When μ_1 is true Probability of accepting			When μ_0 is true Probability of accepting			$\mathrm{ASN}'(\mu_1)$	ε (%)
	μ_{-1}	μ_0	μ_1	μ_{-1}	μ_0	μ_1		
5	0	0.0008	0.9992	0.0005	0.9990	0.0005	4.9907	0.19
10	0	0.0010	0.9990	0.0007	0.9985	0.0008	9.9818	0.18
15	0	0.0010	0.9990	0.0009	0.9983	0.0008	14.9705	0.20
20	0	0.0012	0.9988	0.0010	0.9981	0.0009	19.9525	0.24
25	0	0.0013	0.9987	0.0009	0.9981	0.0010	24.9494	0.20
30	0	0.0012	0.9988	0.0010	0.9980	0.0010	29.9059	0.31

Table 8: Simulated performances for the schemes under asymptotic formulas in Table 4.

N	When μ_1 is true Probability of accepting			When μ_0 is true Probability of accepting			$\mathrm{ASN}'(\mu_1)$	ε (%)
	μ_{-1}	μ_0	μ_1	μ_{-1}	μ_0	μ_1		
5	0	0.0226	0.9774	0.0173	0.9655	0.0172	4.7976	4.05
10	0	0.0313	0.9687	0.0221	0.9560	0.0219	9.4322	5.68
15	0	0.0336	0.9664	0.0241	0.9518	0.0241	14.1221	5.85
20	0	0.0346	0.9654	0.0255	0.9492	0.0253	18.7665	6.17
25	0	0.0367	0.9632	0.0265	0.9471	0.0264	23.3316	6.67
30	0	0.0371	0.9628	0.0270	0.9459	0.0270	27.9989	6.67

propose to find the hypothesis design and test scheme by solving the following system of equations:

$$
\begin{gathered}
P(\text{Accept } H_1 \mid H_0) = \gamma_3, \\
P(\text{Accept } H_{-1} \mid H_0) = \gamma_4, \\
P(\text{Accept } H_0 \mid H_1) = \gamma_1, \\
P(\text{Accept } H_0 \mid H_{-1}) = \gamma_2, \\
\mathrm{ASN}(\theta_{\mathrm{ASN}}) = N.
\end{gathered} \tag{3.1}
$$

Obviously, the key is to find the formulas of the error probabilities and ASN on the left side of the equations in (3.1). Unfortunately, the available approximate formulas cannot meet applicable needs well. For example, Payton and Young [20, 21] adopted the formulas under the continuous-time process and the required minimum sample size before decisions, and obtained some inefficient results. Also, as mentioned in Section 2, Dragalin et al.'s results are restricted to the conditions of small error probabilities and $\theta_{\mathrm{ASN}} = \theta_i$ $(i = -1, 0, 1)$ [22, 23].

To find efficient and applicable designs, we develop the approximate formulas through the numerical quadrature for the three-hypothesis test scheme's performances on the error probabilities and ASN.

To deduce the formulas for the realistic discrete-time situation, we denote n_t as the minimum integer that is not less than n_0. Let L_j and U_j be the values on the two boundaries

DQ and AL in Figure 1 at $n = j$, that is, $L_j = d + j\tan\varphi$, $U_j = a + j\tan\psi$, $j = 1, \ldots, n_t$. Denote $c_L = c + (n_t - n_0)\tan\varphi$, $c_U = c + (n_t - n_0)\tan\psi$, $a' = a + n_t\tan\psi$, and $d' = d + n_t\tan\varphi$. With the decision rule (1.3), we rewrite the system of (3.1) as

$$
\begin{gathered}
R_1(\theta_0) + S_1(\theta_0) - L_1(\theta_0) = \gamma_3, \\
R_{-1}(\theta_0) + S_{-1}(\theta_0) - L_{-1}(\theta_0) = \gamma_4, \\
L_1(\theta_1) + L_{-1}(\theta_1) + L_0(\theta_1) = \gamma_1, \\
L_1(\theta_{-1}) + L_{-1}(\theta_{-1}) + L_0(\theta_{-1}) = \gamma_2, \\
N_0(\theta_{\mathrm{ASN}}) + N_1(\theta_{\mathrm{ASN}}) + N_{-1}(\theta_{\mathrm{ASN}}) + n_t J(\theta_{\mathrm{ASN}}) = N,
\end{gathered}
\tag{3.2}
$$

where $R_1(\theta) = P_\theta(\text{Accept } H_1 \text{ through AL when } n \le n_t)$; $R_{-1}(\theta) = P_\theta(\text{Accept } H_{-1} \text{ through DQ when } n \le n_t)$; $S_1(\theta) = P_\theta(c_U < T_{n_t} < a', L_j < T_j < U_j, j = 1, \ldots, n_t - 1)$; $S_{-1}(\theta) = P_\theta(d' < T_{n_t} < c_L, L_j < T_j < U_j, j = 1, \ldots, n_t - 1)$; $L_1(\theta) = P_\theta(\text{Accept } H_0 \text{ through CM when } n > n_t)$; $L_{-1}(\theta) = P_\theta(\text{Accept } H_0 \text{ through CP when } n > n_t)$; $L_0(\theta) = P_\theta(\text{Accept } H_0 \text{ at } n = n_t)$; $J(\theta) = S_1(\theta) + S_{-1}(\theta) + L_0(\theta)$; $N_0(\theta)$ is the average sample number from a point in (d, a) at $n = 0$ to the point of accepting H_1 or H_{-1} when $n \le n_t$. $N_1(\theta)$ is the average sample number from a point in (c_U, a') at $n = n_t$ to the point of making a decision when $n > n_t$. And $N_{-1}(\theta)$ is the average sample number from a point in (d', c_L) at $n = n_t$ to the point of making a decision when $n > n_t$.

The following theorem provides the approximate formulas through the numerical quadrature for the quantities in (3.2). In fact, these formulas are developed by a stepwise dealing for the continuing sampling area before n_0 and the results by Li and Pu in [24] for the parallel lines areas inside AL//CM and inside CP//DQ, respectively. With such an idea, the proof of Theorem 3.1 is trivial and is neglected here.

Theorem 3.1. *Assume that $X_1, X_2, \ldots$ are i.i.d. observations. Let $f_\theta(x)$ and $F_\theta(x)$ be the p.d.f. and c.d.f. of X, respectively. Assume that $F_\theta^-(x) = P_\theta(X < x)$. Denote $\widetilde{g}_{1\theta}(x) = f_\theta(x)$, and $\widetilde{g}_{j+1\theta}(x) = \sum_{i=1}^{m} \omega(u_i^{(j)})\widetilde{g}_{j\theta}(u_i^{(j)}) f_\theta(x - u_i^{(j)})$, where $u_i^{(j)}$ is the ith numerical quadrature root for $[L_j, U_j]$, $i = 1, \ldots, m$, $j = 1, \ldots, n_t - 1$, and $\omega(u)$ is the corresponding weight for the numerical quadrature root u. Let $u_i^{(n_t)}$ and $u_i^{(n_t)'}$ be the ith numerical quadrature root for $[c_U, a']$ and for $[d', c_L]$, respectively, $i = 1, \ldots, m$.*

Then, the approximate values $\widetilde{R}_1(\theta)$, $\widetilde{R}_{-1}(\theta)$, $\widetilde{S}_1(\theta)$, $\widetilde{S}_{-1}(\theta)$, $\widetilde{L}_1(\theta)$, $\widetilde{L}_{-1}(\theta)$, $\widetilde{L}_0(\theta)$, $\widetilde{N}_0(\theta)$, $\widetilde{N}_1(\theta)$, and $\widetilde{N}_{-1}(\theta)$ for the quantities in (3.2) are the following.

(1)

$$
\widetilde{R}_1(\theta) = 1 - F_\theta^-(U_1) + \sum_{j=2}^{n_t}\sum_{i=1}^{m} \omega\left(u_i^{(j-1)}\right)\widetilde{g}_{j-1\theta}\left(u_i^{(j-1)}\right)\left[1 - F_\theta^-\left(U_j - u_i^{(j-1)}\right)\right]. \tag{3.3}
$$

(2)

$$
\widetilde{R}_{-1}(\theta) = F_\theta(L_1) + \sum_{j=2}^{n_t}\sum_{i=1}^{m} \omega\left(u_i^{(j-1)}\right)\widetilde{g}_{j-1\theta}\left(u_i^{(j-1)}\right)F_\theta\left(L_j - u_i^{(j-1)}\right). \tag{3.4}
$$

(3)

$$\widetilde{S}_1(\theta) = \sum_{i=1}^{m} \omega\left(u_i^{(n_t-1)}\right) \widetilde{g}_{n_t-1\theta}\left(u_i^{(n_t-1)}\right)\left[F_\theta^-\left(a' - u_i^{(n_t-1)}\right) - F_\theta\left(c_U - u_i^{(n_t-1)}\right)\right]. \quad (3.5)$$

(4)

$$\widetilde{S}_{-1}(\theta) = \sum_{i=1}^{m} \omega\left(u_i^{(n_t-1)}\right) \widetilde{g}_{n_t-1\theta}\left(u_i^{(n_t-1)}\right)\left[F_\theta^-\left(c_L - u_i^{(n_t-1)}\right) - F_\theta\left(d' - u_i^{(n_t-1)}\right)\right]. \quad (3.6)$$

(5) *Denote* $\mathbf{q}_\theta^{(1)}(x) = [q_{\theta j}^{(1)}(x)]_{1\times m}$, *where* $q_{\theta j}^{(1)}(x) = \omega(u_j^{(n_t)}) f_\theta(u_j^{(n_t)} + \tan\varphi - x)$, $j = 1, \ldots, m$; $\mathbf{p}_\theta^{(1)} = [p_{\theta i}^{(1)}]_{m\times 1}$, *where* $p_{\theta i}^{(1)} = F_\theta(c_U + \tan\varphi - u_i^{(n_t)})$, $i = 1, \ldots, m$; $\mathbf{Q}_\theta^{(1)} = [q_{\theta ij}^{(1)}]_{m\times m}$, *where* $q_{\theta ij}^{(1)} = \omega(u_j^{(n_t)}) f_\theta(u_j^{(n_t)} + \tan\varphi - u_i^{(n_t)})$, $i, j = 1, \ldots, m$. *Let* $\mathbf{I}$ *be the* $m \times m$ *identity matrix. Then, there is*

$$\widetilde{L}_1(\theta) = \sum_{i=1}^{m} \omega\left(u_i^{(n_t)}\right) \widetilde{g}_{n_t\theta}\left(u_i^{(n_t)}\right) \widetilde{OC}_{1\theta}\left(u_i^{(n_t)}\right), \quad (3.7)$$

where $\widetilde{OC}_{1\theta}(x) = F_\theta(c_U + \tan\varphi - x) + \mathbf{q}_\theta^{(1)}(x)(\mathbf{I} - \mathbf{Q}_\theta^{(1)})^{-1}\mathbf{p}_\theta^{(1)}$.

(6) *Denote* $\mathbf{q}_\theta^{(2)}(x) = [q_{\theta j}^{(2)}(x)]_{1\times m}$, *where* $q_{\theta j}^{(2)}(x) = \omega(u_j^{(n_t)'}) f_\theta(u_j^{(n_t)'} + \tan\varphi - x)$, $j = 1, \ldots, m$; $\mathbf{p}_\theta^{(2)} = [p_{\theta i}^{(2)}]_{m\times 1}$, *where* $p_{\theta i}^{(2)} = F_\theta(d' + \tan\varphi - u_i^{(n_t)'})$, $i = 1, \ldots, m$; $\mathbf{Q}_\theta^{(2)} = [q_{\theta ij}^{(2)}]_{m\times m}$, *where* $q_{\theta ij}^{(2)} = \omega(u_j^{(n_t)'}) f_\theta(u_j^{(n_t)'} + \tan\varphi - u_i^{(n_t)'})$, $i, j = 1, \ldots, m$. *Then, we have*

$$\widetilde{L}_{-1}(\theta) = \widetilde{S}_{-1}(\theta) - \sum_{i=1}^{m} \omega\left(u_i^{(n_t)'}\right) \widetilde{g}_{n_t\theta}\left(u_i^{(n_t)'}\right) \widetilde{OC}_{-1\theta}\left(u_i^{(n_t)'}\right), \quad (3.8)$$

where $\widetilde{OC}_{-1\theta}(x) = F_\theta(d' + \tan\varphi - x) + \mathbf{q}_\theta^{(2)}(x)(\mathbf{I} - \mathbf{Q}_\theta^{(2)})^{-1}\mathbf{p}_\theta^{(2)}$.

(7)

$$\widetilde{L}_0(\theta) = \sum_{i=1}^{m} \omega\left(u_i^{(n_t-1)}\right) \widetilde{g}_{n_t-1\theta}\left(u_i^{(n_t-1)}\right)\left[F_\theta\left(c_U - u_i^{(n_t-1)}\right) - F_\theta^-\left(c_L - u_i^{(n_t-1)}\right)\right]. \quad (3.9)$$

(8)

$$\widetilde{N}_0(\theta) = 1 + \sum_{j=2}^{n_t} j \sum_{i=1}^{m} \omega\left(u_i^{(j-1)}\right) \widetilde{g}_{j-1\theta}\left(u_i^{(j-1)}\right)\left[1 - F_\theta^-\left(U_j - u_i^{(j-1)}\right) + F_\theta\left(L_j - u_i^{(j-1)}\right)\right]. \quad (3.10)$$

Table 9: Simulated performances for the schemes under Gaussian quadrature formulas in Table 1.

N	When μ_1 is true Probability of accepting			When μ_0 is true Probability of accepting			$\mathrm{ASN}'(\mu_0)$	ε (%)
	μ_{-1}	μ_0	μ_1	μ_{-1}	μ_0	μ_1		
5	0	0.0021	0.9979	0.0010	0.9980	0.0010	5.0003	0.01
10	0	0.0020	0.9980	0.0010	0.9980	0.0010	10.0078	0.08
15	0	0.0019	0.9981	0.0010	0.9980	0.0009	14.9949	0.03
20	0	0.0020	0.9980	0.0010	0.9980	0.0010	19.9888	0.06
25	0	0.0020	0.9980	0.0011	0.9979	0.0010	24.9980	0.01
30	0	0.0020	0.9980	0.0010	0.9980	0.0010	29.9972	0.01

(9)

$$\widetilde{N}_1(\theta) = \sum_{i=1}^{m} \omega\left(u_i^{(n_t)}\right) \widetilde{g}_{n_t\theta}\left(u_i^{(n_t)}\right) \widetilde{n}_{1\theta}\left(u_i^{(n_t)}\right), \tag{3.11}$$

where $\widetilde{n}_{1\theta}(x) = 1 + \mathbf{q}_\theta^{(1)}(x)(\mathbf{I} - \mathbf{Q}_\theta^{(1)})^{-1}\mathbf{1}$ *with* $\mathbf{1}$ *being the* $m \times 1$ *vector of 1's.*

(10)

$$\widetilde{N}_{-1}(\theta) = \sum_{i=1}^{m} \omega\left(u_i^{(n_t)'}\right) \widetilde{g}_{n_t\theta}\left(u_i^{(n_t)'}\right) \widetilde{n}_{-1\theta}\left(u_i^{(n_t)'}\right), \tag{3.12}$$

where $\widetilde{n}_{-1\theta}(x) = 1 + \mathbf{q}_\theta^{(2)}(x)(\mathbf{I} - \mathbf{Q}_\theta^{(2)})^{-1}\mathbf{1}$.

Notice that the values on the left side of the equations in (3.2) must be obtained through a computer program with much iterative work, which reveals the method's complexity in computation and impairs the speed of solving the system of (3.2). Nevertheless, the time it costs is tolerable when the accuracy of solving the equations is not too demanded.

Example 1 (Continued). Consider the same problems as those in Example 1 in Section 2. By applying the formulas (3.3)–(3.12) and the 64 Gaussian quadrature roots, we solve the system of (3.2) in a computer program. The hypothesis designs and the corresponding test schemes are listed in Columns 6–9 of Tables 1–4. As a comparison with the method under the asymptotic formulas in Section 2, ε_k in Column 10 of Tables 1–4 records the relative difference between the two hypothesis designs of the two methods. The Monte Carlo simulation study with 1,000,000 replicates in Tables 9, 10, 11, and 12 reveal the schemes' real performances.

The real performances in Tables 9–12 show that the requirements on controlling the error probabilities and ASN may be fully made use of under this method and the numerical quadrature formulas are almost accurate. Therefore, the hypothesis designs and test schemes are highly efficient in terms of, for instance, more efficient designs with smaller k in Tables 1–4 under this method.

To further explain the methods, an example of the airbag quality inspection is provided in the appendix.

Table 10: Simulated performances for the schemes under Gaussian quadrature formulas in Table 2.

N	When μ_1 is true Probability of accepting			When μ_0 is true Probability of accepting			$\text{ASN}'(\mu_0)$	ε (%)
	μ_{-1}	μ_0	μ_1	μ_{-1}	μ_0	μ_1		
5	0	0.0500	0.9500	0.0250	0.9501	0.0249	4.9998	0.00
10	0	0.0499	0.9501	0.0249	0.9499	0.0251	10.0032	0.03
15	0	0.0486	0.9514	0.0271	0.9456	0.0274	15.0038	0.03
20	0	0.0520	0.9480	0.0228	0.9543	0.0229	20.0061	0.03
25	0	0.0510	0.9490	0.0236	0.9530	0.0234	24.9795	0.08
30	0	0.0512	0.9488	0.0235	0.9532	0.0233	29.9926	0.02

Table 11: Simulated performances for the schemes under Gaussian quadrature formulas in Table 3.

N	When μ_1 is true Probability of accepting			When μ_0 is true Probability of accepting			$\text{ASN}'(\mu_1)$	ε (%)
	μ_{-1}	μ_0	μ_1	μ_{-1}	μ_0	μ_1		
5	0	0.0019	0.9981	0.0011	0.9979	0.0010	4.9994	0.01
10	0	0.0020	0.9980	0.0010	0.9980	0.0010	9.9917	0.08
15	0	0.0020	0.9980	0.0010	0.9980	0.0010	15.0154	0.10
20	0	0.0020	0.9980	0.0010	0.9979	0.0010	20.0129	0.06
25	0	0.0020	0.9980	0.0010	0.9980	0.0010	24.9880	0.05
30	0	0.0020	0.9980	0.0010	0.9980	0.0010	30.0151	0.05

Table 12: Simulated performances for the schemes under Gaussian quadrature formulas in Table 4.

N	When μ_1 is true Probability of accepting			When μ_0 is true Probability of accepting			$\text{ASN}'(\mu_1)$	ε (%)
	μ_{-1}	μ_0	μ_1	μ_{-1}	μ_0	μ_1		
5	0	0.0499	0.9501	0.0252	0.9500	0.0248	4.9973	0.05
10	0	0.0499	0.9501	0.0254	0.9496	0.0250	9.9876	0.12
15	0	0.0505	0.9495	0.0250	0.9499	0.0252	15.0181	0.12
20	0	0.0502	0.9498	0.0249	0.9502	0.0249	20.0105	0.05
25	0	0.0498	0.9502	0.0251	0.9500	0.0249	24.9896	0.04
30	0	0.0499	0.9500	0.0250	0.9502	0.0248	30.0180	0.06

4. Conclusions and Remarks

For the three-hypothesis test problems, the methods of designing the hypotheses, together with obtaining the corresponding test schemes, are proposed by adopting asymptotic formulas or numerical quadrature formulas in this paper. As a helpful guide for practitioners, they aid to directly find proper hypotheses under controlled risks and costs in preventing from too many iterative trials on combinations of hypotheses to meet practical needs.

The asymptotic formulas and the numerical quadrature formulas are both alternative tools for the hypothesis designs. Several aspects should be considered when choosing between them in applications.

(1) The method with numerical quadrature formulas outperforms the one under the asymptotic formulas especially when the error probabilities are not very small, as the example shows. In reality, the required error probabilities always range from 0.05 to 0.30 in sequential inspections, which seems to suggest choosing the numerical quadrature formulas to obtain efficient designs.

(2) In computation, the asymptotic formulas provide great convenience for applications, while the numerical quadrature formulas demand much iterative computational work especially when the number of numerical quadrature roots is large. But from the computation with the 64 Gaussian quadrature roots in the example, the time it costs in a common computer is tolerable if the start values for the system of equations are proper. We recommend finding the designs under the asymptotic formulas first, and then apply them as starts to obtain more efficient hypotheses from the numerical quadrature formulas when needed.

(3) When adopting the asymptotic formulas, the expressions for the quantities $D_{\theta_i}, O_{\theta_i}$ $(i = -1, 0, 1)$, and v should be developed for a specific distribution (see [23]). For the use of numerical quadrature formulas, the quadrature roots may be particularly arranged to fit the support points in the discrete distributions (e.g, see Reynolds and Stoumbos [25]). And for the θ_{ASN} out of $(\theta_{-1}, \theta_0, \theta_1)$, only the method with numerical quadrature formulas may take effect.

Actually, the two methods may apply to any distribution out of the Koopman-Darmois family. However, the test schemes under these distributions may be different from that in Figure 1, and the numerical quadrature formulas should be changed according to the test scheme patterns.

For the hypothesis designs asymmetric with the null hypothesis or the multihypothesis test problems, the methods proposed in this paper are still applicable by some extensions of adding more constraints on the designs. The hypothesis design problems under other requests, for example, under the desire of stopping sampling before a limit guaranteed by a provided probability, are still open to scholars and practitioners.

Appendix

Illustration of Airbag Quality Inspection

According to Li et al. [26], the airbag deployment pressure rate per unit of time, which is always assumed to conform with a standard normal distribution after some standardized transformation, is a key index of the airbag quality. The concerned problem here is whether the quality index is zero, positive, or negative. This quality index is measured in a 100 cubic feet testing air tank with sensors and the inspection is destructive. Since the airbag is expensive, the three-hypothesis sequential test scheme is needed to reduce the average inspection costs.

Suppose that the two error probabilities are $\alpha = \beta = 0.05$ and the required $\mathrm{ASN}(\mu_0)$ is no more than 5. Then, the hypothesis designs and test schemes of "$N = 5$" in Table 2 should be taken, that is, $(k, n_0, a, c, d, \tan\psi, \tan\varphi) = (1.6066, 2.0072, 2.0438, 0, -2.0438, 0.8033, -0.8033)$ under the method with asymptotic formulas and $(k, n_0, a, c, d, \tan\psi, \tan\varphi) = (1.3025, 2.1953, 2.2175, 0, -2.2175, 0.6513, -0.6513)$ under the method with Gaussian quadrature formulas.

Table 13: Test process under the test scheme from asymptotic formulas.

j	X_j	T_j	$a + j\tan\psi$	$c + (j - n_0)\tan\psi$ when $j \geq n_0$	$c + (j - n_0)\tan\varphi$ when $j \geq n_0$	$d + j\tan\varphi$	Decision
1	0.5689	0.5689	2.8471	*	*	−2.8471	*
2	−0.2556	0.3133	3.6504	*	*	−3.6504	*
3	−0.3775	−0.0642	4.4537	0.7975	−0.7975	−4.4537	Accept H_0

Table 14: Test process under the test scheme from Gaussian quadrature formulas.

j	X_j	T_j	$a + j\tan\psi$	$c + (j - n_0)\tan\psi$ when $j \geq n_0$	$c + (j - n_0)\tan\varphi$ when $j \geq n_0$	$d + j\tan\varphi$	Decision
1	0.5689	0.5689	2.8688	*	*	−2.8688	*
2	−0.2556	0.3133	3.5201	*	*	−3.5201	*
3	−0.3775	−0.0642	4.1714	0.5241	−0.5241	−4.1714	Accept H_0

Under the method with asymptotic formulas, the hypothesis test problem should be

$$H_{-1} : \mu = -1.6066 \quad \text{vs. } H_0 : \mu = 0 \quad \text{vs. } H_1 : \mu = 1.6066. \tag{A.1}$$

Taking the simulated observations from $N(0,1)$ by Li et al. in [26], we may reach a decision of accepting H_0 when $T_3 = -0.0642$ falls in $[c+(3-n_0)\tan\varphi, c+(3-n_0)\tan\psi] = [-0.7975, 0.7975]$ according to the test process in Table 13.

Under the method with Gaussian quadrature formulas, the hypothesis test problem should be

$$H_{-1} : \mu = -1.3025 \quad \text{vs. } H_0 : \mu = 0 \quad \text{vs. } H_1 : \mu = 1.3025. \tag{A.2}$$

Also taking the simulated observations from $N(0,1)$ by Li et al. in [26], we may accept H_0 after inspecting the third airbag according to the test process in Table 14.

Acknowledgments

This work was partly supported by the National Natural Science Foundation of China (NSFC) under the project Grants numbers 60573125 and 60873264. The authors cordially thank the editor and anonymous referees for their helpful reviews.

References

[1] K. S. Fu, *Sequential Methods in Pattern Recognition and Learning*, Academic Press, New York, NY, USA, 1968.

[2] W. E. Waters, "Sequential sampling in forest insect surveys," *Forest Science*, vol. 1, pp. 68–79, 1955.

[3] B. Lye and R. N. Story, "Spatial dispersion and sequential sampling plan of the southern green stink bug on fresh market tomatoes," *Environmental Entomology*, vol. 18, no. 1, pp. 139–144, 1989.

[4] T. McMillen and P. Holmes, "The dynamics of choice among multiple alternatives," *Journal of Mathematical Psychology*, vol. 50, no. 1, pp. 30–57, 2006.

[5] J. J. Bussgang, "Sequential methods in radar detection," *Proceedings of the IEEE*, vol. 58, no. 5, pp. 731–743, 1970.

[6] E. Grossi and M. Lops, "Sequential along-track integration for early detection of moving targets," *IEEE Transactions on Signal Processing*, vol. 56, no. 8, pp. 3969–3982, 2008.
[7] N. A. Goodman, P. R. Venkata, and M. A. Neifeld, "Adaptive waveform design and sequential hypothesis testing for target recognition with active sensors," *IEEE Journal on Selected Topics in Signal Processing*, vol. 1, no. 1, pp. 105–113, 2007.
[8] S. L. Anderson, "A simple method of comparing the breaking loads of two yarns," *Textile Institute*, vol. 45, pp. 472–479, 1954.
[9] C. Liteanu and I. Rica, *Statistical Theory and Methodology of Trace Analysis*, Halsted, New York, NY, USA, 1980.
[10] A. G. Tartakovsky, B. L. Rozovskii, R. B. Blažek, and H. Kim, "Detection of intrusions in information systems by sequential change-point methods," *Statistical Methodology*, vol. 3, no. 3, pp. 252–293, 2006.
[11] G. B. Wetherill and K. D. Glazebrook, *Sequential Methods in Statistics*, Monographs on Statistics and Applied Probability, Chapman & Hall, London, UK, 3rd edition, 1986.
[12] T.-H. Chen, C.-Y. Chen, H.-C. P. Yang, and C.-W. Chen, "A mathematical tool for inference in logistic regression with small-sized data sets: a practical application on ISW-ridge relationships," *Mathematical Problems in Engineering*, vol. 2008, Article ID 186372, 12 pages, 2008.
[13] T. F. Oliveira, R. B. Miserda, and F. R. Cunha, "Dynamical simulation and statistical analysis of velocity fluctuations of a turbulent flow behind a cube," *Mathematical Problems in Engineering*, vol. 2007, Article ID 24627, 28 pages, 2007.
[14] M. Li and W. Zhao, "Variance bound of ACF estimation of one block of fGn with LRD," *Mathematical Problems in Engineering*, vol. 2010, Article ID 560429, 14 pages, 2010.
[15] M. Li, W.-S. Chen, and L. Han, "Correlation matching method for the weak stationarity test of LRD traffic," *Telecommunication Systems*, vol. 43, no. 3-4, pp. 181–195, 2010.
[16] E. G. Bakhoum and C. Toma, "Relativistic short range phenomena and space-time aspects of pulse measurements," *Mathematical Problems in Engineering*, vol. 2008, Article ID 410156, 20 pages, 2008.
[17] C. Cattani, "Harmonic wavelet approximation of random, fractal and high frequency signals," *Telecommunication Systems*, vol. 43, no. 3-4, pp. 207–217, 2010.
[18] C. Cattani and A. Kudreyko, "Application of periodized harmonic wavelets towards solution of eigenvalue problems for integral equations," *Mathematical Problems in Engineering*, vol. 2010, Article ID 570136, 8 pages, 2010.
[19] P. Armitage, "Sequential analysis with more than two alternative hypotheses, and its relation to discriminant function analysis," *Journal of the Royal Statistical Society. Series B*, vol. 12, pp. 137–144, 1950.
[20] M. E. Payton and L. J. Young, "A sequential procedure for deciding among three hypotheses," *Sequential Analysis*, vol. 13, no. 4, pp. 277–300, 1994.
[21] M. E. Payton and L. J. Young, "A sequential procedure to test three values of a binomial parameter," *Metrika*, vol. 49, no. 1, pp. 41–52, 1999.
[22] V. P. Dragalin, A. G. Tartakovsky, and V. V. Veeravalli, "Multihypothesis sequential probability ratio tests. I. Asymptotic optimality," *IEEE Transactions on Information Theory*, vol. 45, no. 7, pp. 2448–2461, 1999.
[23] V. P. Dragalin, A. G. Tartakovsky, and V. V. Veeravalli, "Multihypothesis sequential probability ratio tests. II. Accurate asymptotic expansions for the expected sample size," *IEEE Transactions on Information Theory*, vol. 46, no. 4, pp. 1366–1383, 2000.
[24] Y. Li and X. L. Pu, "A method on designing three-hypothesis test problems," to appear in *Communications in Statistics—Simulation and Computation*.
[25] M. R. Reynolds Jr. and Z. G. Stoumbos, "The SPRT chart for monitoring a proportion," *IIE Transactions*, vol. 30, no. 6, pp. 545–561, 1998.
[26] Y. Li, X. L. Pu, and F. Tsung, "Adaptive charting schemes based on double sequential probability ratio tests," *Quality and Reliability Engineering International*, vol. 25, no. 1, pp. 21–39, 2009.