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Research Article

Unsteady Unidirectional Flow of Second-Grade Fluid through a Microtube with Wall Slip and Different Given Volume Flow Rate

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The second-grade flows through a microtube with wall slip are solved by Laplace transform technique. The effects of rarefaction and elastic coefficient are considered with an unsteady flow through a microtube for a given but arbitrary inlet volume flow rate with time. Five cases of inlet volume flow rate are as follows: (1) trapezoidal piston motion, (2) constant acceleration, (3) impulsively started flow, (4) impulsively blocked fully developed flow, and (5) oscillatory flow. The results obtained are compared to those solutions under no-slip and slip condition.

1. Introduction

During the past decades, a great deal of literatures used the Navier-Stokes equation to describe the Newtonian fluid. However, the Newtonian fluid is the simplest to be solved and its application is very limited. In practice, many complex fluids such as blood, soaps, clay coatings, certain oils and greases, elastomers, suspensions, and many emulsions have been treated as non-Newtonian fluids. From literatures, the non-Newtonian fluids are mainly classified into three categories on the basis of their behavior in shear. (a) The shear stress of the fluids depends only on the rate of shear. (b) A fluid with a relationship between the shear stress and shear rate. (c) The fluids possess both elastic and viscous properties. One of the most popular models for non-Newtonian fluids is called the second-grade fluid. Erdoğan and İmrak [1] used the second-order Rivlin-Ericksen fluid, or the so-called second-grade fluid to model the non-Newtonian fluid. They solved some unsteady flows, such as unsteady flow over a plane wall, unsteady Couette flow, and unsteady Poiseuille flow. Hayat et al. [2]

analysed the influence of variable viscosity and viscous dissipation on the non-Newtonian flow. Bandelli and Rajagopal [3] solved various startup flows of second-grade fluids in domains with one finite dimension by integral transform method. Some unsteady flows of the fluids of second grade have been investigated by many authors [4–7].

Microfluidics is the significant technologies developed in the engineering field. As the microflow is considered, the no-slip condition is not sufficient for a fluid of second grade. Rarefaction phenomena should be considered when fluid flows in a microtube. The typically flow field can be divided into the four regimes by Knudsen number [8]: $\mathrm{Kn} < 10^{-3}$, continuum flow; $10^{-3} \leq \mathrm{Kn} < 10^{-1}$, slipflow; $10^{-1} \leq \mathrm{Kn} < 10$, transition flow; and $10 \leq \mathrm{Kn}$ free molecular flow. Much research in the literature does not consider the effect of rarefaction in the second-grade fluid. The study examined the effects of rarefaction of an unsteady flow through a microtube by Chen et al. [9].

In practice application, generally the inlet volume flow rate is a given condition instead of pressure gradient. Das and Arakeri [10] solved the unsteady laminar duct flow with a given volume flow rate variation. They discussed the problem with various types of given inlet piston motion in the channel and duct. Also Das and Arakeri [11] verified their earlier experimental work. Chen et al. [12–14] extended Das and Arakeri's work by considering various non-Newtonian fluids. Several studies [15, 16] have suggested the no-slip condition that is deduced as the limiting cases when the slip parameter is equal to zero. Hayat et al. [17] considered the unsteady flow of an incompressible second-grade fluid in a circular duct with a given volume flow rate variation. The effects of Hall current are taken into account. For the above reason, this study considers the wall slip condition and the second-grade fluid with different given volume flow rate. For $\alpha_1 \rightarrow 0$, they reduce to the similar solutions for Newtonian fluids. The results show that the analytical solutions of velocity profile and pressure gradient are affected by the slip conditions and the viscoelastic parameter.

2. Constitutive Equations

The constitutive equation of second-grade Rivlin-Ericksen fluid is in the following form:

$$\vec{T} = -p\,\vec{I} + \mu\,\vec{A}_1 + \alpha_1\,\vec{A}_2 + \alpha_2 \vec{A}_1^{\,2}, \tag{2.1}$$

where $\vec{T}$ is the stress tensor, $\vec{I}$ is identity tensor, p is the static fluid pressure, μ is the dynamic viscosity coefficient, α_1 is the elastic coefficient and α_2 is the transverse viscosity coefficient, and $\vec{A}_1$ and $\vec{A}_2$ represent the Rivlin-Ericksen tensors. Here, μ, α_1 and α_2 are material modules which are assumed constant. The kinematic tensors $\vec{A}_1$ and $\vec{A}_2$ are defined as

$$\begin{gathered}
\vec{A}_1 = \left(\operatorname{grad} \vec{V}\right) + \left(\operatorname{grad} \vec{V}\right)^T, \\
\vec{A}_2 = \frac{d\,\vec{A}_1}{dt} + \vec{A}_1 \left(\operatorname{grad} \vec{V}\right) + \left(\operatorname{grad} \vec{V}\right)^T \vec{A}_1 .
\end{gathered} \tag{2.2}$$

In the above equation, $\vec{V}$ is the velocity and d/dt denotes the material time derivative. Since the fluid is incompressible, it can undergo only isochoric motion, and hence

$$\operatorname{div} \vec{V} = 0, \tag{2.3}$$

and substituting constitutive equation (2.1) into the balance of linear momentum

$$\operatorname{div} \vec{T} + \rho \vec{b} = \rho \frac{d \vec{V}}{dt}, \tag{2.4}$$

where ρ is the density of the fluid and $\vec{b}$ is the body force. In the sense of the Clausius-Duhem inequality and the condition that the Helmholtz free energy is a minimum when the fluid is at rest, then the material modules must be satisfied [7] as follows:

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0. \tag{2.5}$$

In our study, we use the cylindrical polar coordinates (r, θ, x), where r is radial distance from the center of the pipe, θ is the angular direction, and x is the axial direction. Velocity in the x, r, and θ-direction are u, u_r, and u_θ, respectively. We also investigate the fluid rarefaction effect in a microtube, the Knudsen number is an important nondimensional parameter

$$\mathrm{Kn} \equiv \frac{\lambda}{L}, \tag{2.6}$$

where λ is the molecular mean free path, which is defined as the mean secondary collision distance of a gas molecule, and L is the characteristic length.

In order to find the fluid of second grade for unsteady unidirectional flows, we seek a velocity field of the form $u = u(r, t)$, $u_r = 0$, $u_\theta = 0$. The governing equations can be derived from (2.4), which gives

$$\begin{gathered} \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - \rho \frac{\partial u}{\partial t} = \frac{\partial p}{\partial x}, \\ \frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0, \end{gathered} \tag{2.7}$$

where ν is the kinematic viscosity. This implies that the pressure gradient is a function of time only.

3. Methodology of Solution

The problem can be solved if the governing equation, boundary condition, and initial condition are known. This third-order nonhomogeneous partial differential equation is not convenient to use the method of separation of variable to solve. In this paper, we give the

Laplace transform method reducing the two variables into single variable. In other words, we transform PDE into ODE that will effectively reduce the original difficult equation.

The governing equation of motion in x-direction is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(\nu + \theta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{3.1}$$

where $\nu = \mu/\rho$ and $\theta = \alpha_1/\rho$.

With R is the radius of duct, the boundary conditions are

$$u(R,t) = -\beta_\nu\lambda\frac{\partial u(R,t)}{\partial r}, \qquad \frac{\partial u(0,t)}{\partial r} = 0, \tag{3.2}$$

where $\beta_\nu\lambda$ is the velocity slip coefficient and is defined as

$$\beta_\nu = \frac{2 - F_\nu}{F_\nu}, \tag{3.3}$$

and F_ν is the tangential momentum accommodation coefficient that describes the interaction between fluid and wall and is related to constituents of fluid, temperature, velocity, wall temperature, roughness, and chemical status. F_ν is defined as

$$F_\nu \equiv \frac{u_i - u_{re}}{u_i - U_w}, \tag{3.4}$$

where u_i, u_{re}, and U_w are tangential momentum of incoming molecules, reflected molecules, and re-emitted molecules, respectively.

We need an initial condition for the velocity, $u(r,0)$, and the problem can be solved if the pressure gradient function is known. In our case, we determined the pressure gradient indirectly by the volume flow rate, which is given. The velocity is related to the inlet volume flow rate by

$$\int_0^R 2\pi r u(r,t)dr = u_p(t)\pi R^2 = Q(t), \tag{3.5}$$

where u_p is the average inlet velocity.

Using the Laplace transform technique of (3.1), (3.2), and (3.5) yields the following equations:

$$\frac{d^2u(r,s)}{dr^2}+\frac{1}{r}\frac{du(r,s)}{dr}-\frac{s}{\nu+s\theta}u(r,s)=\frac{1}{\rho(\nu+s\theta)}\frac{dP(x,s)}{dx}, \tag{3.6}$$

$$u(R,s)=-\beta_\nu\lambda\frac{du(R,s)}{dr}, \tag{3.7}$$

$$\frac{du(0,s)}{dr}=0, \tag{3.8}$$

$$Q(s)=\int_0^R 2\pi r u(r,s)dr=u_p(s)\pi R^2. \tag{3.9}$$

Equation (3.6) is a second-order inhomogeneous ordinary differential equation. The homogeneous part is the modified Bessel's equation of zeroth order and assuming the particular integral as Ψ_p, the general solution is

$$u(r,s)=C_1I_0(mr)+C_2K_0(mr)+\Psi_p, \tag{3.10}$$

where $m=\sqrt{s/(\nu+s\theta)}$.

Using the boundary conditions (3.7) and (3.8) into (3.6) to solve these two unknown coefficients C_1 and C_2, substituting C_1 and C_2 into (3.6) give

$$u(r,s)=\Psi_p\left(1-\frac{I_0(mr)}{\left[I_0(mR)+\beta_\nu\lambda mI_1(mR)\right]}\right), \tag{3.11}$$

where I_1 is the modified Bessel's equation of the first order.

To solve for the unknown Ψ_p, we substitute (3.11) into (3.9) and Ψ_p is obtained as

$$\Psi_p=\frac{u_p(s)}{\left(1-\left(2I_1(mR)/mR\left[I_0(mR)+\beta_\nu\lambda mI_1(mR)\right]\right)\right)}, \tag{3.12}$$

Substituting Ψ_p into (3.11), we get

$$u(r,s)=\frac{u_p(s)\{[I_0(mR)+\alpha mRI_1(mR)]-I_0(mr)\}}{\{[I_0(mR)+\alpha mRI_1(mR)]-2I_1(mR)/mR\}}, \tag{3.13}$$

or

$$u(r,s)=u_p(s)\cdot G(r,s), \tag{3.14}$$

where

$$G(r,s) = \frac{[I_0(mR) + \alpha mRI_1(mR)] - I_0(mr)}{[I_0(mR) + \alpha mRI_1(mR)] - 2I_1(mR)/mR} \tag{3.15}$$

and $\alpha = \beta_v \lambda / R = \beta_v \mathrm{Kn} \approx \mathrm{Kn}$.

Taking the inverse Laplace transform, the velocity profile is

$$u(r,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} u_p(s)G(r,s)e^{st}ds. \tag{3.16}$$

Furthermore, the pressure gradient is found by substituting (3.11) into (3.6) to obtain

$$\frac{dp(x,s)}{dx} = -u_p(s)\frac{[I_0(mR) - \alpha mRI_1(mR)]}{[I_0(mR) - \alpha mRI_1(mR)] - 2I_1(mR)/mR}, \tag{3.17}$$

We obtain the expressions for the variation of nondimensional pressure gradient with time by taking the inverse transform formula.

4. Illustration of Examples

We consider some examples proposed by Das and Arakeri [10] with the second-grade fluid and the effect of wall-slip conditions on the unsteady flow patterns in a microtube.

For the following case, the velocity moves with a constant acceleration of the piston starting from rest, and the other one, the piston suddenly starts from rest and then keeping this velocity. These two solutions we apply to the trapezoidal motion, that is, the piston has three stages: constant acceleration of the piston starting from rest, a period of constant velocity, and a constant deceleration of the piston to a stop.

4.1. Trapezoidal Piston Motion

We get the solution for the three stages piston velocities which vary with time as follows:

$$u_p(t) = \begin{cases} \dfrac{U_p}{t_0}t, & \text{for } 0 \le t \le t_0, \\ U_p, & \text{for } t_0 \le t \le t_1, \\ U_p\dfrac{(t_2 - t)}{(t_2 - t_1)}, & \text{for } t_1 \le t \le t_2, \\ 0, & \text{for } t_2 \le t \le \infty, \end{cases} \tag{4.1}$$

where U_p is the constant velocity after acceleration, and t_0, t_1, and t_2 are the time periods for changing piston velocity. We use the Heaviside unit step function to describe the piston motion as follows:

$$\begin{aligned} u_p(t) = {} & \frac{U_p}{t_0} t H(t) - \frac{U_p}{t_0} t H(t - t_0) + U_p H(t - t_0) - U_p H(t - t_1) \\ & + U_p \frac{t_2 - t}{t_2 - t_1} H(t - t_1) - U_p \frac{t_2 - t}{t_2 - t_1} H(t - t_2). \end{aligned} \tag{4.2}$$

For the constant acceleration period $(0 \le t \le t_0)$, taking the Laplace transform of $u_p(t) = U_p t H(t) / t_0$, we get

$$u_p(s) = \frac{U_p}{t_0 s^2}. \tag{4.3}$$

From (3.16) expression, the integration is determined using complex variable theory, as discussed by Arparci [18]. We obtain the velocity distribution

$$u(r, t) = \frac{1}{2\pi i} \left[2\pi i \sum_{j=1} R_j \right], \tag{4.4}$$

where R_j is the residual of poles of $U_p e^{st} G(r, s) / t_0 s^2$.

It can be easily observed that $s = 0$ is a pole of order 2. Therefore, the residue at $s = 0$ is

$$\operatorname{Res}(0) = \frac{U_p}{t_0} \left\{ \frac{2t\left[1 - (r/R)^2 + 2\alpha\right]}{(1 + 4\alpha)} + \frac{R^2\left[1 - (r/R)^4 + 4\alpha\right]}{8\nu(1 + 4\alpha)} - \frac{R^2(1 + 6\alpha)\left[1 - (r/R)^2 + 2\alpha\right]}{6\nu(1 + 4\alpha)^2} \right\}. \tag{4.5}$$

The other singular points are the zeroes of

$$I_0(mR) + \alpha m R I_1(mR) - \frac{2 I_1(mR)}{mR} = 0. \tag{4.6}$$

Setting $mR = i\lambda$, we find that

$$\alpha \lambda J_1(\lambda) + J_2(\lambda) = 0. \tag{4.7}$$

If λ_n, $n = 1, 2, 3, \ldots, \infty$ are zeros of (4.7), then $s_n = -\lambda_n^2 \nu / (R^2 + \theta \lambda_n^2)$, $n = 1, 2, 3, \ldots, \infty$ are the simple poles. Since all λ_n are symmetrically placed about zero on the real axis, all the poles

(s_n) lie on the negative real axis. These are simple poles, and residues at all these poles can be obtained as

$$\operatorname{Res}(s_n) = \frac{U_p R^2}{t_0 \nu} \left\{ \frac{2[J_0(\lambda_n) - J_0((r/R)\lambda_n) - \alpha\lambda_n J_1(\lambda_n)]}{[(1+2\alpha)\lambda_n^3 J_1(\lambda_n) + \alpha\lambda_n^4 J_0(\lambda_n)]} e^{-\nu\lambda_n^2 t/(R^2+\theta\lambda_n^2)} \right\}. \tag{4.8}$$

Adding $\operatorname{Res}(0)$ and $\operatorname{Res}(s_n)$, the dimensionless velocity distribution is obtained as

$$\begin{aligned} u^*(c,t^*) &= \frac{1}{t_0^*}\left\{ \frac{2t^*(1-c^2+2\alpha) + (1/8)(1-c^4+4\alpha)}{(1+4\alpha)} - \frac{(1/6)(1+6\alpha)(1-c^2+2\alpha)}{(1+4\alpha)^2} \right\} \\ &\quad + \frac{2}{t_0^*}\sum_{n=1}^{\infty}\left\{ e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} \frac{J_0(\lambda_n) - J_0(c\lambda_n) - \alpha\lambda_n J_1(\lambda_n)}{[(1+2\alpha)\lambda_n^3 J_1(\lambda_n) + \alpha\lambda_n^4 J_0(\lambda_n)]} \right\}, \end{aligned} \tag{4.9}$$

where $u^* = u_p/U_p$, $c = r/R$, $\alpha = \beta_\nu \mathrm{Kn}$, $t^* = t\nu/R^2$, $t_0^* = t_0\nu/R^2$, $\beta = \theta/R^2$.

By the same method, the dimensionless velocity profile during the constant piston velocity $(t_0 \le t \le t_1)$ is obtained as

$$\begin{aligned} u_p(t) &= \frac{U_p}{t_0} t H(t) - \frac{U_p}{t_0} t H(t-t_0) + U_p H(t-t_0), \\ u^*(c,t^*) &= 2\left[\frac{(1-c^2)+2\alpha}{1+4\alpha}\right] + \frac{2}{t_0^*}\sum_{n=1}^{\infty}\left\{ \left[e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} - e^{-\lambda_n^2 (t^*-t_0^*)/(1+\beta\lambda_n^2)} \right] \right. \\ &\qquad \left. \times \frac{J_0(\lambda_n) - J_0(c\lambda_n) - \alpha\lambda_n J_1(\lambda_n)}{[(1+2\alpha)\lambda_n^3 J_1(\lambda_n) + \alpha\lambda_n^4 J_0(\lambda_n)]} \right\}, \end{aligned} \tag{4.10}$$

during the constant deceleration of the piston motion $(t_1 \le t \le t_2)$.

$$\begin{aligned} u_p(t) &= \frac{U_p}{t_0} t H(t) - \frac{U_p}{t_0} t H(t-t_0) + U_p H(t-t_0) - U_p H(t-t_1) + U_p \frac{t_2 - t}{t_2 - t_1} H(t-t_1), \\ \mathrm{u}^*(c,t^*) &= \frac{1}{t_2^* - t_1^*}\left\{ \frac{2(t_2^*-t_1^*)(1-c^2+2\alpha) - (1/8)(1-c^4+4\alpha)}{(1+4\alpha)} + \frac{(1/6)(1+6\alpha)(1-c^2+2\alpha)}{(1+4\alpha)^2} \right\} \\ &\quad + 2\sum_{n=1}^{\infty}\left\{ \left[\frac{e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} - e^{-\lambda_n^2 (t^*-t_0^*)/(1+\beta\lambda_n^2)}}{t_0^*} - \frac{e^{-\lambda_n^2 (t^*-t_1^*)/(1+\beta\lambda_n^2)}}{t_2^* - t_1^*} \right] \right. \\ &\qquad \left. \times \frac{J_0(\lambda_n) - J_0(c\lambda_n) - \alpha\lambda_n J_1(\lambda_n)}{[(1+2\alpha)\lambda_n^3 J_1(\lambda_n) + \alpha\lambda_n^4 J_0(\lambda_n)]} \right\}, \end{aligned} \tag{4.11}$$

where $t_1^* = t_1\nu/R^2$, $t_2^* = t_2\nu/R^2$.

And after the piston has stopped ($t_2 \le t \le \infty$),

$$u_p(t) = \frac{U_p}{t_0} t H(t) - \frac{U_p}{t_0} t H(t-t_0) + U_p H(t-t_0) - U_p H(t-t_1) + U_p \frac{t_2 - t}{t_2 - t_1} H(t-t_1) - U_p \frac{t_2 - t}{t_2 - t_1} H(t-t_2),$$

$$u^*(c,t^*) = 2\sum_{n=1}^{\infty}\left\{\left[\frac{e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} - e^{-\lambda_n^2 (t^*-t_0^*)/(1+\beta\lambda_n^2)}}{t_0^*} - \frac{e^{-\lambda_n^2 (t^*-t_1^*)/(1+\beta\lambda_n^2)} - e^{-\lambda_n^2 (t^*-t_2^*)/(1+\beta\lambda_n^2)}}{t_2^* - t_1^*}\right] \times \frac{J_0(\lambda_n) - J_0(c\lambda_n) - \alpha\lambda_n J_1(\lambda_n)}{\left[(1+2\alpha)\lambda_n^3 J_1(\lambda_n) + \alpha\lambda_n^4 J_0(\lambda_n)\right]}\right\}. \tag{4.12}$$

We also obtain the expressions of the dimensionless pressure gradient during these four different stages. During the constant acceleration period ($0 \le t \le t_0$),

$$\frac{dp^*}{dx^*} = -\frac{1}{t_0^*}\left[\frac{t^* + (\alpha/2) + (1/4)}{(1+4\alpha)} - \frac{(1/12) + (\alpha/2)}{(1+4\alpha)^2}\right] + \frac{1}{4t_0^*}\sum_{n=1}^{\infty}\left\{e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} \frac{J_0(\lambda_n) - \alpha\lambda_n J_1(\lambda_n)}{\left[(1+2\alpha)\lambda_n J_1(\lambda_n) + \alpha\lambda_n^2 J_0(\lambda_n)\right]}\right\}, \tag{4.13}$$

where $P^* = P/(8\mu U_p/R)$, $x^* = x/R$.

During the constant velocity period ($t_0 \le t \le t_1$),

$$\frac{dp^*}{dx^*} = -\frac{1}{1+4\alpha} + \frac{1}{4t_0^*}\sum_{n=1}^{\infty}\left\{\left[e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} - e^{-\lambda_n^2 (t^*-t_0^*)/(1+\beta\lambda_n^2)}\right] \frac{J_0(\lambda_n) - \alpha\lambda_n J_1(\lambda_n)}{\left[(1+2\alpha)\lambda_n J_1(\lambda_n) + \alpha\lambda_n^2 J_0(\lambda_n)\right]}\right\}, \tag{4.14}$$

during the constant deceleration period ($t_1 \le t \le t_2$),

$$\frac{dp^*}{dx^*} = -\frac{1}{t_2^* - t_1^*}\left[\frac{(t_2^* - t_1^*) - ((\alpha/2) + (1/4))}{(1+4\alpha)} + \frac{(1/12) + (\alpha/2)}{(1+4\alpha)^2}\right] + \sum_{n=1}^{\infty}\left\{\left[\frac{e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} - e^{-\lambda_n^2 (t^*-t_0^*)/(1+\beta\lambda_n^2)}}{4t_0^*} - \frac{e^{-\lambda_n^2 (t^*-t_1^*)/(1+\beta\lambda_n^2)}}{4(t_2^* - t_1^*)}\right] \times \frac{J_0(\lambda_n) - \alpha\lambda_n J_1(\lambda_n)}{\left[(1+2\alpha)\lambda_n J_1(\lambda_n) + \alpha\lambda_n^2 J_0(\lambda_n)\right]}\right\}, \tag{4.15}$$

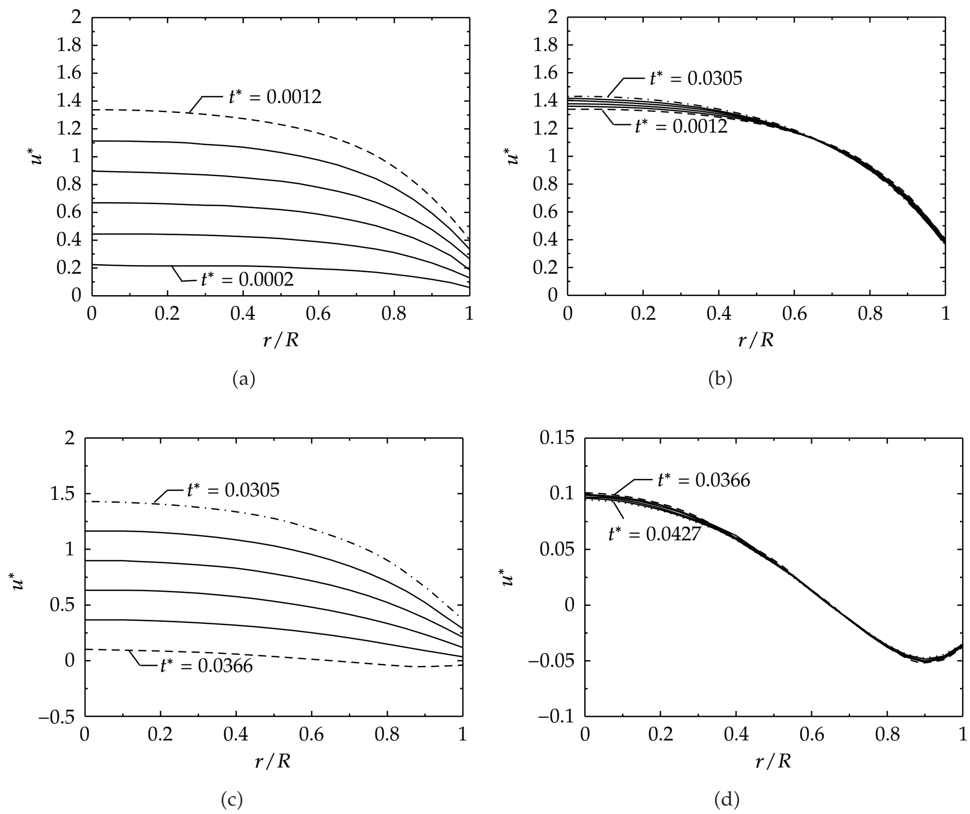


Figure 1: Velocity profiles at different phases at Kn = 0.1 and $\beta = 0.05$ (a) during the acceleration of the piston motion (profiles are shown at time intervals of $t_0^*/6$), (b) when the piston velocity is constant (time intervals of $(t_1^* - t_0^*)/6$), (c) during the deceleration of the piston velocity (time intervals of $(t_2^* - t_1^*)/6$), and (d) after the piston motion has stopped (time intervals of $(0.0427 - t_2^*)/6$).

and after the piston has stopped ($t_2 \le t \le \infty$),

$$\frac{dp^*}{dx^*} = \sum_{n=1}^{\infty}\left\{\left[\frac{e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} - e^{-\lambda_n^2 (t^*-t_0^*)/(1+\beta\lambda_n^2)}}{4t_0^*} - \frac{e^{-\lambda_n^2 (t^*-t_1^*)/(1+\beta\lambda_n^2)} - e^{-\lambda_n^2 (t^*-t_2^*)/(1+\beta\lambda_n^2)}}{4\left(t_2^* - t_1^*\right)}\right]\right.$$
$$\left.\times\frac{J_0(\lambda_n) - \alpha\lambda_n J_1(\lambda_n)}{\left[(1+2\alpha)\lambda_n J_1(\lambda_n) + \alpha\lambda_n^2 J_0(\lambda_n)\right]}\right\}. \tag{4.16}$$

Above these infinite series, equations are convergent and we set the $n = 50$ as enough for the cases. For trapezoidal piston motion with different nondimensional times ($t^* = t\nu/R^2$) are $t_0\nu/R^2$, $t_1\nu/R^2$ and $t_2\nu/R^2 = 0.0012$, 0.0305, and 0.0366, respectively. The velocity profiles at Kn = 0.1 and $\beta = 0.05$ are illustrated in Figure 1.

These values are chosen for the purpose of comparing the results obtained by Das and Arakeri [10] and Chen et al. [9]. When $\alpha \approx \mathrm{Kn} = 0$ (no-slip condition) and $\beta = 0$ (no-elastic effect), the velocity profiles in (4.9) to (4.12) are exactly like Das and Chen's results. Figure 1 shows the second-grade flow with slippage on the microtube wall during four different time periods. The development of velocity is similar to that in Das et al. and Chen et al.'s works. However, the elastic coefficient retarded the change of velocity in the microtube. Because the effect of slippage, the shift of velocity from the wall to centerline is smoother than that in Das et al. and Chen et al.'s works. During the time period when the piston decelerates and stops at time t_2^*, it is observed that the flow reverses its direction near the wall (see Figure 1(c)). After the piston motion ceases, the velocity profile (see Figure 1(d)) continues to have reverse flow near the wall to satisfy the zero mass flow condition. Figure 2 shows the variation of nondimensional pressure gradients with time at $\mathrm{Kn} = 0.1$ and $\beta = 0.05$. During the acceleration and deceleration stages, the pressure gradients are large mainly because of fluid inertia. Finally, when the piston stops, the pressure gradient slowly decays to zero. Figure 3 shows the effect of different β values ($\beta = 0,\ 0.05,\ 0.1$) on the velocity profiles at $\mathrm{Kn} = 0.1$. During the four stages of piston motion, the larger β values the smoother the velocity profile.

The degree of smoothness is proportional to the β value. In the special case, it is worth mentioning that when $\beta \to 0$ (means that $\alpha_1 \to 0$), corresponding to Newtonian fluids, all solutions that have been obtained are going to be those for Newtonian fluids performing the same motions. Figure 4 shows the effect of Kn various values ($\mathrm{Kn} = 0,\ 0.05,\ 0.1$) on the velocity profiles at $\beta = 0.05$. The analytical result demonstrates that a larger Kn value will flatten the velocity profile. It is observed that the slip condition occurs near the wall.

4.2. Constant Acceleration Case

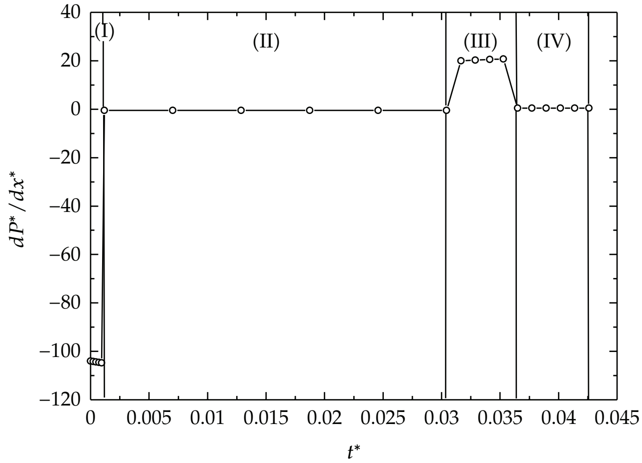
For a piston with constant acceleration can be described by the following equation:

$$u_p(t) = a_p t = \left(\frac{U_p}{t_0}\right)t, \tag{4.17}$$

where a_p is the constant acceleration, U_p is the final velocity after acceleration, and t_0 is the time period of acceleration.

The velocity profile can be obtained by putting $t = t_0$ of (4.9) as follows:

$$\begin{aligned} u^*(c,t^*) = {} & \frac{1}{t^*}\left\{\frac{2t^*(1-c^2+2\alpha)+(1/8)(1-c^4+4\alpha)}{(1+4\alpha)} - \frac{(1/6)(1+6\alpha)(1-c^2+2\alpha)}{(1+4\alpha)^2}\right\} \\ & + \frac{2}{t^*}\sum_{n=1}^{\infty}\left\{e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)}\frac{J_0(\lambda_n)-J_0(c\lambda_n)-\alpha\lambda_n J_1(\lambda_n)}{\left[(1+2\alpha)\lambda_n^3 J_1(\lambda_n)+\alpha\lambda_n^4 J_0(\lambda_n)\right]}\right\}, \end{aligned} \tag{4.18}$$



(I): Piston constant acceleration (III): Piston constant deceleration
(II): Piston constant velocity (IV): Piston stop

Figure 2: The variation of pressure gradient with time for trapezoidal piston motion at Kn = 0.1 and β = 0.05.

and when the pressure gradient, as time approaches infinity, is

$$\frac{dp^*}{dx^*} = -\frac{1}{(1+4\alpha)} - \frac{((\alpha/2)+(1/4))a_p R^2}{(1+4\alpha)\nu u_p} + \frac{((1/12)+(\alpha/2))a_p R^2}{(1+4\alpha)^2 \nu u_p} - \frac{1}{t^*}\left[\frac{t^*+(\alpha/2)+(1/4)}{(1+4\alpha)} - \frac{(1/12)+(\alpha/2)}{(1+4\alpha)^2}\right]. \tag{4.19}$$

4.3. *Suddenly Started Flow*

The solution to the suddenly started flow in a circular duct is as follows:

$$u_p(t) = \begin{cases} 0, & \text{for } t \le 0, \\ U_p, & \text{for } t > 0, \end{cases} \tag{4.20}$$

where U_p is the constant velocity

$$u^*(c,t^*) = 2\left[\frac{(1-c^2)+2\alpha}{1+4\alpha}\right] - 2\sum_{n=1}^{\infty}\left\{e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)}\frac{J_0(\lambda_n)-J_0(c\lambda_n)-\alpha\lambda_n J_1(\lambda_n)}{[\lambda_n J_1(\lambda_n)+2\alpha\lambda_n J_1(\lambda_n)+\alpha\lambda_n^2 J_0(\lambda_n)]}\right\}, \tag{4.21}$$

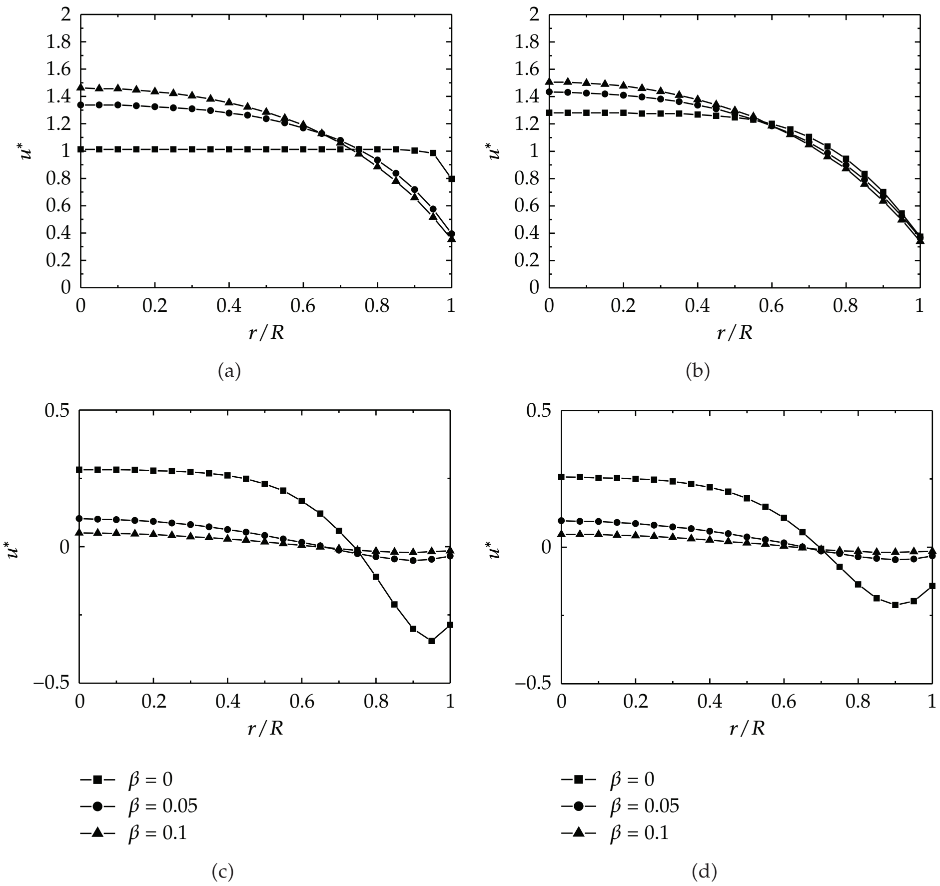


Figure 3: The effect of different β values on the velocity profiles for trapezoidal piston motion at Kn = 0.1: (a) $t^* = 0.0012$, (b) $t^* = 0.0305$, (c) $t^* = 0.0366$, and (d) $t^* = 0.0427$.

and the pressure gradient is

$$\frac{dp^*}{dx^*} = -\frac{1}{1+4\alpha} - \frac{1}{4}\sum_{n=1}^{\infty}\left\{ e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} \frac{\lambda_n [J_0(\lambda_n) - \alpha\lambda_n J_1(\lambda_n)]}{[(1+2\alpha)J_1(\lambda_n) + \alpha\lambda_n J_0(\lambda_n)]} \right\}. \tag{4.22}$$

4.4. Suddenly Blocked Fully Developed Flow

The exact solution of this problem with no-slip wall condition was considered by Weinbaum and Parker [19]. The initial condition for this problem is $u(r,0) = 1 - c^2$, and the mass flow condition is $\int_0^R 2\pi r u dr = 0$. The resulting velocity profile is

$$u^*(c,t^*) = -2\sum_{n=1}^{\infty} e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} \frac{J_0(\lambda_n) - J_0(c\lambda_n)}{\lambda_n J_1(\lambda_n)}. \tag{4.23}$$

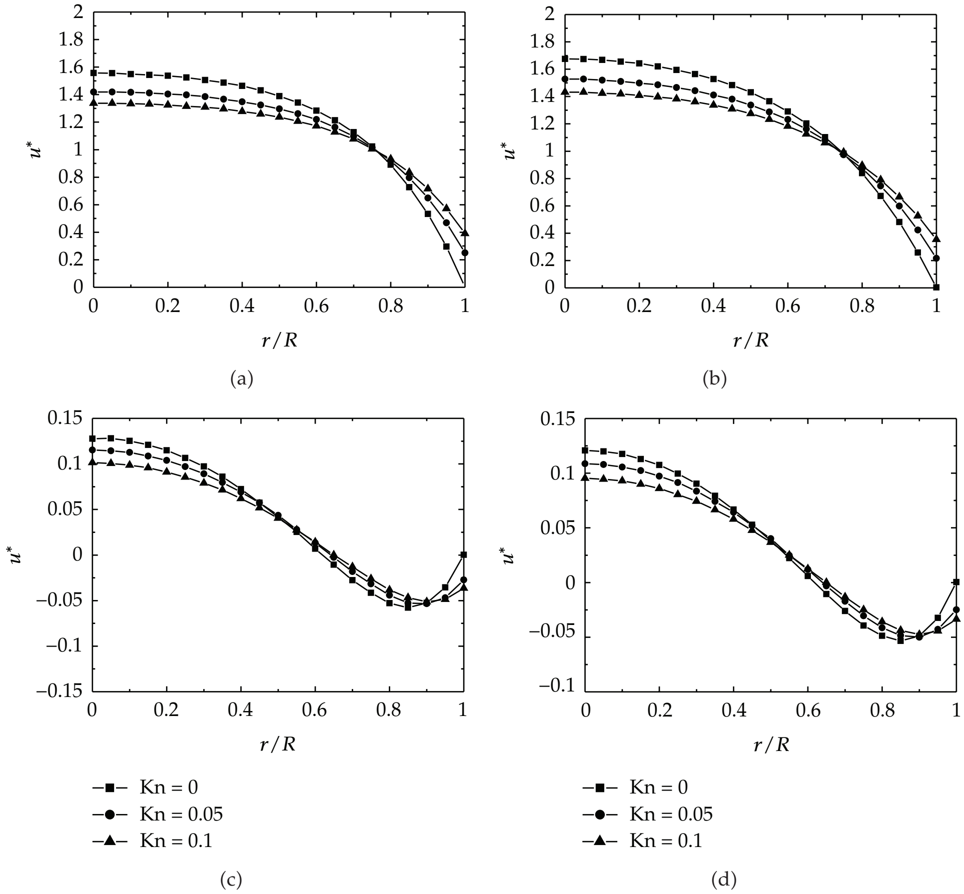


Figure 4: The effect of different Kn values on the velocity profiles for trapezoidal piston motion at $\beta = 0.05$: (a) $t^* = 0.0012$, (b) $t^* = 0.0305$, (c) $t^* = 0.0366$, and (d) $t^* = 0.0427$.

When the wall slip is considered, the corresponding velocity profile and pressure gradient are

$$u^*(c,t^*) = -2\sum_{n=1}^{\infty}\left\{ e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} \frac{J_0(\lambda_n) - J_0(c\lambda_n) - \alpha\lambda_n J_1(\lambda_n)}{\left[\lambda_n J_1(\lambda_n) + 2\alpha\lambda_n J_1(\lambda_n) + \alpha\lambda_n^2 J_0(\lambda_n)\right]} \right\},$$

$$\frac{dp^*}{dx^*} = \frac{1}{4t_0^*}\sum_{n=1}^{\infty}\left\{ e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)} \frac{J_0(\lambda_n) - \alpha\lambda_n J_1(\lambda_n)}{\left[(1+2\alpha)\lambda_n J_1(\lambda_n) + \alpha\lambda_n^2 J_0(\lambda_n)\right]} \right\}. \quad (4.24)$$

5. Oscillatory Flow

Here, the oscillating piston motion starting from rest is considered. The piston motion is defined as

$$u_p(t) = \begin{cases} 0, & \text{for } t \le 0, \\ U_p \sin(\omega t), & \text{for } t > 0. \end{cases} \quad (5.1)$$

Taking the Laplace transform of (5.1), we have

$$u_p(s) = \frac{U_p \omega}{s^2 + \omega^2}. \tag{5.2}$$

Substitute (5.2) into (3.16) to find the velocity profile. The poles are simple poles at $s = \pm i\omega$ and the roots of $\alpha\lambda J_1(\lambda) + J_2(\lambda) = 0$. The solution is

$$\begin{aligned} u^*(c,t^*) &= \frac{i}{2}\left[e^{-i\omega t}G(r,-i\omega) - e^{i\omega t}G(r,i\omega)\right] \\ &\quad + \sum_{n=1}^{\infty}\left\{\frac{e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)}2R^2\nu\omega\lambda_n^4[J_0(\lambda_n) - J_0(c\lambda_n) - \alpha\lambda_n J_1(\lambda_n)]}{\left[\lambda_n^4\nu^2 + \omega^2\left(R^2 + \theta\lambda_n^2\right)^2\right]\left[(1+2\alpha)\lambda_n^3 J_1(\lambda_n) + \alpha\lambda_n^4 J_0(\lambda_n)\right]}\right\}, \end{aligned} \tag{5.3}$$

where $G(r,s)$ is defined by (3.15).

And the pressure gradient is obtained as

$$\begin{aligned} \frac{dp^*}{dx^*} &= -\frac{R^2\omega}{16\nu}\left[e^{i\omega t}\Gamma(i\omega) + e^{-i\omega t}\Gamma(-i\omega)\right] \\ &\quad + \frac{1}{4}\sum_{n=1}^{\infty}\left\{\frac{e^{-\lambda_n^2 t^*/(1+\beta\lambda_n^2)}R^4\omega\lambda_n^4[J_0(\lambda_n) - \alpha\lambda_n J_1(\lambda_n)]}{\rho\left[\lambda_n^4\nu^2 + \omega^2\left(R^2 + \theta\lambda_n^2\right)^2\right]\left[(1+2\alpha)\lambda_n J_1(\lambda_n) + \alpha\lambda_n^2 J_0(\lambda_n)\right]}\right\}, \end{aligned} \tag{5.4}$$

where $\Gamma(r,s) = [I_0(mR) - \alpha mRI_1(mR)]/([I_0(mR) - \alpha mRI_1(mR)] - 2I_1(mR)/mR)$ and $m = \sqrt{s/(\nu + s\theta)}$.

6. Conclusion

In this paper, the second-grade flows through a microtube with wall slip are solved by Laplace transform technique. The analytical solutions of velocity profiles and pressure gradients are compared to those obtained by Das and Arakeri's work [10] for no-slip flow and Chen et al.'s work [9] for wall slip condition. We found that the Kn number represents the degree of rarefaction and the α_1 is characterized as the elastic coefficient. Those two variables play a significant role in influencing the velocity profile.

From the equation we deduced that the larger Kn value decreases the change of velocity distribution. We could find that the wall slip condition moves more fluid at one cross section than at a cross section without slip. Also, the role of elastic coefficient is to retard the development of flow in the microtube.

Nomenclature

a_p: Constant acceleration
$\vec{A}_1$: Rivlin-Ericksen tensor of first order
$\vec{A}_2$: Rivlin-Ericksen tensor of second order
$\vec{b}$: Body force field
c: r/R
C_1, C_2: Arbitrary coefficients
F_v: Tangential momentum accommodation coefficient
$H(t)$: Heaviside unit step function
$\vec{I}$: Identity tensor
I_0, I_1: Modified Bessel's function of the first kind of zeroth and first order
J_0, J_1: Bessel's function of zeroth and first order
Kn: Knudsen number ($\mathrm{Kn} \equiv \lambda/L$)
K_0, K_1: Modified Bessel's function of the second kind of zeroth and first order
L: Characteristic length of the microtube
m: $\sqrt{s/(\nu + s\theta)}$
P: Static pressure
P^*: Nondimensional pressure ($P^* = P/(8\mu U_p/R)$)
Q: Inlet volume flow rate
R: Radius of microtube
r, θ, x: Cylindrical coordinates
s: Parameter of the Laplace transform
t: Time
t_0, t_1, t_2: Time period of acceleration, constant velocity, and deceleration, respectively
$\vec{T}$: Stress tensor
u_r, u_θ, u: Velocity components in the r, θ, and x-directions, respectively
u_{re}: Tangential momentum of reflected molecules
u_i: Tangential momentum of incoming molecules
u_p: Average inlet velocity
u^*: Nondimensional average velocity over cross section
U_p: Constant inlet piston velocity
U_w: Tangential momentum of re-emitted molecules
$\vec{V}$: Velocity vector.

Greek Symbols

α: Nondimensional velocity slip coefficient
α_1: Elastic coefficient
α_2: Transverse viscosity coefficient
β_v: Velocity slip parameter
λ: Molecular mean free path
Ψ_p: Assumed particular solution
ρ: Fluid density
ν: Kinematic viscosity
μ: Dynamic viscosity.

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