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Research Article

Positive Solution for the Elliptic Problems with Sublinear and Superlinear Nonlinearities

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This paper deals with the existence of positive solutions for the elliptic problems with sublinear and superlinear nonlinearities $-\Delta u = \lambda a(x)u^p + b(x)u^q$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where $\lambda > 0$ is a real parameter, $0 < p < 1 < q$. Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, and $a(x)$ and $b(x)$ are some given functions. By means of variational method and super-subsolution method, we obtain some results about existence of positive solutions.

1. Introduction

In this paper, we consider the elliptic problems with sublinear and superlinear nonlinearities

$$\begin{gathered} -\Delta u = \lambda a(x)u^p + b(x)u^q \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{$(1)_\lambda$}$$

where $\lambda > 0$ is a real parameter, $0 < p < 1 < q$. Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, and $a(x)$ and $b(x)$ are some given functions which satisfies the following assumptions:

(H_1) $a(x), b(x) \in L^\infty(\Omega)$, $a(x) \geq c_0$, $b(x) \leq -c_1$, where c_0, c_1 are positive constants,

or

(H_2) $a(x), b(x) \in L^\infty(\Omega)$, $a(x), b(x) \geq c_0$, where c_0 is a positive constant.

For convenience, we denote $((1)_\lambda)$ with hypothesis (H_1) or (H_2) by $(1)_\lambda^-$ and $(1)_\lambda^+$, respectively.

Such problems occur in various branches of mathematical physics and population dynamics, and sublinear analogues or superlinear analogues of $((1)_\lambda)$ have been considered by many authors in recent years (see [1–9] and their references). But most of such studies have been concerned with equations of the type involving sublinear nonlinearity (see [3–6, 8, 9]), with only few references dealing with the elliptic problems with sublinear and superlinear nonlinearities. In [1], Ambrosetti et al. deal with the analogue of $((1)_\lambda)$ with $a(x) = b(x) \equiv 1$. It is known from [2] that there exist $\lambda^* \in (0, \infty)$, such that problem $((1)_\lambda)$ has a solution if $\lambda \le \lambda^*$ and has no solution if $\lambda > \lambda^*$, provided $b(x) \equiv 1$ on Ω.

Our goal in this paper is to show how variational method and super-subsolution method can be used to establish some existence results of problem $((1)_\lambda)$. We work on the Sobolev space $H_0^1(\Omega)$ equipped with the norm $\|x\| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$. For $u \in H_0^1(\Omega)$ we define $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{p+1}\int_\Omega a(x)|u|^{p+1} dx - \frac{1}{q+1}\int_\Omega b(x)|u|^{q+1} dx. \tag{1.1}$$

Let λ_1 be the first eigenvalue of

$$\begin{gathered} -\Delta u = \lambda u, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.2}$$

φ_1 denotes the corresponding eigenfunction satisfying $0 \le \varphi_1(x) \le 1$. $L^p(\Omega)$, $(1 \le p \le \infty)$, denotes Lebesgue spaces, and the norm in L^p is denoted by $\|\cdot\|_p$.

2. The Existence of Positive Solution of $(1)_\lambda^-$

It is well known that

$$\nabla\varphi_1(x) \ne 0, \quad \forall x \in \partial\Omega. \tag{2.1}$$

Define $a = \min_{\partial\Omega}|\nabla\varphi_1|^2$; from (2.1) we know $a > 0$, so we can split the domain Ω into two parts: Ω_ε and $\Omega \setminus \Omega_\varepsilon$, where $\Omega_\varepsilon = \{x \in \Omega : |\nabla\varphi_1|^2 \ge a/2\} \bigcap \{x \in \Omega : \varphi_1(x) \le \varepsilon,\ \varepsilon$ is small enough$\}$. Let $b = \inf_{\Omega\setminus\Omega_\varepsilon}\varphi_1(x)$; we obtain that $b \ge \varepsilon$ by the positivity of φ_1 in Ω, and $\Omega \setminus \Omega_\varepsilon$ is nonempty when ε is small enough.

Theorem 2.1. *Let $a(x)$, $b(x)$ satisfy assumption (H_1), and $0 < p < 1 < q < 2^* - 1$, where $2^* = 2N/(N-2)$ is the limiting exponent in the Sobolev embedding. Then there exists a constant $\widetilde{\lambda} > 0$ such that $(1)_\lambda^-$ possesses at least a weak positive solution $u^*(x) \in H_0^1(\Omega)$ for $\lambda \ge \widetilde{\lambda}$.*

Proof. Let $e(x)$ denote the positive solution of the following equation:

$$\begin{gathered} -\Delta e = 1, \quad x \in \Omega, \\ e = 0, \quad x \in \partial\Omega. \end{gathered} \tag{2.2}$$

Here and hereafter we use the following notations: $A = \|a\|_{\infty}$, $B = \|b\|_{\infty}$, $E = \|e\|_{\infty}$. Since $0 < p < 1$, for all $\lambda \in \mathbb{R}^{+}$, there exists $T = T(\lambda) > 0$ satisfying

$$T \geq \lambda A T^{p} E^{p}. \tag{2.3}$$

Observing that $b(x) \leq -c_1 < 0$, as a consequence, the function Te verifies

$$T = -\Delta(Te) \geq \lambda A (Te)^{p} \geq \lambda a(x)(Te)^{p} + b(x)(Te)^{q}, \tag{2.4}$$

and hence it is a supersolution of $(1)_{\lambda}^{+}$. Let $v(x) = \varphi_1^{l}$, $x \in \Omega$, $l > 1$. For $x \in \Omega$, we have $x \in \Omega_{\varepsilon}$ or $x \in \Omega \setminus \Omega_{\varepsilon}$. We will discuss it from two conditions.

(I) For all $x \in \Omega_{\varepsilon}$, observing that $l > 1$ and when ε is small enough, we have

$$al(l-1)\frac{s^{-2}}{2} - Bs^{l(q-1)} > \lambda_1 l, \quad \forall s \in (0, \varepsilon). \tag{2.5}$$

Since $x \in \Omega_{\varepsilon}$, then it follows that $\varphi_1(x) \leq \varepsilon$, $|\nabla \varphi_1|^2 \geq a/2$. From (2.5) we infer

$$\lambda_1 l \leq l(l-1)\varphi_1^{-2}\left|\nabla \varphi_1\right|^2 - B\varphi_1(x)^{l(q-1)}, \quad \forall x \in \Omega_{\varepsilon}. \tag{2.6}$$

Multiplying (2.6) with φ_1^{l}, we get

$$l(1-l)\varphi_1^{l-2}\left|\nabla \varphi_1\right|^2 + \lambda_1 l \varphi_1^{l} \leq -B\varphi_1^{lq}. \tag{2.7}$$

It follows that

$$-\Delta\left(\varphi_1^{l}\right) \leq \lambda a(x)\left(\varphi_1^{l}\right)^{p} - b(x)\left(\varphi_1^{l}\right)^{q}. \tag{2.8}$$

(II) For all $x \in \Omega \setminus \Omega_{\varepsilon}$, there exists $\tilde{\lambda} > 0$, such that for all $\lambda \geq \tilde{\lambda}$, and we have

$$\lambda c_0 s^{pl} - Bs^{ql} \geq \lambda_1 l s^{l}, \quad \forall s \in R,\ b \leq s \leq 1. \tag{2.9}$$

Since $x \in \Omega \setminus \Omega_{\varepsilon}$, then we have $\varphi_1(x) \geq b$ (and $\varphi_1(x) \leq 1$). From (2.9), it follows that

$$-\Delta\left(\varphi_1^{l}\right) \leq \lambda_1 l \varphi_1^{l} \leq \lambda c_0 \varphi_1^{lp} - Bs^{ql} \leq \lambda a(x)\left(\varphi_1^{l}\right)^{p} + b(x)\left(\varphi_1^{l}\right)^{q}. \tag{2.10}$$

From (2.8) and (2.10), we derive that there exists $\tilde{\lambda} > 0$ such that for all $x \in \Omega$, for all $\lambda \geq \tilde{\lambda}$,

$$-\Delta\left(\varphi_1^{l}\right) \leq \lambda a(x)\left(\varphi_1^{l}\right)^{p} + b(x)\left(\varphi_1^{l}\right)^{q}, \tag{2.11}$$

that is, $v(x) = \varphi_1^{l}(x)$ is a subsolution of $(1)_{\lambda}^{-}$. Taking T as sufficiently large, we also have

$Te > \varphi_1^l$ by minimal principle. Define $w(x) = Te(x)$, and let $K = \{u \in H_0^1(\Omega)\colon v(x) \le u(x) \le w(x),$ for all $x \in \Omega\}$, then K is closed and convex (and weakly closed). Let $f(s) = \lambda a(x)s^p + b(x)s^q$, for all $s \in \mathbb{R}, s > 0$. We consider the function

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \int_0^u f(s)ds\,dx. \tag{2.12}$$

Observe that $b(x) < 0$, $0 < p < 1 < q < 2^* - 1$; we infer that I_λ is coercive, bounded, since it is blow and weakly lower semicontinuous. Using this fact, we conclude that there exists $u^* \in K$, such that $I_\lambda(u^*) = \inf_K I_\lambda$ (see [10]). In the following, we will prove that u^* is a solution of problem $(1)_\lambda^-$.

For $\phi \in K$, define $h : [0,1] \to \mathbb{R}$, such that

$$h(t) = I\left(t\phi + (1-t)u^*\right). \tag{2.13}$$

Clearly, $h(t)$ achieves its minimum at $t = 0$, and

$$h'(t)\big|_{t=0} = \int_\Omega \left[\nabla u^* \nabla\left(\phi - u^*\right)\right] dx - \int_\Omega f(u^*)\left(\phi - u^*\right) dx \ge 0. \tag{2.14}$$

For all $\varphi \in H_0^1(\Omega)$, $\eta > 0$, define

$$\Psi(x) = \begin{cases} v, & \text{when } u^* + \eta\varphi < v, \\ u^* + \eta\varphi, & \text{when } v \le u^* + \eta\varphi \le w, \\ w, & \text{when } u^* + \eta\varphi > w. \end{cases} \tag{2.15}$$

Obviously, $\Psi \in K$, and inserting (2.15) into (2.14), we find

$$\begin{aligned} 0 \le & \int_{v \le u^* + \eta\varphi \le w} \left[\nabla u^* \cdot \nabla(\eta\phi) - f(u^*)(\eta\varphi)\right] dx \\ & + \int_{u^* + \eta\varphi > w} \left[\nabla u^* \nabla(w - u^*) - f(u^*)(w - u^*)\right] dx \\ & + \int_{u^* + \eta\varphi < v} \left[\nabla u^* \nabla(v - u^*) - f(u^*)(v - u^*)\right] dx \end{aligned}$$

$$
\begin{aligned}
&= \eta \int_{v \le u^* + \eta\varphi \le w} \left[\nabla u^* \cdot \nabla \varphi - f(u^*)\varphi\right] dx \\
&\quad + \int_{u^* + \eta\varphi > w} \left[\nabla w \cdot \nabla (w - u^*) - f(w)(w - u^*)\right] dx \\
&\quad + \int_{u^* + \eta\varphi < v} \left[\nabla v \cdot \nabla (v - u^*) - f(v)(v - u^*)\right] dx \\
&\quad - \int_{u^* + \eta\varphi > w} |\nabla w - \nabla u^*|^2 dx - \int_{u^* + \eta\varphi < v} |\nabla v - \nabla u^*|^2 dx \\
&\quad + \int_{u^* + \eta\varphi > w} \left[f(w) - f(u^*)\right](w - u^*) dx \\
&\quad + \int_{u^* + \eta\varphi < v} \left[f(v) - f(u^*)\right](v - u^*) dx.
\end{aligned}
\tag{2.16}
$$

Since $w(x)$ and $v(x)$ are supersolution and subsolution, respectively, then

$$
\begin{aligned}
\int_{u^* + \eta\varphi > w} \left[\nabla w \cdot \nabla (w - u^*) - f(w)(w - u^*)\right] dx &\le \eta \int_{u^* + \eta\varphi > w} \left[\nabla w \cdot \nabla \varphi - f(w)\varphi\right] dx, \\
\int_{u^* + \eta\varphi < v} \left[\nabla v \cdot \nabla (v - u^*) - f(v)(v - u^*)\right] dx &\le \eta \int_{u^* + \eta\varphi < v} \left[\nabla v \cdot \nabla \varphi - f(v)\varphi\right] dx.
\end{aligned}
\tag{2.17}
$$

Observe that $\text{meas}[u^* + \eta\varphi > w] \to 0$, $\text{meas}[u^* + \eta\varphi < v] \to 0$, as $\eta \to 0$,

$$
\begin{aligned}
&\int_{u^* + \eta\varphi > w} \left[\nabla w \cdot \nabla \varphi - f(w)\varphi\right] dx \longrightarrow 0, \\
&\int_{u^* + \eta\varphi < v} \left[\nabla v \cdot \nabla \varphi - f(v)\varphi\right] dx \longrightarrow 0.
\end{aligned}
\tag{2.18}
$$

Since $u^* \in K, b(x) < 0$, it follows that

$$
\begin{aligned}
&\int_{u^* + \eta\varphi > w} \left[f(w) - f(u^*)\right](w - u^*) dx \\
&\quad = \int_{u^* + \eta\varphi > w} \lambda a(x)\left(w^p - u^{*^p}\right)(w - u^*) dx + \int_{u^* + \eta\varphi > w} b(x)\left(w^q - u^{*^q}\right)(w - u^*) dx \\
&\quad \le \int_{u^* + \eta\varphi > w} \lambda a(x)\left(w^p - u^{*^p}\right)(w - u^*) dx.
\end{aligned}
\tag{2.19}
$$

Similar to (2.19), we have

$$\begin{aligned}
&\int_{u^*+\eta\varphi<v} [f(v) - f(u^*)](v-u^*)dx \\
&\quad = \int_{u^*+\eta\varphi<v} \lambda a(x)\left(v^p - u^{*^p}\right)(v-u^*)dx + \int_{u^*+\eta\varphi<v} b(x)\left(v^q - u^{*^q}\right)(v-u^*)dx \\
&\quad \le \int_{u^*+\eta\varphi<v} \lambda a(x)\left(v^p - u^{*^p}\right)(v-u^*)dx.
\end{aligned} \tag{2.20}$$

Similar to (2.18), as $\eta \to 0$, it follows that

$$\begin{aligned}
&\int_{u^*+\eta\varphi>w} \lambda a(x)\left(w^p - u^{*^p}\right)(w-u^*)dx \longrightarrow 0, \\
&\int_{u^*+\eta\varphi<v} \lambda a(x)\left(v^p - u^{*^p}\right)(v-u^*)dx \longrightarrow 0.
\end{aligned} \tag{2.21}$$

As $\eta \to 0$, we also have

$$\int_{v\le u^*+\eta\varphi\le w} [\nabla u^* \cdot \nabla\varphi - f(u^*)\varphi]dx \longrightarrow \int_\Omega [\nabla u^* \cdot \nabla\varphi - f(u^*)\varphi]dx. \tag{2.22}$$

Inserting (2.17), (2.19), and (2.20) into (2.16), we find

$$\begin{aligned}
0 \le \eta\Bigg\{ &\int_{v\le u^*+\eta\varphi\le w} [\nabla u^* \cdot \nabla\varphi - f(u^*)\varphi]dx + \int_{u^*+\eta\varphi>w} [\nabla w \cdot \nabla\varphi - f(w)\varphi]dx \\
&+ \int_{u^*+\eta\varphi<v} [\nabla v \cdot \nabla\varphi - f(v)\varphi]dx \Bigg\} \\
&+ \int_{u^*+\eta\varphi>w} \lambda a(x)\left(w^p - u^{*^p}\right)(w-u^*)dx \\
&+ \int_{u^*+\eta\varphi<v} \lambda a(x)\left(v^p - u^{*^p}\right)(v-u^*)dx.
\end{aligned} \tag{2.23}$$

Dividing by η and letting $\eta \to 0$, using (2.18), (2.21), and (2.22), we derive

$$\int_\Omega [\nabla u^* \cdot \nabla\varphi - f(u^*)\varphi]dx \ge 0. \tag{2.24}$$

Noting that φ is arbitrary, this holds equally for $-\varphi$, and it follows that u^* is indeed a weak solution of $(1)_\lambda^-$, and the strong maximum principle yields $u^* > \varphi_1^l$, in Ω. Therefore it is a weak positive solution of $(1)_\lambda^-$. □

3. The Existence of Positive Solution of $(1)_\lambda^+$

Theorem 3.1. *Let $a(x)$, $b(x)$ satisfy assumption (H_2), and $0 < p < 1 < q < +\infty$. Then there exists $\Lambda \in \mathbb{R}$, $\Lambda > 0$, such that*

(i) *for all $\lambda \in (0, \Lambda)$ problem $(1)_\lambda^+$ has a minimal solution u_λ such that $I_\lambda(u_\lambda) < 0$. Moreover u_λ is increasing with respect to λ;*

(ii) *for $\lambda = \Lambda$ problem $(1)_\lambda^+$ has at least one weak solution $u \in H \cap L^{p+1}$;*

(iii) *for all $\lambda > \Lambda$ problem $(1)_\lambda^+$ has no solution.*

To prove Theorem 3.1, let us define

$$\Lambda = \sup\left\{\lambda > 0 : (1)_\lambda^+ \text{ has a solution}\right\}. \tag{3.1}$$

First of all we prove a useful lemma.

Lemma 3.2. *One has $0 < \Lambda < +\infty$.*

Proof. Let $e(x)$ denote the solution of the following equation:

$$\begin{aligned} -\Delta e = 1, \quad & x \in \Omega, \\ e = 0, \quad & x \in \partial\Omega. \end{aligned} \tag{3.2}$$

Since $0 < p < 1 < q$, we can find $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$ there exists $T = T(\lambda) > 0$ satisfying

$$T \geq \lambda A T^p E^p + B T^q E^q. \tag{3.3}$$

As a consequence, the function Te verifies

$$T = -\Delta(Te) \geq \lambda A (Te)^p + B(Te)^q \geq \lambda a(x)(Te)^p + b(x)(Te)^q, \tag{3.4}$$

and hence it is a supersolution of $(1)_\lambda^+$. Moreover, let u_0 denote the solution of the following problem:

$$\begin{aligned} -\Delta u = \lambda a(x) u_0^p, \quad & x \in \Omega, \\ u_0 = 0, \quad & x \in \partial\Omega. \end{aligned} \tag{3.5}$$

(From [3] we know that u_0 exists.) Then εu_0 is a subsolution of $(1)_\lambda^+$, provided

$$-\Delta(\varepsilon u_0) = \lambda \varepsilon a(x) u_0^p \leq \lambda a(x)(\varepsilon u_0)^p + b(x)(\varepsilon u_0)^q, \tag{3.6}$$

which is satisfied for all $\varepsilon > 0$ small enough and all λ. Taking ε as possibly smaller, we also have

$$\varepsilon u_0 < Te. \tag{3.7}$$

It follows that $(1)_\lambda^+$ has a solution u, $\varepsilon u_0 \le u \le Te$ whenever $\lambda \le \lambda_0$, and thus $\Lambda \ge \lambda_0$.

Next, let λ^* be such that

$$c_0(\lambda^* t^p + t^q) > \lambda_1 t, \quad \forall t > 0. \tag{3.8}$$

If λ is such that $(1)_\lambda^+$ has a solution u, multiplying $(1)_\lambda^+$ by φ_1 and integrating over Ω we find

$$\lambda_1 \int_\Omega u\varphi_1 dx = \lambda \int_\Omega a(x)u^p \varphi_1 dx + \int_\Omega b(x)u^q \varphi_1 dx \ge c_0 \left[\int_\Omega (\lambda u^p \varphi_1 + u^q \varphi_1) dx \right]. \tag{3.9}$$

This and (3.5) immediately imply that $\lambda < \lambda^*$ and show that $\Lambda \le \lambda^*$, hence $0 < \Lambda < +\infty$. □

We are now ready to give the proof of Theorem 3.1.

Proof. (i) From the proof of lemma, it follows that, for all $\lambda \in (0, \Lambda)$, problem $(1)_\lambda^+$ has a solution u_λ. Let u_0 satisfy (3.5); the iteration

$$-\Delta u_{n+1} = \lambda a(x) u_n^p + b(x) u_n^q \tag{3.10}$$

satisfies $u_n \uparrow u_\lambda$ by making use of Lemma 3.3 of [1] and maximum principle. It is easy to check that u_λ is a minimal solution of $(1)_\lambda^+$. Indeed, if u is any solution of $(1)_\lambda^+$, then $u \ge u_0$ and u is a supersolution of $(1)_\lambda^+$. Thus $u_n \le u$, for all n, by induction, and $u_\lambda \le u$. Next, we will prove that $I_\lambda(u_\lambda) < 0$. Indeed,

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{p+1} \int_\Omega a(x)|u|^{p+1} dx - \frac{1}{q+1} \int_\Omega b(x)|u|^{q+1} dx. \tag{3.11}$$

Since u_λ is a solution of $(1)_\lambda^+$ we have

$$\int_\Omega |\nabla u_\lambda|^2 dx = \int_\Omega \lambda a(x) u_\lambda^{p+1} dx + \int_\Omega b(x) u_\lambda^{q+1} dx. \tag{3.12}$$

From Lemma 3.5 of [1], we know

$$\int_\Omega \left[\left|\nabla \varphi\right|^2 - \left(\lambda p a(x) u_\lambda^{p-1} + q b(x) u_\lambda^{q-1}\right) \varphi^2 \right] dx \ge 0, \quad \forall \varphi \in H_0^1. \tag{3.13}$$

In particular with $\varphi = u_\lambda$, we infer

$$\int_\Omega |\nabla u_\lambda|^2 dx - \lambda p \int_\Omega a(x) u_\lambda^{p+1} dx - q \int_\Omega b(x) u_\lambda^{q+1} dx \ge 0. \tag{3.14}$$

Combining (3.12) and (3.14), we obtain

$$\begin{aligned} I_\lambda(u_\lambda) &= \lambda\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega a(x)u_\lambda^{p+1}dx + \left(\frac{1}{2}-\frac{1}{q+1}\right)\int_\Omega b(x)u_\lambda^{q+1}dx \\ &\leq \frac{1-p}{2}\left(-\frac{1}{p+1}+\frac{1}{q+1}\right)\int_\Omega a(x)u_\lambda^{p+1}dx < 0. \end{aligned} \tag{3.15}$$

To complete the proof of (i), it remains to show that

$$u_\lambda < u_{\lambda_1} \quad \text{whenever } \lambda < \lambda_1. \tag{3.16}$$

Indeed, if $\lambda < \lambda_1$ then u_{λ_1} is a supersolution of $(1)_\lambda^+$. Since, for $\varepsilon > 0$ small, εu_0 is a subsolution of $(1)_\lambda^+$ and $\varepsilon u_0 < u_{\lambda_1}$, then $(1)_\lambda^+$ possesses a solution v, with

$$(\varepsilon u_0 \leq) v \leq u_{\lambda_1}. \tag{3.17}$$

Since u_λ is the minimal solution of $(1)_\lambda^+$, we infer that $u_\lambda \leq v \leq u_{\lambda_1}$. Moreover

$$\begin{aligned} -\Delta(u_{\lambda_1}-u_\lambda) &= \lambda_1 a(x)u_{\lambda_1}^p + b(x)u_{\lambda_1}^q - \left(\lambda a(x)u_\lambda^p + b(x)u_\lambda^q\right) \\ &\geq \lambda a(x)u_{\lambda_1}^p + b(x)u_{\lambda_1}^q - a(x)u_\lambda^p - b(x)u_\lambda^q \geq 0. \end{aligned} \tag{3.18}$$

Since $u_{\lambda_1} \neq u_\lambda$ (because $\lambda < \lambda_1$), then the Hopf Maximum principle yields $u_\lambda < u_{\lambda_1}$.

(ii) Let λ_n be a sequence such that $\lambda_n \uparrow \Lambda$; then from $I_{\lambda_n}(u_{\lambda_n}) < 0$ we deduce that there exists $C > 0$ such that

$$\begin{aligned} &\|\nabla u_n\|^2 \leq C, \\ &\|u_n\|_{p+1}^{p+1} \leq C. \end{aligned} \tag{3.19}$$

Then there exists $u^* \in H_0^1$ such that $u_n \longrightarrow u^* > 0$ a.e. in Ω, strongly in L^{p+1} and weakly in H_0^1. Such a u^* is thus a weak solution of $(1)_\lambda^+$ for $\lambda = \Lambda$.

(iii) This follows from the definition of Λ. □

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References

[1] A. Ambrosetti, H. Brezis, and G. Cerami, "Combined effects of concave and convex nonlinearities in some elliptic problems," *Journal of Functional Analysis*, vol. 122, no. 2, pp. 519–543, 1994.
[2] Z. Tan and Z. Yao, "Existence of multiple solutions for semilinear elliptic equation," *Advances in Mathematics*, vol. 31, no. 4, pp. 343–354, 2002.
[3] H. Brezis and S. Kamin, "Sublinear elliptic equations in $\mathbb{R}^N$," *Manuscripta Mathematica*, vol. 74, no. 1, pp. 87–106, 1992.
[4] H. Brezis and L. Oswald, "Remarks on sublinear elliptic equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 10, no. 1, pp. 55–64, 1986.
[5] K. Taira and K. Umezu, "Positive solutions of sublinear elliptic boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 29, no. 7, pp. 761–771, 1997.
[6] M. Ghergu and V. Rădulescu, "Nonradial blow-up solutions of sublinear elliptic equations with gradient term," *Communications on Pure and Applied Analysis*, vol. 3, no. 3, pp. 465–474, 2004.
[7] M. Ghergu and V. D. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, vol. 37 of *Oxford Lecture Series in Mathematics and Its Applications*, The Clarendon Press, Oxford University Press, Oxford, UK, 2008.
[8] R. Kajikiya, "Comparison theorem and uniqueness of positive solutions for sublinear elliptic equations," *Archiv der Mathematik*, vol. 91, no. 5, pp. 427–435, 2008.
[9] R. Kajikiya, "A priori estimates of positive solutions for sublinear elliptic equations," *Transactions of the American Mathematical Society*, vol. 361, no. 7, pp. 3793–3815, 2009.
[10] M. Struwe, *Variational Methods*, Springer, Berlin, Germany, 1990.