

Research Article

An Inverse Eigenvalue Problem of Hermite-Hamilton Matrices in Structural Dynamic Model Updating

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Received 11 February 2010; Accepted 27 April 2010

Academic Editor: Angelo Luongo

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We first consider the following inverse eigenvalue problem: given $X \in C^{n \times m}$ and a diagonal matrix $\Lambda \in C^{m \times m}$, find $n \times n$ Hermite-Hamilton matrices K and M such that $KX = MX\Lambda$. We then consider an optimal approximation problem: given $n \times n$ Hermitian matrices K_a and M_a , find a solution (K, M) of the above inverse problem such that $\|K - K_a\|^2 + \|M - M_a\|^2 = \min$. By using the Moore-Penrose generalized inverse and the singular value decompositions, the solvability conditions and the representations of the general solution for the first problem are derived. The expression of the solution to the second problem is presented.

1. Introduction

Throughout this paper, we will adopt the following notations. Let $C^{m \times n}$, $HC^{n \times n}$, and $UC^{n \times n}$ stand for the set of all $m \times n$ matrices, $n \times n$ Hermitian matrices, and unitary matrices over the complex field C , respectively. By $\|\cdot\|$ we denote the Frobenius norm of a matrix. The symbols A^T , A^* , A^{-1} , and A^\dagger denote the transpose, conjugate transpose, inverse, and Moore-Penrose generalized inverse of A , respectively.

Definition 1.1. Let $J_n = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, $n = 2k$, and $A \in C^{n \times n}$. If $A = A^*$ and $J_n A J_n = A^*$, then the matrix A is called Hermite-Hamilton matrix.

We denote by $HHC^{n \times n}$ the set of all $n \times n$ Hermite-Hamilton matrices.

Vibrating structures such as bridges, highways, buildings, and automobiles are modeled using finite element techniques. These techniques generate structured matrix second-order differential equations:

$$M_a \ddot{z}(t) = K_a z(t), \quad (1.1)$$

where M_a, K_a are analytical mass and stiffness matrices. It is well known that all solutions of the above differential equation can be obtained via the algebraic equation $K_a x = \lambda M_a x$. But such finite element model is rarely available in practice, because its natural frequencies and mode shapes often do not match very well with experimentally measured ones obtained from a real-life vibration test [1]. It becomes necessary to update the original model to attain consistency with empirical results. The most common approach is to modify K_a and M_a to satisfy the dynamic equation with the measured model data. Let $X \in C^{n \times m}$ be the measured model matrix and $\Lambda = \text{diag}(\delta_1, \delta_2, \dots, \delta_m) \in C^{m \times m}$ the measured natural frequencies matrix, where $n \geq m$. The measured mode shapes and frequencies are assumed correct and have to satisfy

$$KX = MX\Lambda, \quad (1.2)$$

where $M, K \in C^{n \times n}$ are the mass and stiffness matrices to be corrected. To date, many techniques for model updating have been proposed. For undamped systems, various techniques have been discussed by Berman [2] and Wei [3]. Theory and computation of damped systems were proposed by authors of [4, 5]. Another line of thought is to update damping and stiffness matrices with symmetric low-rank correction [6]. The system matrices are adjusted globally in these methods. As model errors can be localized by using sensitivity analysis [7], residual force approach [8], least squares approach [9], and assigned eigenstructure [10], it is usual practice to adjust partial elements of the system matrices using measured response data.

The model updating problem can be regarded as a special case of the inverse eigenvalue problem which occurs in the design and modification of mass-spring systems and dynamic structures. The symmetric inverse eigenvalue problem and generalized inverse eigenvalue problem with submatrix constraint in structural dynamic model updating have been studied in [11] and [12], respectively. Hamiltonian matrices usually arise in the analysis of dynamic structures [13]. However, the inverse eigenvalue problem for Hermite-Hamilton matrices has not been discussed. In this paper, we will consider the following inverse eigenvalue problem and an associated optimal approximation problem.

Problem 1. Given that $X \in C^{n \times m}$ and a diagonal matrix $\Lambda \in C^{m \times m}$, find $n \times n$ Hermite-Hamilton matrices K and M such that

$$KX = MX\Lambda. \quad (1.3)$$

Problem 2. Given that $K_a, M_a \in HC^{n \times n}$, let S_E be the solution set of Problem 1. Find $(\widehat{K}, \widehat{M}) \in S_E$ such that

$$\|\widehat{K} - K_a\|^2 + \|\widehat{M} - M_a\|^2 = \min_{(K, M) \in S_E} (\|K - K_a\|^2 + \|M - M_a\|^2). \quad (1.4)$$

We observe that, when $M = I$, Problem 1 can be reduced to the following inverse eigenproblem:

$$KX = X\Lambda, \quad (1.5)$$

which has been solved for different classes of structured matrices. For example, Xie et al. considered the problem for the case of symmetric, antipersymmetric, antisymmetric, and persymmetric matrices in [14, 15]. Bai and Chan studied the problem for the case of centrosymmetric and centroskew matrices in [16]. Trench investigated the case of generalized symmetry or skew symmetry matrices for the problem in [17] and Yuan studied R-symmetric matrices for the problem in [18].

The paper is organized as follows. In Section 2, using the Moore-Penrose generalized inverse and the singular value decompositions of matrices, we give explicit expressions of the solution for Problem 1. In Section 3, the expressions of the unique solution for Problem 2 are given and a numerical example is provided.

2. Solution of Problem 1

Let

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ -iI_k & iI_k \end{pmatrix}. \quad (2.1)$$

Lemma 2.1. *Let $A \in C^{n \times n}$. Then $A \in HHC^{n \times n}$ if and only if there exists a matrix $N \in C^{k \times k}$ such that*

$$A = U \begin{pmatrix} 0 & N \\ N^* & 0 \end{pmatrix} U^*, \quad (2.2)$$

where U is the same as in (2.1).

Proof. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$, and let each block of A be square. From Definition 1.1 and (2.1), it can be easily proved. \square

Lemma 2.2 (see [19]). *Let $A \in C^{m \times n}$, $B \in C^{p \times q}$, and $E \in C^{m \times q}$. Then the matrix equation $AXB = E$ has a solution $X \in C^{n \times p}$ if and only if $AA^\dagger EB^\dagger B = E$; in this case the general solution of the equation can be expressed as $X = A^\dagger EB^\dagger + Y - A^\dagger AYBB^\dagger$, where $Y \in C^{n \times p}$ is arbitrary.*

Let the partition of the matrix U^*X be

$$U^*X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad X_1, X_2 \in C^{k \times m}, \quad (2.3)$$

where U is defined as in (2.1).

We assume that the singular value decompositions of the matrices X_1 and X_2 are

$$X_1 = R \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} S^*, \quad X_2 = W \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^*, \quad (2.4)$$

where $R = (R_1, R_2) \in UC^{k \times k}$, $S = (S_1, S_2) \in UC^{m \times m}$, $D = \text{diag}(d_1, \dots, d_l) > 0$, $l = \text{rank}(X_1)$, $R_1 \in C^{k \times l}$, $S_1 \in C^{m \times l}$, and $W = (W_1, W_2) \in UC^{k \times k}$, $V = (V_1, V_2) \in UC^{m \times m}$, $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_s) > 0$, $s = \text{rank}(X_2)$, $W_1 \in C^{k \times s}$, $V_1 \in C^{m \times s}$.

Let the singular value decompositions of the matrices $X_2 \Lambda V_2$ and $X_1 \Lambda S_2$ be

$$X_2 \Lambda V_2 = P \begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} Q^*, \quad X_1 \Lambda S_2 = T \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} H^*, \quad (2.5)$$

where $P = (P_1, P_2) \in UC^{k \times k}$, $Q = (Q_1, Q_2) \in UC^{(m-s) \times (m-s)}$, $\Omega = \text{diag}(\omega_1, \dots, \omega_t) > 0$, $t = \text{rank}(X_2 \Lambda V_2)$, $P_1 \in C^{k \times t}$, $Q_1 \in C^{(m-s) \times t}$, and $T = (T_1, T_2) \in UC^{k \times k}$, $H \in UC^{(m-l) \times (m-l)}$, $\Delta = \text{diag}(a_1, \dots, a_g) > 0$, $g = \text{rank}(X_1 \Lambda S_2)$, $T_1 \in C^{k \times g}$.

Theorem 2.3. Suppose that $X \in C^{n \times m}$ and $\Lambda \in C^{m \times m}$ is a diagonal matrix. Let the partition of $U^* X$ be (2.3), and let the singular value decompositions of X_1 , X_2 , $X_2 \Lambda V_2$, and $X_1 \Lambda S_2$ be given in (2.4) and (2.5), respectively. Then (1.3) is solvable and its general solution can be expressed as

$$M = U \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix} U^*, \quad K = U \begin{pmatrix} 0 & FX_2 \Lambda X_2^\dagger + GW_2^* \\ (FX_2 \Lambda X_2^\dagger + GW_2^*)^* & 0 \end{pmatrix} U^*, \quad (2.6)$$

where

$$F = T_2 J P_2^*, \quad G = (X_1 \Lambda X_1^\dagger)^* F W_2 + R_2 Y, \quad (2.7)$$

with $J \in C^{(k-g) \times (k-t)}$, $Y \in C^{(k-l) \times (k-s)}$ being arbitrary matrices, and U is the same as in (2.1).

Proof. By Lemma 2.1, we know that (K, M) is a solution to Problem 1 if and only if there exist matrices $N, F \in C^{k \times k}$ such that

$$K = U \begin{pmatrix} 0 & N \\ N^* & 0 \end{pmatrix} U^*, \quad M = U \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix} U^*, \quad (2.8)$$

$$U \begin{pmatrix} 0 & N \\ N^* & 0 \end{pmatrix} U^* X = U \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix} U^* X \Lambda.$$

Using (2.3), the above equation is equivalent to the following two equations:

$$NX_2 = FX_2 \Lambda, \quad (2.9)$$

$$N^* X_1 = F^* X_1 \Lambda, \quad \text{i.e., } X_1^* N = (X_1 \Lambda)^* F. \quad (2.10)$$

By the singular value decomposition of X_2 , then the relation (2.9) becomes

$$0 = FX_2 \Lambda V_2, \quad (2.11)$$

$$NW_1 \Sigma = FX_2 \Lambda V_1. \quad (2.12)$$

Clearly, (2.11) with respect to unknown matrix F is always solvable. By Lemma 2.2 and (2.5), we get

$$F = LP_2^*, \quad (2.13)$$

where $L \in C^{k \times (k-t)}$ is an arbitrary matrix. Substituting $F = LP_2^*$ into (2.12), we get

$$NW_1 = (LP_2^*)X_2\Lambda V_1\Sigma^{-1}. \quad (2.14)$$

Since W_1 is of full column rank, then the above equation with respect to unknown matrix N is always solvable, and the general solution can be expressed as

$$\begin{aligned} N &= (LP_2^*X_2\Lambda V_1\Sigma^{-1})W_1^* + GW_2^* \\ &= LP_2^*X_2\Lambda X_2^\dagger + GW_2^*, \end{aligned} \quad (2.15)$$

where $G \in C^{k \times (k-s)}$ is an arbitrary matrix.

Substituting $F = LP_2^*$ and (2.15) into (2.10), we get

$$X_1^*(LP_2^*X_2\Lambda X_2^\dagger + GW_2^*) = (X_1\Lambda)^*LP_2^*. \quad (2.16)$$

By the singular value decomposition of X_1 , then the relation (2.16) becomes

$$0 = S_2^*(X_1\Lambda)^*LP_2^*, \quad (2.17)$$

$$DR_1^*(LP_2^*X_2\Lambda X_2^\dagger + GW_2^*) = S_1^*(X_1\Lambda)^*LP_2^*. \quad (2.18)$$

Clearly, (2.17) with respect to unknown matrix L is always solvable. From Lemma 2.2 and (2.5), we have

$$\begin{aligned} L &= J_1 - (X_1\Lambda S_2)(X_1\Lambda S_2)^\dagger J_1 P_2^* P_2 \\ &= J_1 - (X_1\Lambda S_2)(X_1\Lambda S_2)^\dagger J_1 \\ &= T_2 J, \end{aligned} \quad (2.19)$$

where $J \in C^{(k-g) \times (k-t)}$ is arbitrary. Substituting $L = T_2 J$ into (2.18), we get

$$DR_1^*GW_2^* = (X_1\Lambda S_1)^*T_2 J P_2^* - DR_1^*T_2 J P_2^* X_2\Lambda X_2^\dagger. \quad (2.20)$$

Then, we have

$$R_1^*GW_2^* = D^{-1}(X_1\Lambda S_1)^*T_2 J P_2^* - R_1^*T_2 J P_2^* X_2\Lambda X_2^\dagger. \quad (2.21)$$

Since R_1^* is of full row rank, then the above equation with respect to GW_2^* is always solvable. By Lemma 2.2, we have

$$GW_2^* = \left(X_1 \Lambda X_1^\dagger\right)^* T_2 J P_2^* - R_1 R_1^* T_2 J P_2^* X_2 \Lambda X_2^\dagger + (I - R_1 R_1^*) Y_1, \quad (2.22)$$

where $Y_1 \in C^{k \times k}$ is arbitrary. Then, we get

$$\begin{aligned} G &= \left(X_1 \Lambda X_1^\dagger\right)^* T_2 J P_2^* W_2 - R_1 R_1^* T_2 J P_2^* X_2 \Lambda X_2^\dagger W_2 + (I - R_1 R_1^*) Y_1 W_2, \\ &= \left(X_1 \Lambda X_1^\dagger\right)^* T_2 J P_2^* W_2 + R_2 Y, \end{aligned} \quad (2.23)$$

where $Y \in C^{(k-l) \times (k-s)}$ is arbitrary.

Finally, we have

$$F = T_2 J P_2^*, \quad N = F X_2 \Lambda X_2^\dagger + G W_2^*, \quad (2.24)$$

where $G = \left(X_1 \Lambda X_1^\dagger\right)^* F W_2 + R_2 Y$. The proof is completed. \square

From Lemma 2.1, we have that if the mass matrix $M \in HHC^{n \times n}$, then M is not positive definite. If M is symmetric positive definite and K is a symmetric matrix, then (1.3) can be reformulated as the following form:

$$AX = X\Lambda, \quad (2.25)$$

where $A = M^{-1}K$. From [20, Theorem 7.6.3], we know that A is a diagonalizable matrix, all of whose eigenvalues are real. Thus, $\Lambda \in R^{m \times m}$ and X is of full column rank. Assume that X is a real $n \times m$ matrix. Let the singular value decomposition of X be

$$X = \tilde{U} \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} \tilde{V}^T, \quad \tilde{U} \in OR^{n \times n}, \quad \tilde{V} \in OR^{m \times m}, \quad \Gamma = \text{diag}(\gamma_1, \dots, \gamma_m) > 0, \quad (2.26)$$

where $OR^{n \times n}$ denotes the set of all orthogonal matrices. The solution of (2.25) can be expressed as

$$A = \tilde{U} \begin{pmatrix} \Gamma \tilde{V}^T \Lambda \tilde{V} \Gamma^{-1} & Z_{12} \\ 0 & Z_{22} \end{pmatrix} \tilde{U}^T, \quad (2.27)$$

where $Z_{12} \in R^{m \times (n-m)}$ is an arbitrary matrix and $Z_{22} \in R^{(n-m) \times (n-m)}$ is an arbitrary diagonalizable matrix (see [21, Theorem 3.1]).

Let $\Lambda = \text{diag}(\lambda_1 I^{k_1}, \dots, \lambda_q I^{k_q})$ with $\lambda_1 < \lambda_2 < \dots < \lambda_q$. Choose $Z_{22} = \bar{G} \Lambda_2 \bar{G}^{-1}$, where $\bar{G} \in R^{(n-m) \times (n-m)}$ is an arbitrary nonsingular matrix and $\Lambda_2 = \text{diag}(\lambda_{q+1} I^{k_{q+1}}, \dots, \lambda_p I^{k_p})$ with $\lambda_p > \dots > \lambda_{q+1} > \lambda_q$. The solutions to (1.3) with respect to unknown matrices $M > 0$ and $K = K^T$ are presented in the following theorem.

Theorem 2.4 (see [21]). *Given that $X \in R^{n \times m}$, $\text{rank}(X) = m$, and $\Lambda = \text{diag}(\lambda_1 I^{k_1}, \dots, \lambda_q I^{k_q}) \in R^{m \times m}$, let the singular value decomposition of X be (2.26). Then the symmetric positive-definite solution M and symmetric solution K to (1.3) can be expressed as*

$$M = \tilde{U} \tilde{F}^T \tilde{F} \tilde{U}^T, \quad K = \tilde{U} \tilde{F}^T \Delta \tilde{F} \tilde{U}^T, \quad (2.28)$$

where $\Delta = \text{diag}(\Lambda, \Lambda_2)$, $\tilde{F} = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix}$, $F_{11} = \text{diag}(L_1, \dots, L_q) V \Gamma^{-1} \in R^{m \times m}$, and $F_{22} = \text{diag}(L_{q+1}, \dots, L_p) \bar{G}^{-1} \in R^{(n-m) \times (n-m)}$, where $L_i \in R^{k_i \times k_i}$ is an arbitrary nonsingular matrix ($i = 1, 2, \dots, p$). The matrix F_{12} satisfies the equation $\Lambda F_{12} \bar{G} - F_{12} \bar{G} \Lambda_2 = F_{11} Z_{12} \bar{G}$.

3. Solution of Problem 2

Lemma 3.1 (see [22]). *Given that $A \in C^{m \times n}$, $B \in C^{p \times q}$, $C \in C^{l \times n}$, $D \in C^{p \times t}$, $E \in C^{m \times q}$, and $H \in C^{l \times t}$, let*

$$S_a = \left\{ Z \mid Z \in C^{n \times p}, \|[AZB - E, CZD - H]\|^2 = \min \right\}, \quad (3.1)$$

$$S_b = \left\{ Z \mid Z \in C^{n \times p}, A^*AZBB^* + C^*CZDD^* = A^*EB^* + C^*HD^* \right\}.$$

Then $Z \in S_a$ if and only if $Z \in S_b$.

For the given matrices $K_a, M_a \in HC^{n \times n}$, let

$$U^* M_a U = \begin{pmatrix} C_1 & C_2 \\ C_2^* & C_3 \end{pmatrix}, \quad U^* K_a U = \begin{pmatrix} K_1 & K_2 \\ K_2^* & K_3 \end{pmatrix}. \quad (3.2)$$

From Theorem 2.3, we know that $S_E \neq \emptyset$. The following theorem is for the best approximation solution of Problem 2.

Theorem 3.2. *Given that $X \in C^{n \times m}$, $\Lambda \in C^{m \times m}$, and $K_a, M_a \in HC^{n \times n}$, then Problem 2 has a unique solution and the solution can be expressed as*

$$\hat{M} = U \begin{pmatrix} 0 & \hat{F} \\ \hat{F}^* & 0 \end{pmatrix} U^*, \quad \hat{K} = U \begin{pmatrix} 0 & \hat{F} X_2 \Lambda X_2^\dagger + K_2 W_2 W_2^* \\ (\hat{F} X_2 \Lambda X_2^\dagger + K_2 W_2 W_2^*)^* & 0 \end{pmatrix} U^*, \quad (3.3)$$

where

$$\hat{F} = \left(C_2 + K_2 (X_2 \Lambda X_2^\dagger)^* \right) \left(I + (X_2 \Lambda X_2^\dagger) (X_2 \Lambda X_2^\dagger)^* \right)^{-1}. \quad (3.4)$$

Proof. It is easy to verify that S_E is a closed convex subset of $HHC^{n \times n} \times HHC^{n \times n}$. From the best approximation theorem, we know that there exists a unique solution $(\widehat{K}, \widehat{M})$ in S_E such that (1.4) holds. From Theorem 2.3 and the unitary invariant of the Frobenius norm, we have

$$\begin{aligned} & \|M_a - M\|^2 + \|K_a - K\|^2 \\ &= \left\| \begin{pmatrix} C_1 & C_2 \\ C_2^* & C_3 \end{pmatrix} - \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix} \right\|^2 + \left\| \begin{pmatrix} K_1 & K_2 \\ K_2^* & K_3 \end{pmatrix} - \begin{pmatrix} 0 & FX_2\Lambda X_2^\dagger + GW_2^* \\ (FX_2\Lambda X_2^\dagger + GW_2^*)^* & 0 \end{pmatrix} \right\|^2, \end{aligned} \quad (3.5)$$

where $G = (X_1\Lambda X_1^\dagger)^*FW_2 + R_2Y$. Hence, $\|M_a - M\|^2 + \|K_a - K\|^2 = \min$ is equivalent to

$$\|F - C_2\|^2 + \left\| FX_2\Lambda X_2^\dagger + (X_1\Lambda X_1^\dagger)^*FW_2W_2^* + R_2YW_2^* - K_2 \right\|^2 = \min. \quad (3.6)$$

Let

$$f = \|F - C_2\|^2 + \left\| FX_2\Lambda X_2^\dagger + (X_1\Lambda X_1^\dagger)^*FW_2W_2^* + R_2YW_2^* - K_2 \right\|^2. \quad (3.7)$$

Then from the unitary invariant of the Frobenius norm, we have

$$\begin{aligned} f &= \|F - C_2\|^2 \\ &+ \left\| FX_2\Lambda X_2^\dagger(W_1, W_2) + (X_1\Lambda X_1^\dagger)^*FW_2W_2^*(W_1, W_2) + R_2YW_2^*(W_1, W_2) - K_2(W_1, W_2) \right\|^2 \\ &= \|F - C_2\|^2 + \left\| (FX_2\Lambda X_2^\dagger W_1, 0) + (0, (X_1\Lambda X_1^\dagger)^*FW_2) + (0, R_2Y) - (K_2W_1, K_2W_2) \right\|^2 \\ &= \|F - C_2\|^2 + \left\| FX_2\Lambda X_2^\dagger W_1 - K_2W_1 \right\|^2 + \left\| (X_1\Lambda X_1^\dagger)^*FW_2 + R_2Y - K_2W_2 \right\|^2. \end{aligned} \quad (3.8)$$

Let $h = \|(X_1\Lambda X_1^\dagger)^*FW_2 + R_2Y - K_2W_2\|^2$. It is not difficult to see that, when

$$R_2Y = K_2W_2 - (X_1\Lambda X_1^\dagger)^*FW_2, \quad (3.9)$$

that is, $Y = R_2^* K_2 W_2 - R_2^* (X_1 \Lambda X_1^\dagger)^* F W_2$, we have $h = 0$. In other words, we can always find Y such that $h = 0$. Let

$$g = \|F - C_2\|^2 + \|FX_2 \Lambda X_2^\dagger W_1 - K_2 W_1\|^2 = \left\| \left[F - C_2, FX_2 \Lambda X_2^\dagger W_1 - K_2 W_1 \right] \right\|^2. \quad (3.10)$$

Then, we have that $f = \min$ is equivalent to $g = \min$. According to Lemma 3.1 and (3.10), we get the following matrix equation:

$$F + F(X_2 \Lambda X_2^\dagger W_1)(X_2 \Lambda X_2^\dagger W_1)^* = C_2 + K_2 W_1(X_2 \Lambda X_2^\dagger W_1)^*, \quad (3.11)$$

and its solution is $\hat{F} = (C_2 + K_2(X_2 \Lambda X_2^\dagger)^*)(I + (X_2 \Lambda X_2^\dagger)(X_2 \Lambda X_2^\dagger)^*)^{-1}$. Again from Lemma 3.1, we have that, when $F = \hat{F}$, g attains its minimum, which gives $\hat{Y} = R_2^* K_2 W_2 - R_2^* (X_1 \Lambda X_1^\dagger)^* \hat{F} W_2$, and $\hat{G} = (X_1 \Lambda X_1^\dagger)^* \hat{F} W_2 + R_2 \hat{Y} = K_2 W_2$. Then, the unique solution of Problem 2 given by (3.3) is obtained. \square

Now, we give an algorithm to compute the optimal approximate solution of Problem 2.

Algorithm.

- (1) Input K_a, M_a, X, Λ , and U .
- (2) Compute X_2 according to (2.3).
- (3) Find the singular value decomposition of X_2 according to (2.4).
- (4) Calculate \hat{F} by (3.4).
- (5) Compute (\hat{M}, \hat{K}) by (3.3).

Example 1. Let $n = 6$, $m = 3$, and the matrices M_a, K_a, X , and Λ be given by

$$M_a = \begin{pmatrix} 1.56 & 0.66 & 0.54 & -0.39 & 0 & 0 \\ 0.66 & 0.36 & 0.39 & -0.27 & 0 & 0 \\ 0.54 & 0.39 & 3.12 & 0 & 0.54 & -0.39 \\ -0.39 & -0.27 & 0 & 0.72 & 0.39 & -0.27 \\ 0 & 0 & 0.54 & 0.39 & 3.12 & 0 \\ 0 & 0 & -0.39 & -0.27 & 0 & 0.72 \end{pmatrix},$$

$$K_a = \begin{pmatrix} 2 & 3 & -2 & 3 & 0 & 0 \\ 3 & 6 & -3 & 3 & 0 & 0 \\ -2 & -3 & 4 & 0 & -2 & 3 \\ 3 & 3 & 0 & 12 & -3 & 3 \\ 0 & 0 & -2 & -3 & 4 & 0 \\ 0 & 0 & 3 & 3 & 0 & 12 \end{pmatrix},$$

$$X = \begin{pmatrix} 0.0347 + 0.1507i & -0.6975i & 0.0003 + 0.0858i \\ 0.6715i & 0.0277 + 0.0760i & -0.0846 - 0.0101i \\ -0.0009 + 0.1587i & -0.0814 + 0.0196i & 0.6967 \\ -0.1507 + 0.0347i & 0.6975 & -0.0858 + 0.0003i \\ -0.6715 & -0.0760 + 0.0277i & 0.0101 - 0.0846i \\ -0.1587 - 0.0009i & -0.0196 - 0.0814i & 0.6967i \end{pmatrix},$$

$$\Lambda = \text{diag}(0.3848 + 0.0126i, 2.5545 + 0.4802i, 2.5607). \quad (3.12)$$

From the Algorithm, we obtain the unique solution of Problem 2 as follows:

$$\hat{F} = \begin{pmatrix} -1.4080 + 1.1828i & 1.0322 + 0.4732i & -0.8111 - 0.0874i \\ 0.9537 + 0.2935i & -0.7529 - 0.0137i & -0.6596 - 0.3106i \\ -0.6624 + 0.1982i & -0.3566 - 0.0051i & -1.0958 + 1.0040i \end{pmatrix},$$

$$\hat{N} = \begin{pmatrix} -4.3706 + 2.1344i & 1.6264 - 0.3128i & -2.2882 - 0.3290i \\ 2.4251 + 1.2137i & -0.5229 + 0.0005i & -1.4620 - 0.7688i \\ -1.6669 + 0.1663i & 0.6991 - 0.6057i & -2.6437 + 2.5190i \end{pmatrix}, \quad (3.13)$$

$$\hat{M} = U \begin{pmatrix} 0 & \hat{F} \\ \hat{F}^* & 0 \end{pmatrix} U^*, \quad \hat{K} = U \begin{pmatrix} 0 & \hat{N} \\ \hat{N} & 0 \end{pmatrix} U^*,$$

where $U = (1/\sqrt{2}) \begin{pmatrix} I_3 & I_3 \\ -iI_3 & iI_3 \end{pmatrix}$. It is easy to calculate $\|\hat{K}X - \hat{M}X\Lambda\| = 2.1121e - 015$, and $\|[\hat{M} - M_a, \hat{K} - K_a]\| = 19.7467$.

Acknowledgments

This paper was granted financial support from National Natural Science Foundation (10901056) and Shanghai Natural Science Foundation (09ZR1408700), NSFC grant (10971070). The authors would like to thank the referees for their valuable comments and suggestions.

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