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*Research Article*

# The Extended Tanh Method and the Exp-Function Method to Solve a Kind of Nonlinear Heat Equation

**Weimin Zhang**

*School of Mathematics, Jiaying University, Meizhou, Guangdong 514015, China*

Correspondence should be addressed to Weimin Zhang, wm-zh@163.com

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We study a kind of nonlinear heat equation with temperature-dependent thermal properties by the aid of the extended Tanh method and the Exp-function method. We obtain abundant new exact solutions of the equation. By comparing both of the methods, we find that the Exp-function method gives more solutions in this problem.

## 1. Introduction

The classical heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

also known as the diffusion equation, describes in typical applications of the evolution in time of the density $u = u(x, t)$ of some quantities such as heat and chemical concentration [1, page 44]. In this case, the thermal diffusivity and thermal conductivity of the medium are assumed to be constant. However, in some media such as gases, the parameters are proportional to the temperature of the medium giving rise to a nonlinear heat equation of the following form [2]:

$$C(x)\frac{\partial u}{\partial t} = \lambda \frac{\partial}{\partial x}\left(ku\frac{\partial u}{\partial x}\right), \tag{1.2}$$

where $C = C(x)$ is the conductivity, $k$ is diffusivity, and $\lambda$ is a constant. When the diffusivity is proportional to $u^{\alpha}$, a more general nonlinear heat equation reads as

$$C(x)\frac{\partial u}{\partial t} = \lambda\frac{\partial}{\partial x}\left(u^{\alpha}\frac{\partial u}{\partial x}\right). \tag{1.3}$$

In a recent paper [3], using the Adomian decomposition method, the author discussed the following nonlinear heat equation with temperature dependent diffusivity:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u}{\partial x}\right), \tag{1.4}$$

where $f(u) = u^{m}$and $m = 2, -2, 1/2$.

In this paper we are interested in the following nonlinear heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-1}\frac{\partial u}{\partial x}\right) \tag{1.5}$$

and discuss its traveling wave solutions. As we know, a solution $u$ of the form

$$u(x,t) = U(\xi), \quad \xi = k(x - ct) \tag{1.6}$$

is called a traveling wave (with wavefront normal to $k$, velocity $c/|k|$, and profile $U$) [1, page 172].

Here we employ, for the first time, the extended Tanh method and Exp-function method for solving (1.5), and abundant new exact solutions of (1.5) are presented. We compare both of the methods and find that the Exp-function method is more efficient than the extended Tanh method in this problem.

## 2. The Extended Tanh Method

We now describe the extended Tanh method for the given partial differential equations. The Tanh method was defined by Malfliet [4] and Fan and Hon [5]. The Tanh method was successfully applied to nonlinear evolution equations [6, 7], and so on. The extended Tanh method was presented in [8] to solve breaking solitary equation. Wazwaz summarized the main steps introduced for using this method as follows [9].

We consider first a general form of nonlinear partial differential equation involving the two variables $t, x$

$$P(u, u_t, u_x, u_{xx}, \ldots) = 0. \tag{2.1}$$

In this paper we only discuss the traveling wave solutions.

(1) To find the traveling wave solution of (2.1), make the transformation

$$u(x,t) = U(\xi), \quad \xi = k(x - ct), \tag{2.2}$$

where $k, c$ are constants to be determined later. From this reason, we use the following changes:

$$\frac{\partial}{\partial t} = -kc\frac{d}{d\xi}, \qquad \frac{\partial}{\partial x} = -k\frac{d}{d\xi},$$
$$\frac{\partial^2}{\partial x^2} = k^2\frac{d^2}{d\xi^2}, \qquad \frac{\partial^3}{\partial x^3} = k^3\frac{d^3}{d\xi^3}, \ldots, \tag{2.3}$$

and so on for the other derivates. Using (2.3) changes the NLPDE (2.1) to an ODE

$$P(U, U', U'', U''', \ldots) = 0. \tag{2.4}$$

(2) If all terms of the resulting ODE contain derivatives in $\xi$, then by integrating this equation, by considering the constant of integration to be zero, we obtain a simplified ODE.

(3) We then introduce a new independent variable

$$Y = \tanh(\xi) \quad \text{or} \quad Y = \coth(\xi) \tag{2.5}$$

that leads to the change of derivates

$$\frac{d}{d\xi} = \left(1 - Y^2\right)\frac{d}{dY},$$
$$\frac{d^2}{d\xi^2} = \left(1 - Y^2\right)\left(-2Y\frac{d}{dY} + \left(1 - Y^2\right)\frac{d^2}{dY^2}\right), \tag{2.6}$$
$$\frac{d^3}{d\xi^3} = \left(1 - Y^2\right)\left(\left(6Y^2 - 2\right)\frac{d}{dY} - 6Y\left(1 - Y^2\right)\frac{d^2}{dY^2} + \left(1 - Y^2\right)^2\frac{d^3}{dY^3}\right),$$

where other derivatives can be derived in a similar manner. We use a new independent variable [9]

$$Y = \tan(\xi) \quad \text{or} \quad Y = -\cot(\xi) \tag{2.7}$$

that leads to the change of derivates

$$\frac{d}{d\xi} = \left(1 + Y^2\right)\frac{d}{dY},$$
$$\frac{d^2}{d\xi^2} = \left(1 + Y^2\right)\left(-2Y\frac{d}{dY} + \left(1 + Y^2\right)\frac{d^2}{dY^2}\right), \tag{2.8}$$
$$\frac{d^3}{d\xi^3} = \left(1 + Y^2\right)\left(\left(6Y^2 + 2\right)\frac{d}{dY} + 6Y\left(1 + Y^2\right)\frac{d^2}{dY^2} + \left(1 + Y^2\right)^2\frac{d^3}{dY^3}\right),$$

where other derivatives can be derived.

(4) Introduce the ansatz

$$U(\xi) = \sum_{s=-n}^{m} a_s Y^s, \tag{2.9}$$

where $m, n$ are nonnegative integers, in most cases, that will be determined. Substituting (2.6) and (2.7) into the ODE (2.4) yields an equation in powers of $Y$.

(5) To determine the parameter$m, n$, we usually balance linear derivative term of the highest order in the resulting equation with the highest order nonlinear terms [8, 9]. With $m, n$ determined, equate the coefficients of powers of $Y$ to zero in the resulting equation. This will give a system of algebraic equations involving the$a_s$, $(s = -n, \ldots, 0, 1, \ldots, m)$. Having determined these parameters, knowing that it is a positive integer in most cases, using (2.9) we obtain an analytic solution in a closed form.

It is worthy notice if $n = 0$ in (2.9), then the extended Tanh method reduces to the Tanh method, so the Tanh method is a special case of the extended Tanh method.

## 3. The Exp-Function Method

Recently, He and Wu [10] proposed a straightforward and concise method called Exp-function method to obtain exact solutions of NLEEs. The Exp-function method leads to both generalized solitary solutions and periodic solutions [11–14] and was successfully applied to KdV equation with variable coefficients [15], to the combine KdV-mKdV equations with variable coefficients [16], to difference-differential equations [17, 18], and so forth. This paper applies the Exp-function method with the help of Mathematica computation to a kind of nonlinear heat equation with temperature-dependent thermal properties; abundant new exact solutions are hereby constructed. We consider a general nonlinear PDE in the form

$$P(u, u_t, u_x, u_{xx}, \ldots) = 0. \tag{3.1}$$

Using a transformation

$$u(x,t) = U(\xi), \quad \xi = \alpha(x - \beta t), \tag{3.2}$$

where $\alpha, \beta$ are constants, we can rewrite (3.1) in the following nonlinear ODE:

$$Q(U, U', U'', U''', \ldots) = 0, \tag{3.3}$$

where the prime denotes the derivation with respect to $\xi$.

According to Exp-function method, we assume that the solution can be expressed in the form [10]

$$U(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}, \tag{3.4}$$

where $c, d, p$, and $q$ are positive integers which could be freely chosen and $a_n$ and $b_m$ are unknown constants to be determined. In order to determine the values of $c$ and $p$, we balance the linear derivative term of highest order in (3.3) with the highest order nonlinear term [10]. Similarly, to determine the values of $d$ and $q$, we balance the linear derivative term of lowest order in (3.3) with the lowest order nonlinear term [10].

## 4. The Extended Tanh Method to the Nonlinear Heat Equation (1.5)

As described in Section 2, we make the transformation

$$u(x,t) = U(\xi), \quad \xi = r(x - ct), \tag{4.1}$$

and (1.5) becomes

$$-cU^2U' + r(U')^2 - rUU'' = 0. \tag{4.2}$$

By balancing the nonlinear terms $U^2U', (U')^2, UU''$,we have

$$\begin{gathered} 2m + (m+1) = 2(m+1) = m + (m+2), \\ -2n - (n+1) = -2(n+1) = -n - (n+1), \end{gathered} \tag{4.3}$$

which yields $m = n = 1$. Therefore by the use of the Tanh method, we may choose a solution of (4.2) in the form

$$U = U(\xi) = a_{-1}Y^{-1} + a_0 + a_1Y, \tag{4.4}$$

where $Y = \tanh(\xi)$ or $Y = \coth(\xi)$ and $a_j$ $(j = -1, 0, 1)$ are constants to be determined later. Substituting (4.4) into (4.2) we have

$$Y^{-4}\left(A_0 + A_1Y + A_2Y^2 + A_3Y^3 + A_4Y^4 + A_5Y^5 + A_6Y^6 + A_7Y^7 + A_8Y^8\right) = 0, \tag{4.5}$$

where

$$\begin{gathered} A_0 \equiv -ra_{-1}^2 + ca_{-1}^3, \qquad A_1 \equiv -2ra_{-1}a_0 + 2ca_{-1}^2a_0, \\ A_2 \equiv -ca_{-1}^3 + ca_{-1}a_0^2 - 4ra_{-1}a_1 + ca_{-1}^2a_1, \\ A_3 \equiv 2ra_{-1}a_0 - 2ca_{-1}^2a_0, \\ A_4 \equiv ra_{-1}^2 - ca_{-1}a_0^2 + 8ra_{-1}a_1 - ca_{-1}^2a_1 - ca_0^2a_1 + ra_1^2 - ca_{-1}a_1^2, \\ A_5 \equiv 2ra_0a_1 - 2ca_0a_1^2, \qquad A_6 \equiv -4ra_{-1}a_1 + ca_0^2a_1 + ca_{-1}a_1^2 - ca_1^3, \\ A_7 \equiv -2ka_0a_1 + 2ca_0a_1^2, \qquad A_8 = -ka_1^2 + ca_1^3. \end{gathered} \tag{4.6}$$

Solving the following algebraic equation system with the aid of the Mathematica Package

$$\{A_0 = 0,\ A_1 = 0,\ A_2 = 0,\ A_3 = 0,\ A_4 = 0,\ A_5 = 0,\ A_6 = 0,\ A_7 = 0,\ A_8 = 0\}, \tag{4.7}$$

we then get the following results:

(1) $\{a_{-1} = (r/c),\quad a_0 = -(2r/c),\quad a_1 = (r/c)\}$,

(2) $\{a_{-1} = (r/c),\quad a_0 = (2r/c),\quad a_1 = (r/c)\}$,

(3) $\{a_{-1} = 0,\quad a_0 = -(r/c),\quad a_1 = (r/c)\}$,

(4) $\{a_{-1} = (r/c),\quad a_0 = -(r/c),\quad a_1 = 0\}$,

(5) $\{a_{-1} = 0,\quad a_0 = (r/c),\quad a_1 = (r/c)\}$,

(6) $\{a_{-1} = (r/c),\quad a_0 = (r/c),\quad a_1 = 0\}$,

where $r, c$ are nonzero free parameters. Substituting these results into (4.4) and then changing to exponential form, we obtain the following exact solutions:

$$\begin{aligned}
u_{11}(x,t) &= \frac{r}{c}(\coth[r(x-ct)] - 2 + \tanh[r(x-ct)]) = \frac{4r}{c(-1+\exp[4r(x-ct)])},\\
u_{12}(x,t) &= \frac{r}{c}(\coth[r(x-ct)] + 2 + \tanh[r(x-ct)]) = \frac{4r\exp[4r(x-ct)]}{c(-1+\exp[4r(x-ct)])},\\
u_{13}(x,t) &= \frac{r}{c}(-1 + \tanh[r(x-ct)]) = -\frac{2r}{c(1+\exp[2r(x-ct)])},\\
u_{14}(x,t) &= \frac{r}{c}(-1 + \coth[r(x-ct)]) = \frac{2r}{c(-1+\exp[2\mathrm{r}(x-ct)])},\\
u_{15}(x,t) &= \frac{r}{c}(1 + \tanh[r(x-ct)]) = \frac{2r\exp[2r(x-ct)]}{c(1+\exp[2\mathrm{r}(x-ct)])},\\
u_{16}(x,t) &= \frac{r}{c}(1 + \coth[r(x-ct)]) = \frac{2r\exp[2r(x-ct)]}{c(-1+\exp[2r(x-ct)])}.
\end{aligned} \tag{4.8}$$

If we choose the solution forms of (2.7) and insert them into (4.4) and (4.2), we have

$$Y^{-4}\left(B_0 + B_1Y + B_2Y^2 + B_3Y^3 + B_4Y^4 + B_5Y^5 + B_6Y^6 + B_7Y^7 + B_8Y^8\right) = 0, \tag{4.9}$$

where

$$
\begin{gathered}
B_0 \equiv -ra_{-1}^2 + ca_{-1}^3, \qquad B_1 \equiv -2ra_{-1}a_0 + 2ca_{-1}^2 a_0, \\
B_2 \equiv ca_{-1}^3 + ca_{-1}a_0^2 - 4ra_{-1}a_1 + ca_{-1}^2 a_1, \qquad B_3 \equiv -2ra_{-1}a_0 + 2ca_{-1}^2 a_0, \\
B_4 \equiv ra_{-1}^2 + ca_{-1}a_0^2 - 8ra_{-1}a_1 + ca_{-1}^2 a_1 - ca_0^2 a_1 + ra_1^2 - ca_{-1}a_1^2, \\
B_5 \equiv -2ra_0a_1 - 2ca_0a_1^2, \qquad B_6 \equiv -4ra_{-1}a_1 - ca_0^2 a_1 - ca_{-1}a_1^2 - ca_1^3, \\
B_7 \equiv -2ra_0a_1 - 2ca_0a_1^2, \qquad B_8 \equiv -ra_1^2 - ca_1^3.
\end{gathered} \tag{4.10}
$$

Solving the following algebraic equation system with the aid of the Mathematica Package

$$\{B_0 = 0,\ B_1 = 0,\ B_2 = 0,\ B_3 = 0,\ B_4 = 0,\ B_5 = 0,\ B_6 = 0,\ B_7 = 0,\ B_8 = 0\}, \tag{4.11}$$

we then get the following results:

(1) $\{a_{-1} = (r/c),\ a_0 = -(2ir/c),\ a_1 = -(r/c)\}$,

(2) $\{a_{-1} = (r/c),\ a_0 = (2ir/c),\ a_1 = -(r/c)\}$,

(3) $\{a_{-1} = 0,\ a_0 = -(ir/c),\ a_1 = -(r/c)\}$,

(4) $\{a_{-1} = 0,\ a_0 = (ir/c),\ a_1 = -(r/c)\}$,

(5) $\{a_{-1} = (r/c),\ a_0 = -(ir/c), a_1 = 0\}$,

(6) $\{a_{-1} = (r/c),\ a_0 = (ir/c),\ a_1 = 0\}$,

where $r, c$ are nonzero free parameters and $i^2 = -1$. Substituting these results into (4.4) and changing to exponential form, we obtain the following exact solutions:

$$
\begin{aligned}
u_{21}(x,t) &= \frac{r}{c}(\cot[r(x-ct)] - 2i - \tan[r(x-ct)]) = \frac{4ri\exp[4ri(x-ct)]}{c(-1+\exp[4ri(x-ct)])}, \\
u_{22}(x,t) &= \frac{r}{c}(\cot[r(x-ct)] + 2i - \tan[r(x-ct)]) = \frac{4ri}{c(-1+\exp[4ri(x-ct)])}, \\
u_{23}(x,t) &= \frac{r}{c}(-i - \tan[r(x-ct)]) = -\frac{2ri}{c(1+\exp[2ri(x-ct)])}, \\
u_{24}(x,t) &= \frac{r}{c}(i - \tan[r(x-ct)]) = \frac{2ri\exp[2ri(x-ct)]}{c(1+\exp[2ri(x-ct)])}, \\
u_{25}(x,t) &= \frac{r}{c}(-i + \cot[r(x-ct)]) = \frac{2ri}{c(-1+\exp[2ri(x-ct)])}, \\
u_{26}(x,t) &= \frac{r}{c}(i + \cot[r(x-ct)]) = \frac{2ri\exp[2ri(x-ct)]}{c(-1+\exp[2ri(x-ct)])}.
\end{aligned} \tag{4.12}
$$

## 5. The Exp-Function Method to the Nonlinear Heat Equation (1.5)

In this section, the Exp-function method is applied to the nonlinear heat equation (1.5).

Using the transformation

$$u(x,t) = U(\xi), \quad \xi = k(x - ct), \tag{5.1}$$

(1.5) becomes

$$-cU^2U' + k(U')^2 - kUU'' = 0. \tag{5.2}$$

Here we assume that the solution of (5.2) can be expressed in the following form [10]:

$$U(\xi) = \frac{\sum_{i=-m}^{n} a_i \exp(i\xi)}{\sum_{j=-s}^{t} b_m \exp(j\xi)} = \frac{a_{-m}\exp(-m\xi) + \cdots + a_n \exp(n\xi)}{b_{-s}\exp(-s\xi) + \cdots + b_t \exp(t\xi)}, \tag{5.3}$$

where $a_i, b_j (i, j \in \mathbb{Z})$ are unknown constants and $m, n, s, t$ are nonnegative integers to be further determined. Here take notice of nonlinear term in (5.2), and we can balance $U^2U', (U')^2$ and $UU''$ by the idea of the Exp-function method [10] to determine the values of $m, ns, t$. By simple calculation, we have

$$\begin{aligned}
UU' &= \frac{c_1 \exp[-(2m+3s)\xi] + \cdots + c_2 \exp[(2n+3t)\xi]}{d_1 \exp[-5s\xi] + \cdots + d_2 \exp[5t\xi]}, \\
(U')^2 &= \frac{c_3 \exp[-(2m+3s)\xi] + \cdots + c_4 \exp[(2n+3t)\xi]}{d_3 \exp[-5s\xi] + \cdots + d_4 \exp[5t\xi]}, \\
UU'' &= \frac{c_5 \exp[-(2m+3s)\xi] + \cdots + c_6 \exp[(2n+3t)\xi]}{d_5 \exp[-5s\xi] + \cdots + d_6 \exp[5t\xi]}.
\end{aligned} \tag{5.4}$$

According to (5.4), we find that $m, n, s, t$ are arbitrary nonnegative integers. This provides great freedom to choose $m, n,\ s, t$and may be get more abundant solutions of (1.5). For simplicity, we only discuss the following one case, that is $m = s = 1$ and $n = t = 1$. In this case, (5.3) reduces to

$$U = U(\xi) = \frac{a_{-1}\exp(-\xi) + a_0 + a_1 \exp(\xi)}{b_{-1}\exp(-\xi) + b_0 + b_1 \exp(\xi)}. \tag{5.5}$$

Substituting (5.5) into (5.2) by help of Mathematica package computation yields

$$A\left(A_1 e^{\xi} + A_2 e^{2\xi} + A_3 e^{3\xi} + A_4 e^{4\xi} + A_5 e^{5\xi} + A_6 e^{6\xi} + A_7 e^{7\xi}\right) = 0, \tag{5.6}$$

where

$$
\begin{aligned}
A &= \left(b_{-1} + b_0 e^{\xi} + b_1 e^{2\xi}\right)^4, \\
A_1 &= -ca_{-1}^2 a_0 b_{-1} - ka_{-1} a_0 b_{-1}^2 + ca_{-1}^3 b_0 + ka_{-1}^2 b_{-1} b_0, \\
A_2 &= -2ca_{-1} a_0^2 b_{-1} - 2ca_{-1}^2 a_1 b_{-1} - 4ka_{-1} a_1 b_{-1}^2 + 2ca_{-1}^2 a_0 b_0 + 2ca_{-1}^3 b_1 + 4ka_{-1}^2 b_{-1} b_1, \\
A_3 &= -ca_0^2 b_{-1} - 6ca_{-1} a_0 a_1 b_{-1} - ka_0 a_1 b_{-1}^2 + ca_{-1} a_0^2 b_0 + ca_{-1}^2 a_1 b_0 \\
&\quad + ka_0^2 b_{-1} b_0 - 6ka_{-1} a_1 b_{-1} b_0 - ka_{-1} a_0 b_0^2 + 5ca_{-1}^2 a_0 b_1 + 6ka_{-1} a_0 b_{-1} b_1 + ka_{-1}^2 b_0 b_1, \\
A_4 &= -4ca_0^2 a_1 b_{-1} - 4ca_{-1} a_1^2 b_{-1} - 4ka_{-1} a_1 b_0^2 + 4ca_{-1} a_0^2 b_1 + 4ca_{-1}^2 a_1 b_1 + 4ka_0^2 b_{-1} b_1, \\
A_5 &= -5ca_0 a_1^2 b_{-1} - ca_0^2 a_1 b_0 - ca_{-1} a_1^2 b_0 + ka_1^2 b_{-1} b_0 - ka_0 a_1 b_0^2 \\
&\quad + ca_0^3 b_1 + 6ca_{-1} a_0 a_1 b_1 + 6ka_0 a_1 b_{-1} b_1 + ka_0^2 b_0 b_1 - 6ka_{-1} a_1 b_0 b_1 - ka_{-1} a_0 b_1^2, \\
A_6 &= -2ca_1^3 b_{-1} - 2ca_0 a_1^2 b_0 + 2ca_0^2 a_1 b_1 + 2ca_{-1} a_1^2 b_1 + 4ka_1^2 b_{-1} b_1 - 4ka_{-1} a_1 b_1^2, \\
A_7 &= -ca_1^3 b_0 + ca_0 a_1^2 b_1 + ka_1^2 b_0 b_1 - ka_0 a_1 b_1^2.
\end{aligned}
\tag{5.7}
$$

Equating the coefficients of $e^{n\xi}$ $(n = 1, 2, \ldots, 7)$ to zero, we get a set of algebraic equations

$$\{A_1 = 0, A_2 = 0, A_3 = 0, A_4 = 0, A_5 = 0, A_6 = 0, A_7 = 0\}. \tag{5.8}$$

Solving the above system by using Mathematica Package, we can get the solution as follows:

(1) $\{a_{-1} = 0, a_0 = a_0, a_1 = 0, b_{-1} = 0, b_0 = -(ca_0/k), b_1 = b_1\}$.

(2) $\{a_{-1} = 0, a_0 = 0, a_1 = a_1, b_{-1} = 0, b_0 = b_0, b_1 = (ca_1/k)\}$.

(3) $\{a_{-1} = a_{-1}, a_0 = 0, a_1 = 0, b_{-1} = -(ca_{-1}/k), b_0 = b_0, b_1 = 0\}$.

(4) $\{a_{-1} = a_{-1}, a_0 = 0, a_1 = 0, b_{-1} = -(ca_{-1}/2k), b_0 = 0, b_1 = b_1\}$.

(5) $\{a_{-1} = 0, a_0 = 0, a_1 = a_1, b_{-1} = 0, b_0 = 0, b_1 = (ca_1/2k)\}$.

(6) $\{a_{-1} = 0, a_0 = a_0, a_1 = 0, b_{-1} = b_{-1}, b_0 = (ca_0/k), b_1 = 0\}$.

(7) $\{a_{-1} = a_{-1}, a_0 = a_0, a_1 = 0, b_{-1} = -(ca_{-1}/k), b_0 = b_0, b_1 = a_0(ca_0 + kb_0)/ka_{-1}\}$.

(8) $\{a_{-1} = 0, a_0 = a_0, a_1 = a_1, b_{-1} = -(a_0(ca_0 - kb_0)/ka_1), b_0 = b_0, b_1 = (ca_1/k)\}$.

(9) $\{a_{-1} = a_{-1}, a_0 = a_0, a_1 = 0, b_{-1} = -(ca_{-1}/k), b_0 = -(ca_0/2k), b_1 = (ca_0^2/2ka_{-1})\}$.

(10) $\{a_{-1} = 0, a_0 = a_0, a_1 = a_1, b_{-1} = -(ca_0^2/2ka_1), b_0 = (ca_0/2k), b_1 = (ca_1/k)\}$.

Substituting cases (1)–(10) into (5.5) yields

$$\begin{aligned}
u_{31} &= \frac{a_0}{-(ca_0/k) + b_1\exp(\xi)} = \frac{ka_0}{-ca_0 + kb_1\exp(k(x-ct))}.\\
u_{32} &= \frac{a_1\exp(\xi)}{(ca_1/k)\exp(\xi) + b_0} = \frac{ka_1\exp(k(x-ct))}{kb_0 + ca_1\exp(k(x-ct))}.\\
u_{33} &= \frac{a_{-1}\exp(-\xi)}{-(ca_{-1}/k)\exp(-\xi) + b_0} = \frac{ka_{-1}\exp[-k(x-ct)]}{-ca_{-1}\exp[-k(x-ct)] + kb_0}.\\
u_{34} &= \frac{a_{-1}\exp(-\xi)}{-(ca_{-1}/2k)\exp(-\xi) + b_1\exp(\xi)} = \frac{2ka_{-1}\exp[-k(x-ct)]}{-ca_{-1}\exp[-k(x-ct)] + 2kb_1\exp[k(x-ct)]}.\\
u_{35} &= \frac{a_1\exp(\xi)}{b_{-1}\exp(-\xi) + (ca_{-1}/2k)\exp(\xi)} = \frac{2ka_1\exp[k(x-ct)]}{2kb_{-1}\exp[-k(x-ct)] + ca_1\exp[k(x-ct)]}.\\
u_{36} &= \frac{a_0}{b_{-1}\exp(-\xi) + (ca_0/k)} = \frac{ka_0}{kb_{-1}\exp[-k(x-ct)] + ca_0}.\\
u_{37} &= \frac{a_{-1}\exp(-\xi) + a_0}{-(ca_{-1}/k)\exp(-\xi) + b_0 + (a_0(ca_0+kb_0)/ka_{-1})\exp(\xi)}\\
&= \frac{ka_{-1}^2\exp[-k(x-ct)] + ka_{-1}a_0}{-ca_{-1}^2\exp[-k(x-ct)] + ka_{-1}b_0 + a_0(ca_0+kb_0)\exp[k(x-ct)]}.\\
u_{38} &= \frac{a_0 + a_1\exp(\xi)}{-(a_0(ca_0-kb_0)/ka_1)\exp(-\xi) + b_0 + (ca_1/k)\exp(\xi)}\\
&= \frac{ka_1a_0 + ka_1^2\exp[k(x-ct)]}{a_0(kb_0-ca_0)\exp[-k(x-ct)] + ka_1b_0 + ca_1^2\exp[k(x-ct)]}.\\
u_{39} &= \frac{a_0 + a_{-1}\exp(-\xi)}{-(ca_{-1}/k)\exp(-\xi) - (ca_0/2k) + \left(ca_0^2/2ka_{-1}\right)\exp(\xi)}\\
&= \frac{2ka_{-1}a_0 + 2ka_{-1}^2\exp[-k(x-ct)]}{-2ca_{-1}^2\exp[-k(x-ct)] - ca_{-1}a_0 + ca_0^2\exp[k(x-ct)]}.\\
u_{3,10} &= \frac{a_0 + a_1\exp(\xi)}{-\left(ca_0^2/2ka_1\right)\exp(-\xi) + (ca_0/2k) + (ca_1/k)\exp(\xi)}\\
&= \frac{2ka_1a_0 + 2ka_1^2\exp[k(x-ct)]}{-ca_0^2\exp[-k(x-ct)] + ca_1a_0 + 2ca_1^2\exp[k(x-ct)]}.
\end{aligned} \tag{5.9}$$

## 6. Comparison and Discussion

In this section we make comparison between the Tanh method's solutions and the Exp-function method solutions of (1.5). We can obtain the following results.

(1) If we set $k = 4r, a_0 = 1, b_1 = (c/4r)$ in the equation $u_{31}$, then $u_{31} = u_{11}$,

(2) If we set $k = 4r, a_1 = 1, b_0 = -(c/4r)$ in the equation $u_{32}$, then $u_{32} = u_{12}$,

(3) If we set $k = 2r, a_0 = -1, b_1 = -(c/2r)$ in the equation $u_{31}$, then $u_{31} = u_{13}$,

(4) If we set $k = 2r, a_0 = 1, b_1 = (c/2r)$ in the equation $u_{31}$, then $u_{31} = u_{14}$,

(5) If we set $k = 2r, a_1 = 1, b_0 = (c/2r)$ in the equation $u_{32}$, then $u_{32} = u_{15}$,

(6) If we set $k = 2r, a_1 = 1, b_0 = -(c/2r)$ in the equation $u_{32}$, then $u_{32} = u_{16}$,

(7) If we set $k = 4ri, a_1 = 1, b_0 = -(c/4r)$ in the equation $u_{32}$, then $u_{32} = u_{21}$,

(8) If we set $k = 4ri, a_0 = 1, b_1 = (c/4ri)$ in the equation $u_{31}$, then $u_{31} = u_{22}$,

(9) If we set $k = 2ri, a_0 = 1, b_1 = -(c/2ri)$ in the equation $u_{31}$, then $u_{31} = u_{23}$,

(10) If we set $k = 2ri, a_1 = 1, b_0 = (c/2ri)$ in the equation $u_{32}$, then $u_{32} = u_{24}$,

(11) If we set $k = 2ri, a_0 = 1, b_1 = (c/2ri)$ in the equation $u_{31}$, then $u_{31} = u_{25}$,

(12) If we set $k = 2ri, a_1 = 1, b_0 = -(c/2ri)$ in the equation $u_{32}$, then $u_{32} = u_{26}$.

where $i^2 = -1$. The above obtained results show that the Exp-function method can obtain more abundant explicit solutions than Tanh method for (1.5). If we use the method of separation of variables [1, page 167], the rational solution of (1.5) can be constructed as follows:

$$u(x,t) = \frac{2a^2(kt+b)}{k(ax+c)^2}, \tag{6.1}$$

where $k, a, b, c$ are constants.

## 7. Conclusion

Nonlinear phenomena appear in a wide variety of scientific fields, such as applied mathematics, physics and engineering problems. However, solving nonlinear differential equations corresponding to the nonlinear problems are often complicate. Especially, obtaining their explicit solutions is even more difficult. Up to now, a lot of new methods for solving nonlinear differential equations are developed, for example, Bäcklund transformation method, inverse scattering method, Darboux transformation method, Hirota's bilinear method, and so forth. But, generally speaking, all of the above methods have their own advantages and shortcomings, respectively. In this paper, by applying the Exp-function method and the extended Tanh method with the help of Mathematica computation to the nonlinear heat equation with temperature-dependent thermal properties, we obtain abundant exact solutions. The obtained results show that the Exp-function method and the extended Tanh method are simple and effective methods to solve nonlinear differential equations. By comparison, we find that the Exp-function method is more effective in finding exact solutions than the extended Tanh method for (1.5).

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