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Research Article

Warped Product Semi-Invariant Submanifolds of Nearly Cosymplectic Manifolds

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We study warped product semi-invariant submanifolds of nearly cosymplectic manifolds. We prove that the warped product of the type $M_\perp \times_f M_T$ is a usual Riemannian product of $M_\perp$ and M_T, where $M_\perp$ and M_T are anti-invariant and invariant submanifolds of a nearly cosymplectic manifold $\overline{M}$, respectively. Thus we consider the warped product of the type $M_T \times_f M_\perp$ and obtain a characterization for such type of warped product.

1. Introduction

The notion of warped product manifolds was introduced by Bishop and O'Neill in 1969 as a natural generalization of the Riemannian product manifolds. Later on, the geometrical aspect of these manifolds has been studied by many researchers (cf., [1–3]). Recently, Chen [1] (see also [4]) studied warped product CR-submanifolds and showed that there exists no warped product CR-submanifolds of the form $M = M_\perp \times_f M_T$ such that $M_\perp$ is a totally real submanifold and M_T is a holomorphic submanifold of a Kaehler manifold $\overline{M}$. Therefore he considered warped product CR-submanifold in the form $M = M_T \times_f M_\perp$ which is called CR-warped product, where M_T and $M_\perp$ are holomorphic and totally real submanifolds of a Kaehler manifold $\overline{M}$. Motivated by Chen's papers, many geometers studied CR-warped product submanifolds in almost complex as well as contact setting (see [3, 5, 6]).

Almost contact manifolds with Killing structure tensors were defined in [7] as nearly cosymplectic manifolds, and it was shown that normal nearly cosymplectic manifolds are cosymplectic (see also [8]). Later on, Blair and Showers [9] studied nearly cosymplectic structure (ϕ, ξ, η, g) on a manifold $\overline{M}$ with η closed from the topological viewpoint.

In this paper, we have generalized the results of Chen' [1] in this more general setting of nearly cosymplectic manifolds and have shown that the warped product in the form $M = M_{\perp} \times_f M_T$ is simply Riemannian product of $M_{\perp}$ and M_T where $M_{\perp}$ is an anti-invariant submanifold and M_T is an invariant submanifold of a nearly cosymplectic manifold $\overline{M}$. Thus we consider the warped product submanifold of the type $M = M_T \times_f M_{\perp}$ by reversing the two factors $M_{\perp}$ and M_T and simply will be called *warped product semi-invariant submanifold*. Thus, we derive the integrability of the involved distributions in the warped product and obtain a characterization result.

2. Preliminaries

A $(2n+1)$-dimensional C^{∞} manifold $\overline{M}$ is said to have an *almost contact structure* if there exist on $\overline{M}$ a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying [9]

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact manifold $\overline{M}$ satisfying the following compatibility condition:

$$\eta(X) = g(X, \xi), \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

where X and Y are vector fields on $\overline{M}$ [9].

An almost contact structure (ϕ, ξ, η) is said to be *normal* if the almost complex structure J on the product manifold $\overline{M} \times \mathbb{R}$ given by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi,\ \eta(X)\frac{d}{dt}\right), \tag{2.3}$$

where f is a C^{∞}-function on $\overline{M} \times \mathbb{R}$, has no torsion, that is, J is integrable, and the condition for normality in terms of ϕ, ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on $\overline{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Finally the *fundamental 2-form* Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be *cosymplectic*, if it is normal and both Φ and η are closed [9]. The structure is said to be *nearly cosymplectic* if ϕ is Killing, that is, if

$$\left(\overline{\nabla}_X \phi\right)Y + \left(\overline{\nabla}_Y \phi\right)X = 0, \tag{2.4}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ is the tangent bundle of $\overline{M}$ and $\overline{\nabla}$ denotes the Riemannian connection of the metric g. Equation (2.4) is equivalent to $(\overline{\nabla}_X \phi)X = 0$, for each $X \in T\overline{M}$. The structure is said to be *closely cosymplectic* if ϕ is Killing and η is closed. It is well known that an almost contact metric manifold is *cosymplectic* if and only if $\overline{\nabla}\phi$ vanishes identically, that is, $(\overline{\nabla}_X \phi)Y = 0$ and $\overline{\nabla}_X \xi = 0$.

Proposition 2.1 (see [9]). *On a nearly cosymplectic manifold, the vector field ξ is Killing.*

From the above proposition we have $\overline{\nabla}_X \xi = 0$, for any vector field X tangent to $\overline{M}$, where $\overline{M}$ is a nearly cosymplectic manifold.

Let M be submanifold of an almost contact metric manifold $\overline{M}$ with induced metric g, and if ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively, then, Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for each $X, Y \in TM$ and $N \in T^{\perp}M$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\overline{M}$. They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\overline{M}$ as well as being induced on M.

For any $X \in TM$, we write

$$\phi X = TX + FX, \tag{2.8}$$

where TX is the tangential component and FX is the normal component of ϕX.

Similarly for any $N \in T^{\perp}M$, we write

$$\phi N = BN + CN, \tag{2.9}$$

where BN is the tangential component and CN is the normal component of ϕN. The covariant derivatives of the tensor fields P and F are defined as

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.10}$$

$$\left(\overline{\nabla}_X F\right)Y = \nabla^{\perp}_X FY - F\nabla_X Y \tag{2.11}$$

for all $X, Y \in TM$.

Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\overline{M}$. then for every $x \in M$ there exists a maximal invariant subspace denoted by $\mathfrak{D}_x$ of the tangent space $T_x M$ of M. If the dimension of $\mathfrak{D}_x$ is the same for all values of $x \in M$, then $\mathfrak{D}_x$ gives an invariant distribution $\mathfrak{D}$ on M.

A submanifold M of an almost contact metric manifold $\overline{M}$ is called *semi-invariant* submanifold if there exists on M a differentiable invariant distribution $\mathfrak{D}$ whose orthogonal complementary distribution $\mathfrak{D}^{\perp}$ is anti-invariant, that is,

(i) $TM = \mathfrak{D} \oplus \mathfrak{D}^{\perp} \oplus \langle \xi \rangle$,

(ii) $\phi(\mathfrak{D}_x) \subseteq D_x$,

(iii) $\phi(\mathfrak{D}_x^\perp) \subset T_x^\perp M$

for any $x \in M$, where $T_x^\perp M$ denotes the orthogonal space of $T_x M$ in $T_x\overline{M}$. A semi-invariant submanifold is called *anti-invariant* if $\mathfrak{D}_x = \{0\}$ and *invariant* if $\mathfrak{D}_x^\perp = \{0\}$, respectively, for any $x \in M$. It is called the *proper semi-invariant* submanifold if neither $\mathfrak{D}_x = \{0\}$ nor $\mathfrak{D}_x^\perp = \{0\}$, for every $x \in M$.

Let M be a semi-invariant submanifold of an almost contact metric manifold $\overline{M}$. Then, $F(T_x M)$ is a subspace of $T_x^\perp M$. Then for every $x \in M$, there exists an invariant subspace ν_x of $T_x\overline{M}$ such that

$$T_x^\perp M = F(T_x M) \oplus \nu_x. \tag{2.12}$$

A semi-invariant submanifold M of an almost contact metric manifold $\overline{M}$ is called *Riemannian product* if the invariant distribution $\mathfrak{D}$ and anti-invariant distribution $\mathfrak{D}^\perp$ are totally geodesic distributions in M.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, and let f be a positive differentiable function on M_1. The *warped product* of M_1 and M_2 is the product manifold $M_1 \times_f M_2 = (M_1 \times M_2, g)$, where

$$g = g_1 + f^2 g_2, \tag{2.13}$$

where f is called the *warping function* of the warped product. The warped product $N_1 \times_f N_2$ is said to be *trivial* or simply Riemannian product if the warping function f is constant. This means that the Riemannian product is a special case of warped product.

We recall the following general results obtained by Bishop and O'Neill [10] for warped product manifolds.

Lemma 2.2. *Let $M = M_1 \times_f M_2$ be a warped product manifold with the warping function f. Then*

(i) $\nabla_X Y \in TM_1$, *for each* $X, Y \in TM_1$,

(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$, *for each* $X \in TM_1$ *and* $Z \in TM_2$,

(iii) $\nabla_Z W = \nabla_Z^{M_2} W - (g(Z, W)/f)\, \mathrm{grad} f$,

where ∇ and ∇^{M_2} denote the Levi-Civita connections on M and M_2, respectively.

In the above lemma $\mathrm{grad} f$ is the gradient of the function f defined by $g(\mathrm{grad} f, U) = Uf$, for each $U \in TM$. From the Lemma 2.2, we have that on a warped product manifold $M = M_1 \times_f M_2$

(i) M_1 is totally geodesic in M;

(ii) M_2 is totally umbilical in M.

Now, we denote by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ the tangential and normal parts of $(\overline{\nabla}_X \phi)Y$, that is,

$$\left(\overline{\nabla}_X \phi\right) Y = \mathcal{P}_X Y + \mathcal{Q}_X Y \tag{2.14}$$

for all $X, Y \in TM$. Making use of (2.5), (2.6), and (2.8)–(2.11), the following relations may easily be obtained

$$\mathcal{P}_X Y = (\nabla_X T)Y - A_{FY}X - Bh(X, Y), \tag{2.15}$$

$$Q_X Y = \left(\overline{\nabla}_X F\right)Y + h(X, TY) - Ch(X, Y). \tag{2.16}$$

It is straightforward to verify the following properties of $\mathcal{P}$ and Q, which we enlist here for later use:

(p_1) (i) $\mathcal{P}_{X+Y}W = \mathcal{P}_X W + \mathcal{P}_Y W$, (ii) $Q_{X+Y}W = Q_X W + Q_Y W$,

(p_2) (i) $\mathcal{P}_X(Y + W) = \mathcal{P}_X Y + \mathcal{P}_X W$, (ii) $Q_X(Y + W) = Q_X Y + Q_X W$,

(p_3) $g(\mathcal{P}_X Y, W) = -g(Y, \mathcal{P}_X W)$

for all $X, Y, W \in TM$.

On a submanifold M of a nearly cosymplectic manifold $\overline{M}$, we obtain from (2.4) and (2.14) that

$$\text{(i) } \mathcal{P}_X Y + \mathcal{P}_Y X = 0, \qquad \text{(ii) } Q_X Y + Q_Y X = 0 \tag{2.17}$$

for any $X, Y \in TM$.

3. Warped Product Semi-Invariant Submanifolds

Throughout the section we consider the submanifold M of a nearly cosymplectic manifold $\overline{M}$ such that the structure vector field ξ is tangent to M. First, we prove that the warped product $M = M_1 \times_f M_2$ is trivial when ξ is tangent to M_2, where M_1 and M_2 are Riemannian submanifolds of a nearly cosymplectic manifold $\overline{M}$. Thus, we consider the warped product $M = M_1 \times_f M_2$, when ξ is tangent to the submanifold M_1. We have the following nonexistence theorem.

Theorem 3.1. *A warped product submanifold $M = M_1 \times_f M_2$ of a nearly cosymplectic manifold $\overline{M}$ is a usual Riemannian product if the structure vector field ξ is tangent to M_2, where M_1 and M_2 are the Riemannian submanifolds of $\overline{M}$.*

Proof. For any $X \in TM_1$ and ξ tangent to M_2, we have

$$\overline{\nabla}_X \xi = \nabla_X \xi + h(X, \xi). \tag{3.1}$$

Using the fact that ξ is Killing on a nearly cosymplectic manifold (see Proposition 2.1) and Lemma 2.2(ii), we get

$$0 = \left(X \ln f\right)\xi + h(X, \xi). \tag{3.2}$$

Equating the tangential component of (3.2), we obtain $X \ln f = 0$, for all $X \in TM_1$, that is, f is constant function on M_1. Thus, M is Riemannian product. This proves the theorem. □

Now, the other case of warped product $M = M_1 \times_f M_2$ when $\xi \in TM_1$, where M_1 and M_2 are the Riemannian submanifolds of $\overline{M}$. For any $X \in TM_2$, we have

$$\overline{\nabla}_X \xi = \nabla_X \xi + h(X, \xi). \tag{3.3}$$

By Proposition 2.1, and Lemma 2.2(ii), we obtain

$$\text{(i)}\xi \ln f = 0, \quad \text{(ii)}h(X, \xi) = 0. \tag{3.4}$$

Thus, we consider the warped product semi-invariant submanifolds of a nearly cosymplectic manifold $\overline{M}$ of the types:

(i) $M = M_\perp \times_f M_T$,

(ii) $M = M_T \times_f M_\perp$,

where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\overline{M}$, respectively. In the following theorem we prove that the warped product semi-invariant submanifold of the type (i) is CR-product.

Theorem 3.2. *The warped product semi-invariant submanifold $M = M_\perp \times_f M_T$ of a nearly cosymplectic manifold $\overline{M}$ is a usual Riemannian product of $M_\perp$ and M_T, where $M_\perp$ and M_T are anti-invariant and invariant submanifolds of $\overline{M}$, respectively.*

Proof. When $\xi \in TM_T$, then by Theorem 3.1, M is a Riemannian product. Thus, we consider $\xi \in TM_\perp$. For any $X \in TM_T$ and $Z \in TM_\perp$, we have

$$\begin{aligned} g(h(X, \phi X), FZ) &= g(h(X, \phi X), \phi Z) = g\left(\overline{\nabla}_X \phi X, \phi Z\right) \\ &= g\left(\phi \overline{\nabla}_X X, \phi Z\right) + g\left(\left(\overline{\nabla}_X \phi\right) X, \phi Z\right). \end{aligned} \tag{3.5}$$

From the structure equation of nearly cosymplectic, the second term of right hand side vanishes identically. Thus from (2.2), we derive

$$\begin{aligned} g(h(X, \phi X), FZ) &= g\left(\overline{\nabla}_X X, Z\right) - \eta(Z) g\left(\overline{\nabla}_X X, \xi\right) \\ &= -g\left(X, \overline{\nabla}_X Z\right) + \eta(Z) g\left(X, \overline{\nabla}_X \xi\right). \end{aligned} \tag{3.6}$$

Then from (2.5), Lemma 2.2(ii), and Proposition 2.1, we obtain

$$g(h(X, \phi X), FZ) = -(Z \ln f)\|X\|^2. \tag{3.7}$$

Interchanging X by ϕX in (3.7) and using the fact that $\xi \in TM_\perp$, we obtain

$$g(h(X, \phi X), FZ) = (Z \ln f)\|X\|^2. \tag{3.8}$$

It follows from (3.7) and (3.8) that $Z \ln f = 0$, for all $Z \in TM_{\perp}$. Also, from (3.4) we have $\xi \ln f = 0$. Thus, the warping function f is constant. This completes the proof of the theorem. □

From the above theorem we have seen that the warped product of the type $M = M_{\perp} \times_f M_T$ is a usual Riemannian product of an anti-invariant submanifold $M_{\perp}$ and an invariant submanifold M_T of a nearly cosymplectic manifold $\overline{M}$. Since both $M_{\perp}$ and M_T are totally geodesic in M, then M is CR-product. Now, we study the warped product semi-invariant submanifold $M = M_T \times_f M_{\perp}$ of a nearly cosymplectic manifold $\overline{M}$.

Theorem 3.3. *Let $M = M_T \times_f M_{\perp}$ be a warped product semi-invariant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then the invariant distribution $\mathfrak{D}$ and the anti-invariant distribution $\mathfrak{D}^{\perp}$ are always integrable.*

Proof. For any $X, Y \in \mathfrak{D}$, we have

$$F[X, Y] = F\nabla_X Y - F\nabla_Y X. \tag{3.9}$$

Using (2.11), we obtain

$$F[X, Y] = \left(\overline{\nabla}_X F\right)Y - \left(\overline{\nabla}_Y F\right)X. \tag{3.10}$$

Then by (2.16), we derive

$$F[X, Y] = Q_X Y - h(X, TY) + Ch(X, Y) - Q_Y X + h(Y, TX) - Ch(X, Y). \tag{3.11}$$

Thus from (2.17)(ii), we get

$$F[X, Y] = 2Q_X Y + h(Y, TX) - h(X, TY). \tag{3.12}$$

Now, for any $X, Y \in D$, we have

$$h(X, TY) + \nabla_X TY = \overline{\nabla}_X TY = \overline{\nabla}_X \phi Y. \tag{3.13}$$

Using the covariant derivative property of $\overline{\nabla}\phi$, we obtain

$$h(X, TY) + \nabla_X TY = \left(\overline{\nabla}_X \phi\right)Y + \phi\overline{\nabla}_X Y. \tag{3.14}$$

Then by (2.5) and (2.14), we get

$$h(X, TY) + \nabla_X TY = P_X Y + Q_X Y + \phi(\nabla_X Y + h(X, Y)). \tag{3.15}$$

Since M_T is totally geodesic in M (see Lemma 2.2(i)), then using (2.8) and (2.9), we obtain

$$h(X,TY)+\nabla_X TY=\mathcal{P}_X Y+\mathcal{Q}_X Y+T\nabla_X Y+Bh(X,Y)+Ch(X,Y). \tag{3.16}$$

Equating the normal components of (3.16), we get

$$h(X,TY)=\mathcal{Q}_X Y+Ch(X,Y). \tag{3.17}$$

Similarly, we obtain

$$h(Y,TX)=\mathcal{Q}_Y X+Ch(X,Y). \tag{3.18}$$

Then from (3.17) and (3.18), we arrive at

$$h(Y,TX)-h(X,TY)=\mathcal{Q}_Y X-\mathcal{Q}_X Y. \tag{3.19}$$

Hence, using (2.17)(ii), we get

$$h(Y,TX)-h(X,TY)=-2\mathcal{Q}_X Y. \tag{3.20}$$

Thus, it follows from (3.12) and (3.20) that $F[X,Y]=0$, for all $X,Y\in D$. This proves the integrability of D. Now, for the integrability of $D^{\perp}$, we consider any $X\in D$ and $Z,W\in D^{\perp}$, and we have

$$\begin{aligned} g([Z,W],X)&=g\left(\overline{\nabla}_Z W-\overline{\nabla}_W Z,X\right).\\ &=-g(\nabla_Z X,W)+g(\nabla_W X,Z). \end{aligned} \tag{3.21}$$

Using Lemma 2.2(ii), we obtain

$$g([Z,W],X)=-(X\ln f)g(Z,W)+(X\ln f)g(Z,W)=0. \tag{3.22}$$

Thus from (3.22), we conclude that $[Z,W]\in\mathfrak{D}^{\perp}$, for each $Z,W\in\mathfrak{D}^{\perp}$. Hence, the theorem is proved completely. □

Lemma 3.4. *Let $M=M_T\times_f M_{\perp}$ be a warped product submanifold of a nearly cosymplectic manifold $\overline{M}$. If $X,Y\in TM_T$ and $Z,W\in TM_{\perp}$, then*

(i) $g(\mathcal{P}_X Y,Z)=g(h(X,Y),FZ)=0$,

(ii) $g(\mathcal{P}_X Z,W)=g(h(X,Z),FW)-g(h(X,W),FZ)=-(\phi X\ln f)g(Z,W)-g(h(X,Z),FW)$,

(iii) $g(h(\phi X,Z),FZ)=(X\ln f)\|Z\|^2$.

Proof. For a warped product manifold $M = M_T \times_f M_\perp$, we have that M_T is totally geodesic in M; then by (2.10), $(\overline{\nabla}_X T)Y \in TM_T$, for any $X, Y \in TM_T$, and therefore from (2.15), we get

$$g(\mathcal{P}_X Y, Z) = -g(Bh(X,Y), Z) = g(h(X,Y), FZ). \tag{3.23}$$

The left-hand side of (3.23) is skew symmetric in X and Y whereas the right hand side is symmetric in X and Y, which proves (i). Now, from (2.10) and (2.15), we have

$$\mathcal{P}_X Z = -T\nabla_X Z - A_{FZ}X - Bh(X,Z) \tag{3.24}$$

for any $X \in TM_T$ and $Z \in TM_\perp$. Using Lemma 2.2 (ii), the first term of right-hand side is zero. Thus, taking the product with $W \in TM_\perp$, we obtain

$$g(\mathcal{P}_X Z, W) = -g(A_{FZ}X, W) - g(Bh(X,Z), W), \tag{3.25}$$

Then by (2.2) and (2.7), we get

$$g(\mathcal{P}_X Z, W) = -g(h(X,W), FZ) + g(h(X,Z), FW). \tag{3.26}$$

which proves the first equality of (ii). Again, from (2.10) and (2.15), we have

$$\mathcal{P}_Z X = \nabla_Z TX - T\nabla_Z X - Bh(X,Z). \tag{3.27}$$

Thus using Lemma 2.2(ii), we derive

$$\mathcal{P}_Z X = (TX \ln f)Z - Bh(X,Z). \tag{3.28}$$

Taking inner product with $W \in TM_\perp$ and using (2.2), we obtain

$$g(\mathcal{P}_Z X, W) = (\phi X \ln f) g(Z,W) + g(h(X,Z), FW). \tag{3.29}$$

Then from (2.17)(i), we get

$$g(\mathcal{P}_X Z, W) = -(\phi X \ln f) g(Z,W) - g(h(X,Z), FW). \tag{3.30}$$

This is the second equality of (ii). Now, from (3.24) and (3.28), we have

$$\mathcal{P}_X Z + \mathcal{P}_Z X = -T\nabla_X Z - A_{FZ}X + (TX \ln f)Z - 2Bh(X,Z). \tag{3.31}$$

Left-hand side and the first term of right-hand side are zero on using (2.17)(i) and Lemma 2.2(i), respectively. Thus the above equation takes the form

$$(TX \ln f)Z = A_{FZ}X + 2Bh(X,Z). \tag{3.32}$$

Taking the product with Z and on using (2.2) and (2.7), we get

$$(\phi X \ln f)\|Z\|^2 = g(h(X,Z),FZ) - 2g(h(X,Z),FZ) = -g(h(X,Z),FZ). \tag{3.33}$$

Interchanging X by ϕX and using (2.1), we obtain

$$\{-X + \eta(X)\xi\} \ln f\|Z\|^2 = -g(h(\phi X, Z), FZ). \tag{3.34}$$

Thus by (3.4)(i), the above equation reduces to

$$(X \ln f)\|Z\|^2 = g(h(\phi X, Z), FZ). \tag{3.35}$$

This proves the lemma completely. □

Theorem 3.5. *A proper semi-invariant submanifold M of a nearly cosymplectic manifold $\overline{M}$ is locally a semi-invariant warped product if and only if the shape operator of M satisfies*

$$A_{\phi Z} X = -(\phi X \mu) Z, \quad X \in \mathfrak{D} \oplus \langle \xi \rangle, \; Z \in \mathfrak{D}^{\perp} \tag{3.36}$$

for some function μ on M satisfying $V(\mu) = 0$ for each $V \in \mathfrak{D}^{\perp}$.

Proof. If $M = M_T \times_f M_{\perp}$ is a warped product semi-invariant submanifold, then by Lemma 3.4 (iii), we obtain (3.36). In this case $\mu = \ln f$.

Conversely, suppose M is a semi-invariant submanifold of a nearly cosymplectic manifold $\overline{M}$ satisfying (3.36). Then

$$g(h(X,Y), \phi Z) = g(A_{\phi Z} X, Y) = -(\phi X \mu) g(Y, Z) = 0. \tag{3.37}$$

Now, from (2.5) and the property of covariant derivative of $\overline{\nabla}$, we have

$$\begin{aligned} g(h(X,Y), \phi Z) &= g\left(\overline{\nabla}_X Y, \phi Z\right) = -g\left(\phi \overline{\nabla}_X Y, Z\right) \\ &= -g\left(\overline{\nabla}_X \phi Y, Z\right) + g\left(\left(\overline{\nabla}_X \phi\right) Y, Z\right). \end{aligned} \tag{3.38}$$

Then from (2.5), (2.14), and (3.37), the above equation takes the form

$$g(\nabla_X TY, Z) = g(P_X Y, Z). \tag{3.39}$$

Using (2.10) and (2.15), we obtain

$$g(\nabla_X TY, Z) = g(\nabla_X TY, Z) - g(T \nabla_X Y, Z) - g(Bh(X,Y), Z). \tag{3.40}$$

Thus by (2.2), the above equation reduces to

$$g(T\nabla_X Y, Z) = g(h(X, Y), \phi Z). \tag{3.41}$$

Hence using (2.7) and (3.36), we get

$$g(T\nabla_X Y, Z) = g(A_{\phi Z} X, Y) = 0, \tag{3.42}$$

which implies $\nabla_X Y \in \mathfrak{D} \oplus \langle \xi \rangle$, that is, $\mathfrak{D} \oplus \langle \xi \rangle$ is integrable and its leaves are totally geodesic in M. Now, for any $Z, W \in \mathfrak{D}^{\perp}$ and $X \in \mathfrak{D} \oplus \langle \xi \rangle$, we have

$$\begin{aligned} g(\nabla_Z W, \phi X) &= g\left(\overline{\nabla}_Z W, \phi X\right) = -g\left(\phi \overline{\nabla}_Z W, X\right) \\ &= g\left(\left(\overline{\nabla}_Z \phi\right) W, X\right) - g\left(\overline{\nabla}_Z \phi W, X\right). \end{aligned} \tag{3.43}$$

Then, using (2.6) and (2.14), we obtain

$$g(\nabla_Z W, \phi X) = g(\mathcal{P}_Z W, X) + g(A_{\phi W} Z, X). \tag{3.44}$$

Thus from (2.7) and the property (p_3), we arrive at

$$g(\nabla_Z W, \phi X) = -g(W, \mathcal{P}_Z X) + g(h(Z, X), \phi W). \tag{3.45}$$

Again using (2.7) and (2.17)(i), we get

$$g(\nabla_Z W, \phi X) = g(\mathcal{P}_X Z, W) + g(A_{\phi W} X, Z). \tag{3.46}$$

On the other hand, from (2.10) and (2.15), we have

$$P_X Z = -T\nabla_X Z - A_{FZ} X - Bh(X, Z). \tag{3.47}$$

Taking the product with $W \in D^{\perp}$ and using (3.36), we obtain

$$g(\mathcal{P}_X Z, W) = -g(T\nabla_X Z, W) + (\phi X\mu) g(Z, W) + g(h(X, Z), FW). \tag{3.48}$$

The first term of right-hand side of above equation is zero using the fact that $TW = 0$, for any $W \in \mathfrak{D}^{\perp}$. Again using (2.7), we get

$$g(\mathcal{P}_X Z, W) = (\phi X\mu) g(Z, W) + g(A_{\phi W} X, Z). \tag{3.49}$$

Thus from (3.36), we derive

$$g(\mathcal{P}_X Z, W) = (\phi X\mu) g(Z, W) - (\phi X\mu) g(Z, W) = 0. \tag{3.50}$$

Then from (3.36), (3.46), and (3.50), we obtain

$$g(\nabla_Z W, \phi X) = -(\phi X \mu) g(Z, W). \tag{3.51}$$

Let $M_\perp$ be a leaf of $\mathfrak{D}^\perp$, and let $h^\perp$ be the second fundamental form of the immersion of $M_\perp$ into M. Then for any $Z, W \in \mathfrak{D}^\perp$, we have

$$g\left(h^\perp(Z, W), \phi X\right) = g(\nabla_Z W, \phi X). \tag{3.52}$$

Hence, from (3.51) and (3.52), we conclude that

$$g\left(h^\perp(Z, W), \phi X\right) = -(\phi X \mu) g(Z, W). \tag{3.53}$$

This means that integral manifold $M_\perp$ of $\mathfrak{D}^\perp$ is totally umbilical in M. Since the anti-invariant distribution $\mathfrak{D}^\perp$ of a semi-invariant submanifold M is always integrable (Theorem 3.3) and $V(\mu) = 0$ for each $V \in \mathfrak{D}^\perp$, which implies that the integral manifold of $\mathfrak{D}^\perp$ is an extrinsic sphere in M; that is, it is totally umbilical and its mean curvature vector field is nonzero and parallel along $M_\perp$. Hence by virtue of results obtained in [11], M is locally a warped product $M_T \times_f M_\perp$, where M_T and $M_\perp$ denote the integral manifolds of the distributions $\mathfrak{D} \oplus \langle \xi \rangle$ and $\mathfrak{D}^\perp$, respectively and f is the warping function. Thus the theorem is proved. □

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