Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 412565, 16 pages
doi:10.1155/2011/412565

Research Article

The Intensity Model for Pricing Credit Securities with Jump Diffusion and Counterparty Risk

Ruili Hao and Zhongxing Ye

Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, China

Correspondence should be addressed to Ruili Hao, haoruili13@163.com

Received 15 January 2011; Accepted 21 February 2011

Academic Editor: Moran Wang

We present an intensity-based model with counterparty risk. We assume the default intensity of firm depends on the stochastic interest rate driven by the jump-diffusion process and the default states of counterparty firms. Furthermore, we make use of the techniques in Park (2008) to compute the conditional distribution of default times and derive the explicit prices of bond and CDS. These are extensions of the models in Jarrow and Yu (2001).

1. Introduction

As credit securities are actively traded and the financial market becomes complex, the valuation of credit securities has called for more effective models according to the real market. Until now, there have been mainly two basic models: the structural model and the reduced-form model. In the first model, the firm's default is governed by the value of its assets and debts, while the default in the reduced-form model is governed by the exogenous factor.

The structural approach was pioneered by Merton [1], then extended by Black and Cox [2] and Longstaff and Schwartz [3], assuming the default before the maturity date and others. In the above models, the asset process was all driven by the Brownian motion. Since the asset value may suffer a sudden drop for the reason of some events in the economy, Zhou [4] provided a jump-diffusion model with credit risk in which jump amplitude followed a log-normal distribution and valuated defaultable securities. In his model, Zhou gave the explicit expressions of defaultable securities' prices when the default occurred at the maturity date T, but only gave a tractably simulating approach when the firm defaulted before time T. For the first-passage-time models of credit risk with jump-diffusion process, Steve and Amir [5] and Zhang and Melnik [6] used the approach of Brownian bridge to estimate the jump-diffusion process and priced barrier options. Kou and Wang [7] and Kou et al. [8] made use of the

Laplace transform to valuate the credit risk after estimating the jump-diffusion process with an infinitesimal generator. For the problem of the valuation of credit derivatives involving jump-diffusion process, it is still difficult to get explicit results in the event of defaulting before the maturity date, despite using the above approaches. However, it is more convenient to use reduced-form approach for valuating the credit risk in such situation.

Comparing with the structural approach, the reduced-form approach is flexible and tractable in the real market. It is pioneered by Jarrow et al. [9] and Duffie and Singleton [10]. They introduced exogenous mechanism of firm's default. Their models considered the default as a random event which is controlled by a exogenous intensity process.

Davis and Lo [11] firstly proposed the model of credit contagion to account for concentration risk in large portfolios of defaultable securities (DL Model). Later, motivated by a series of events such as the South Korean banking crisis, Long Term Capital Management's potential default and so on, Jarrow and Yu [12] thought the traditionally structural and reduced-form models were full of problems because they all ignored the firm's specific source of credit risk. They made use of the Davis's contagious model and introduced the concept of counterparty risk which is from the default of firm's counterparties. In their models, they paid more attention to the primary-secondary framework in which the intensity of default was influenced by the economy-wide state variables and the default state of the counterparty. Later, there are also other similar applications such as Leung and Kwork [13], Bai et al. [14] and so on. In recent years, some authors applied this into portfolio credit securities such as Yu [15] and Leung and Kwok [16]. Nevertheless, the stochastic interest rate in the above models still was driven by diffusion processes.

At present, aggregate credit risk is still one of the most pervasive threats in the financial markets, which is from the contagious risk caused by business counterparties. In this paper, we mainly discuss the pricing of defaultable securities in primary-secondary framework, extending the models in Jarrow and Yu [12]. We consider that the macroeconomic variables contain the risk-free interest rate which shows the interaction between credit risk and market risk. However, the interest rate may drop suddenly due to some events in the modern economy. Therefore, we allow the stochastic interest rate to follow a jump-diffusion process rather than the continuous diffusion process in Jarrow and Yu [12]. Thus, our model not only reflects the real market much better, but more precisely to identify the impact of counterparty risk on the valuation of credit securities. Moreover, we apply the techniques in Park [17] to the pricing of bond and CDS, so that we avoid solving the PDEs.

2. Model

2.1. The Setting of Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^{T^*}, P)$ be the filtered probability space satisfying the usual conditions, where $\mathcal{F} = \mathcal{F}_{T^*}$, T^* is large enough but finite and P is an equivalent martingale measure under which discounted securities' prices are martingales.

On $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^{T^*}, P)$, there is an $\mathcal{R}^d$-valued process $X = (X^1, \ldots, X^d)$, where $\{X^i\}_{i=1}^d$ are Markov processes and represent d economy-wide state variables. Beside these, there are n companies with n point processes $\{N^i\}_{i=1}^n (N_0^i = 0)$ which represent the default processes of n companies, respectively. When N^i first jumps from 0 to 1, we call the company i defaults and denote τ^i be the default time of company i. Thus, $N_t^i = 1_{\{\tau^i \le t\}}$, where $1_{\{\cdot\}}$ is the indicator function.

The filtration $\mathcal{F}$ is generated by the state variables and the default processes of n companies as follows

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^n, \tag{2.1}$$

where

$$\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t), \qquad \mathcal{H}_t^i = \sigma\left(N_s^i, 0 \leq s \leq t\right). \tag{2.2}$$

Denote that

$$\mathcal{H}_t = \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^n, \qquad \mathcal{H}_t^{-i} = \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^{i-1} \vee \mathcal{H}_t^{i+1} \vee \cdots \vee \mathcal{H}_t^n. \tag{2.3}$$

We assume the default time τ^i $(i = 1, \ldots, n)$ possesses a strictly positive $\mathcal{F}_{T^*}^X \vee \mathcal{H}_{T^*}^{-i}$-adapted intensity process λ_t^i satisfying $\int_0^t \lambda_s^i ds < \infty$, P-a.s. for all $t \in [0, T^*]$. The intensity process λ_t^i shows the local default probability in the sense that the default probability of company i over a small interval $(t, t + \Delta t)$ is equal to $\lambda_t^i \Delta t$. These N^i, $1 \leq i \leq n$ generate the defaults of n companies. Their intensity processes λ^i, $1 \leq i \leq n$ depend on state variables and the default states of all other companies. Due to the counterparty risk, $\{\tau^i\}_{i=1}^n$ may no longer be assumed independent conditionally on $\mathcal{F}^X$.

2.2. *Primary-Secondary Framework*

We divide n firms into two mutually exclusive types: l primary firms and $n - l$ secondary firms. Primary firms' default processes only depend on state variables, while secondary firms' default processes depend on the state variables and the default states of the primary firms. This model was proposed by Jarrow and Yu [12]. Now, we provide some assumptions of the model.

Assumption 1 (economy-wide state variables). The state variable X_t may contain the risk-free spot rate r_t or other economical variables in the economy environment which may impact on the default probability of the companies.

Assumption 2 (the default times). On $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^{T^*}, P)$, we add several independent unit exponential random variables $\{E^i, 1 \leq i \leq l\}$ which are independent of X under probability measure P. The default times of l primary firms can be defined as

$$\tau^i = \inf\left\{t : \int_0^t \lambda_s^i ds \geq E^i\right\}, \quad 1 \leq i \leq l, \tag{2.4}$$

where λ_t^i is adapted to $\mathcal{F}_t^X$. Then, we add another series independent unit exponential random variables $\{E^j, l+1 \leq j \leq n\}$ which are independent of X and τ^i, $1 \leq i \leq l$. The default times of $n-l$ secondary firms can be defined as

$$\tau^j = \inf\left\{ t : \int_0^t \lambda_s^j ds \geq E^j \right\}, \quad 1+l \leq j \leq n, \tag{2.5}$$

where λ_t^j is adapted to $\mathcal{F}_t^X \vee \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^l$.

Assumption 3 (the default probability). Because E^i $(1 \leq i \leq l)$ is independent of state variables X, the conditional and unconditional survival probability distributions of primary firm i are given by

$$P\left(\tau^i > t \mid \mathcal{F}_{T^*}^X\right) = \exp\left(-\int_0^t \lambda_s^i ds \right), \tag{2.6}$$

$$P\left(\tau^i > t\right) = E\left[\exp\left(-\int_0^t \lambda_s^i ds \right) \right], \quad t \in [0, T^*]. \tag{2.7}$$

Similarly, since E^j $(1+l \leq j \leq n)$ is independent of X and τ^i, $1 \leq i \leq l$, we have the conditional and unconditional survival probability distributions of secondary firm j

$$\begin{gathered} P\left(\tau^j > t \mid \mathcal{F}_{T^*}^X \vee \mathcal{H}_{T^*}^1 \vee \cdots \vee \mathcal{H}_{T^*}^l\right) = \exp\left(-\int_0^t \lambda_s^j ds \right), \\ P\left(\tau^j > t\right) = E\left[\exp\left(-\int_0^t \lambda_s^j ds \right) \right], \quad t \in [0, T^*]. \end{gathered} \tag{2.8}$$

Assumption 4 (the default intensity). Because the Primary firms' default processes only depend on macrovariables, we denote their default intensities by

$$\lambda_t^i = \Lambda_{0,t}^i, \quad 1 \leq i \leq l. \tag{2.9}$$

In addition, secondary firms' default processes depend on the macrovariables and the default processes of the primary firms. We denote the intensities by

$$\lambda_t^j = \Lambda_{0,t}^j + \Sigma_{k=1}^l \Lambda_{k,t}^j 1_{\{\tau^k \leq t\}}, \quad 1+l \leq j \leq n, \tag{2.10}$$

where $\Lambda_{k,t}^j$ is adapted to $\mathcal{F}_{T^*}^X$ for all k. $\Lambda_{0,t}^i$ and $\Lambda_{0,t}^j$ can be constants or stochastic processes which are correlated with the state variables.

Assumption 5 (the risk-free interest rate). The risk-free interest rate r_t in this framework is stochastic which may follow CIR model, HJM model, Vasicek model or their extensions. It has effect on the defaults of n companies.

3. The Pricing of Credit Securities

In this section, we price the defaultable bonds and credit default swap (CDS) in the primary-secondary framework satisfying Assumption 1 to Assumption 5. To obtain some explicit results, we give another specific assumptions. We assume that the state variable X_t only contains the risk-free spot rate r_t and the default of a firm is correlated with the default-free term structure. Namely, we will present a one-factor model for credit risk. Furthermore, we mainly consider single counterparty. There are one primary firm and one secondary firm in our pricing model. Counterparty risk may occur when secondary firm holds large amounts of debt issued by the primary firm.

We suppose that the risk-free interest rate follows the jump-diffusion process

$$dr_t = \alpha(K - r_t)dt + \sigma dW_t + q_t dY_t, \tag{3.1}$$

where W_t is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ and Y_t is a Possion process under P with intensity μ. q_t is a deterministic function and α, σ, K are constants. We assume W_t and Y_t are mutually independent.

Remark 3.1. In fact, from Park [17], we know (3.1) has the explicit solution as follows:

$$r_t = r_0 e^{-\alpha t} + \alpha K \int_0^t e^{-\alpha(t-s)} ds + \sigma \int_0^t e^{-\alpha(t-s)} dW_s + \int_0^t q_s e^{-\alpha(t-s)} dY_s. \tag{3.2}$$

Moreover, in accordance with the properties of W_s and Y_s, we can check that r_t is a $\mathcal{F}^r$-Markov process, which plays an important role in the following.

3.1. *Defaultable Bonds' Pricing*

We first give some general pricing formulas for bonds in the primary-secondary framework described in Section 2.2. Suppose that the face value of bond i $(i = 1, \ldots, n)$ is 1 dollar. Under the equivalent martingale measure P, the default-free and defaultable bond's prices are, respectively, given by

$$p(t,T) = E_t\left[\exp\left(-\int_t^T r_s ds\right)\right], \tag{3.3}$$

$$V^i(t,T) = E_t\left[e^{-\int_t^T r_s ds}\left(\beta^i 1_{\{\tau^i \le T\}} + 1_{\{\tau^i > T\}}\right)\right], \tag{3.4}$$

where $E_t[\cdot]$ represents the conditional expectation with respect to $\mathcal{F}_t$, β^i is the recovery rate of defaultable bond i, and $T(< T^*)$ is the maturity date.

Lemma 3.2 (see [12]). *The defaultable bond price can also be expressed as*

$$V^i(t,T) = \beta^i p(t,T) + 1_{\{\tau^i > t\}}\left(1 - \beta^i\right) E_t\left[\exp\left(-\int_t^T \left(r_s + \lambda_s^i\right) ds\right)\right], \quad t \le T. \tag{3.5}$$

In the following, we only consider the case with two firms. Firm A is the primary firm whose default is independent of the default risk of secondary firm B but depends on the interest rate r, while firm B's default is correlated with the state of firm A and the risk-free interest rate. One assumes their intensity processes, respectively, satisfy some linear relations below:

$$\lambda_t^A = b_0^A + b_1^A r_t, \tag{3.6}$$

$$\lambda_t^B = b_0^B + b_1^B r_t + b 1_{\{\tau^A \le t\}}, \tag{3.7}$$

where b_0^A, b_1^A, b_0^B, b_1^B, and b are positive constants.

Remark 3.3. The interest rate r_t in our model is an extension of Vasicek model. It may cause negative intensity. We use the similar method in Jarrow and Yu [12] to avoid this case. We can assume $\lambda_t^A = \max\{b_0^A + b_1^A r_t, 0\}$, $\lambda_t^B = \max\{b_0^B + b_1^B r_t + b 1_{\{\tau^A \le t\}}, 0\}$, we will discuss it in other paper.

We price the bonds issued by firm A and firm B. To be convenient, we use time-t forward interest rate instead of time-0 forward interest rate in (3.2). Let $f(0,u) = r_0 e^{-\alpha u}$, then for $u \ge t$, (3.2) can be expressed as

$$\begin{aligned} r_u &= f(0,u) + \left(\int_0^t + \int_t^u\right) \alpha K e^{\alpha(v-u)} dv + \left(\int_0^t + \int_t^u\right) \sigma e^{\alpha(v-u)} dW_v + \left(\int_0^t + \int_t^u\right) q_v e^{\alpha(v-u)} dY_v \\ &= f(t,u) + \int_t^u \alpha K e^{\alpha(v-u)} dv + \int_t^u \sigma e^{\alpha(v-u)} dW_v + \int_t^u q_v e^{\alpha(v-u)} dY_v, \end{aligned} \tag{3.8}$$

where

$$f(t,u) = f(0,u) + \int_0^t \alpha K e^{\alpha(v-u)} dv + \int_0^t \sigma e^{\alpha(v-u)} dW_v + \int_0^t q_v e^{\alpha(v-u)} dY_v. \tag{3.9}$$

Now, we present an important theorem in the pricing process of credit securities.

Theorem 3.4. *Suppose that r_t follows* (3.1) *and $R_{t,T} = \int_t^T r_s ds$ be the cumulative interest from time t to T. Let $E_t[e^{-aR_{t,T}}] = g(a,t,T)$ for all $a \in \mathcal{R}$, then one obtains*

$$g(a,t,T) = \exp\left(\int_t^T \left[-af(t,u) + \frac{1}{2}\sigma^2 a^2 c_T^2(u) + \mu\left(e^{-aq_u c_T(u)} - 1\right)\right] du - aK(T-t) + aKc_T(t)\right), \tag{3.10}$$

where

$$c_v(u) = -\frac{1}{\alpha}\left(e^{\alpha(u-v)} - 1\right), \quad 0 \le v,\ u \le T. \tag{3.11}$$

Proof. The proving ideas are similar to Jarrow and Yu [12] and Park [17]. Since r_t follows (3.1), it has an explicit expression as (3.8). Then, we have

$$\begin{aligned} -a\int_t^T r_u du &= -\int_t^T af(t,u)du - \int_t^T du\int_t^u a\alpha K e^{\alpha(v-u)}dv \\ &\quad - \int_t^T du\int_t^u \sigma a e^{\alpha(v-u)}dW_v - \int_t^T du\int_t^u q_v a e^{\alpha(v-u)}dY_v \\ &\triangleq h_1(t,T) + h_2(t,T) + h_3(t,T) + h_4(t,T), \end{aligned} \tag{3.12}$$

where $a \in \mathcal{R}$ and

$$h_1(t,T) = -a\int_t^T f(t,u)du, \tag{3.13}$$

$$h_2(t,T) = -\int_t^T du\int_t^u a\alpha K e^{\alpha(v-u)}dv = -aK(T-t) - \frac{aK}{\alpha}\left(e^{\alpha(t-T)}-1\right), \tag{3.14}$$

$$h_3(t,T) = -\int_t^T du\int_t^u \sigma a e^{\alpha(v-u)}dW_v, \tag{3.15}$$

$$h_4(t,T) = -\int_t^T du\int_t^u q_v a e^{\alpha(v-u)}dY_v. \tag{3.16}$$

By the Markov property of r, we have

$$E_t\left[e^{-aR_{t,T}}\right] = E\left[e^{-aR_{t,T}} \mid r_t\right]. \tag{3.17}$$

Hence, we mainly need to obtain $E[e^{h_3(t,T)} \mid r_t]$ and $E[e^{h_4(t,T)} \mid r_t]$. By *Fübini*'s theorem, (3.15) and (3.16) become

$$\begin{aligned} h_3(t,T) &= -\int_t^T dW_v\int_v^T \sigma a e^{\alpha(v-u)}du = -\int_t^T \sigma a c_T(v)dW_v, \\ h_4(t,T) &= -\int_t^T dY_v\int_v^T q_v a e^{\alpha(v-u)}du = -\int_t^T a q_v c_T(v)dY_v, \end{aligned} \tag{3.18}$$

where $c_T(v)$ is given by (3.11). Moreover, $h_3(t,T)$ follows the normal distribution with mean 0 and variance $\sigma^2a^2\int_t^T c_T^2(v)dv$. Therefore, by the independent increments of the diffusion process,

$$E\left[\exp(A+B)\right] = \exp\left(\frac{(\mathrm{Var}[A]+\mathrm{Var}[B])}{2}\right). \tag{3.19}$$

So

$$E\left[e^{h_3(t,T)}\right] = \exp\left(\frac{1}{2}\sigma^2 a^2 \int_t^T c_T^2(v)dv\right). \tag{3.20}$$

In addition, using the results in Park [17], based on independent increments for the jump process,

$$E\left[e^{h_4(t,T)}\right] = \exp\left(\mu \int_t^T \left(e^{-aq_v c_T(v)} - 1\right)dv\right). \tag{3.21}$$

We substitute (3.13), (3.14), (3.20) and (3.21) into $E_t[e^{-aR_{t,T}}]$ and deduce (3.10). □

Thus, from Lemma 3.2 and Theorem 3.4, we can derive the pricing formulas of defaultable bonds.

Theorem 3.5. *In the primary-secondary framework described as above, the bonds issued by firm A and B have the same maturity date T and recovery rate $\beta^A = \beta^B = 0$. If the intensity processes λ_t^A and λ_t^B satisfy* (3.6) *and* (3.7) *and no defaults occur up to time t, then the time-t price of bond issued by primary firm A is*

$$V^A(t,T) = g\left(1 + b_1^A, t, T\right)\exp\left(-b_0^A(T-t)\right) \tag{3.22}$$

and the time-t price of bond issued by secondary firm B is

$$\begin{aligned} V^B(t,T) = {} & g\left(1 + b_1^B, t, T\right)e^{-(b_0^B+b)(T-t)} + be^{-(K+Kb_1^B+b_0^B+b)T+(1+b_0^B+b_0^A+b_1^B+b_1^A)t} \\ & \cdot \int_t^T e^{-(1+b_1^B+b_1^A)\int_t^s f(t,u)du+(b-b_1^AK-b_0^A)s+(1+b_1^B+b_1^A)Kc_s(t)+(1+b_1^B)(K-r_0)d(s,T,0)-f_1(t,s)+M(s)}ds, \end{aligned} \tag{3.23}$$

where for for all $k, v, u \in [0,T]$, $c_v(u)$ is defined as (3.11)

$$d(k,v,u) = -\frac{1}{\alpha}e^{\alpha u}\left(e^{-\alpha v} - e^{-\alpha k}\right), \tag{3.24}$$

$$f_1(t,s) = \int_0^t \sigma\left(1 + b_1^B\right)d(s,T,u)dW_u + \int_0^t \left(1 + b_1^B\right)q_u d(s,T,u)dY_u, \tag{3.25}$$

$$\begin{aligned} M(s) = {} & \int_s^T \left[\frac{1}{2}\sigma^2\left(1 + b_1^B\right)^2 c_T^2(u) + \mu\left(e^{-(1+b_1^B)q_u c_T(u)} - 1\right)\right]du \\ & + \int_t^s \frac{1}{2}\sigma^2\left[\left(1 + b_1^B + b_1^A\right)c_s(u) + \left(1 + b_1^B\right)d(s,T,u)\right]^2 du \\ & + \int_t^s \mu\left[e^{-q_u((1+b_1^B+b_1^A)c_s(u)+(1+b_1^B)d(s,T,u))} - 1\right]du. \end{aligned} \tag{3.26}$$

Proof. Firstly, from Lemma 3.2 and Theorem 3.4, we can easily show that (3.22) holds. Secondly, according to Lemma 3.2, (3.7), and the properties of conditional expectation, we obtain the price of bond issued by firm B at time t

$$
\begin{aligned}
V^B(t,T) &= E_t\left[\exp\left(-\int_t^T \left(r_s+\lambda_s^B\right)ds\right)\right] \\
&= E_t\left[\exp\left(-b_0^B(T-t)-\left(1+b_1^B\right)R_{t,T}-b\left(T-\tau^A\right)1_{\{\tau^A\le T\}}\right)\right] \\
&= E_t\left[\exp\left(-b_0^B(T-t)-\left(1+b_1^B\right)R_{t,T}\right)\cdot E\left[\exp\left(-b\left(T-\tau^A\right)1_{\{\tau^A\le T\}}\right) \mid \mathcal{F}_t\vee\mathcal{F}_{T^*}^r\right]\right].
\end{aligned}
\tag{3.27}
$$

By (2.6), the property of conditional expectation and the law of integration by parts, we check that

$$
\begin{aligned}
&E\left[\exp\left(-b\left(T-\tau^A\right)1_{\{\tau^A\le T\}}\right) \mid \mathcal{F}_t\vee\mathcal{F}_{T^*}^r\right] \\
&\quad= \left(\int_t^T+\int_T^\infty\right)e^{-b(T-s)1_{\{s\le T\}}}d\left(1-e^{-b_0^A(s-t)-b_1^A R_{t,s}}\right) \\
&\quad= e^{-b(T-t)}\left(1+b\int_t^T e^{-(b_0^A-b)(s-t)-b_1^A R_{t,s}}ds\right).
\end{aligned}
\tag{3.28}
$$

Hence,

$$
\begin{aligned}
V^B(t,T) &= e^{-(b_0^B+b)(T-t)}E_t\left[e^{-(1+b_1^B)R_{t,T}}\right]+be^{-(b_0^B+b)T+(b_0^B+b_0^A)t} \\
&\quad\cdot\int_t^T e^{-(b_0^A-b)s}E_t\left[e^{-(1+b_1^B+b_1^A)R_{t,s}-(1+b_1^B)R_{s,T}}\right]ds,
\end{aligned}
\tag{3.29}
$$

where (3.29) involves the interchange of the expectation and the integral. Further, using the law of iterated conditional expectations, we have

$$
\begin{aligned}
&E_t\left[e^{-(1+b_1^B+b_1^A)R_{t,s}}\cdot e^{-(1+b_1^B)R_{s,T}}\right] \\
&\quad= E_t\left[e^{-(1+b_1^B+b_1^A)R_{t,s}}\cdot E_s\left[e^{-(1+b_1^B)R_{s,T}}\right]\right] \\
&\quad= E_t\left[e^{-(1+b_1^B+b_1^A)R_{t,s}}\cdot g\left(1+b_1^B,s,T\right)\right] \\
&\quad\triangleq E_t\left[e^I\right].
\end{aligned}
\tag{3.30}
$$

Denote $1 + b_1^B + b_1^A = m_1$ and $1 + b_1^B = m_2$. Then, from Theorem 3.4, we show that

$$
\begin{aligned}
I &\triangleq -m_1 \int_t^s r_u du - m_2 \int_s^T f(s,u)du - m_2 K(T-s) + K m_2 c_T(s) \\
&\quad + \int_s^T \left[\frac{1}{2}\sigma^2 m_2^2 c_T^2(u) + \mu\left(e^{-m_2 q_u c_T(u)} - 1\right)\right] du \\
&\triangleq I_1 + I_2,
\end{aligned}
\tag{3.31}
$$

where

$$
\begin{aligned}
I_1 &= -m_1 \int_t^s r_u du - m_2 \int_s^T f(s,u)du, \\
I_2 &= \int_s^T \left[\frac{1}{2}\sigma^2 m_2^2 c_T^2(u) + \mu\left(e^{-m_2 q_u c_T(u)} - 1\right)\right] du \\
&\quad - m_2 K(T-s) + K m_2 c_T(s).
\end{aligned}
\tag{3.32}
$$

Again, by (3.9), we have

$$
\begin{aligned}
-m_2 \int_s^T f(s,u)du &= -\int_s^T m_2 f(0,u)du - \int_s^T du \int_0^s m_2 \alpha K e^{\alpha(v-u)} dv \\
&\quad - \int_s^T du \int_0^s \sigma m_2 e^{\alpha(v-u)} dW_v - \int_s^T du \int_0^s m_2 q_v e^{\alpha(v-u)} dY_v \\
&\triangleq h_1'(s,T) + h_2'(s,T) + h_3'(s,T) + h_4'(s,T),
\end{aligned}
\tag{3.33}
$$

where

$$
\begin{aligned}
h_1'(s,T) &= -m_2 \int_s^T f(0,u)du, \\
h_2'(s,T) &= -\int_s^T du \int_0^s m_2 \alpha K e^{\alpha(v-u)} dv, \\
h_3'(s,T) &= -\int_s^T du \int_0^s \sigma m_2 e^{\alpha(v-u)} dW_v, \\
h_4'(s,T) &= -\int_s^T du \int_0^s m_2 q_v e^{\alpha(v-u)} dY_v.
\end{aligned}
\tag{3.34}
$$

We easily check that

$$
\begin{aligned}
h_1'(s,T) &= -m_2 r_0 d(s,T,0), \\
h_2'(s,T) &= -m_2 K c_T(s) + m_2 K d(s,T,0),
\end{aligned}
\tag{3.35}
$$

where $d(s,T,u)$ is given by (3.24). Moreover, using *Fübini*'s theorem, we obtain

$$h_3'(s,T) = -\int_0^s \sigma m_2 d(s,T,u)dW_u,$$
$$h_4'(s,T) = -\int_0^s m_2 q_u d(s,T,u)dY_u. \tag{3.36}$$

Therefore, we give a different expression below:

$$\begin{aligned}-m_2\int_s^T f(s,u)du &= -m_2 r_0 d(s,T,0) - m_2 K c_T(s) + m_2 K d(s,T,0)\\ &\quad - \int_0^s \sigma m_2 d(s,T,u)dW_u - \int_0^s m_2 q_u d(s,T,u)dY_u.\end{aligned} \tag{3.37}$$

Then, from (3.12) and (3.37), we find that

$$\begin{aligned}I_1 &= -m_1\int_t^s r_u du - m_2\int_s^T f(s,u)du\\ &= -m_1\int_t^s f(t,u)du - m_1 K(s-t) + m_1 K c_s(t) + m_2(K-r_0)d(s,T,0)\\ &\quad - m_2 K c_T(s) - \int_0^t \sigma m_2 d(s,T,u)dW_u - \int_0^t m_2 q_u d(s,T,u)dY_u\\ &\quad - \int_t^s \sigma(m_1 c_s(u) + m_2 d(s,T,u))dW_u - \int_t^s q_u(m_1 c_s(u) + m_2 d(s,T,u))dY_u.\end{aligned} \tag{3.38}$$

In addition, applying (3.19) and the results in Park [17], we have

$$E\left[e^{-\int_t^s \sigma(m_1 c_s(u)+m_2 d(s,T,u))dW_u}\right] = \exp\left(\frac{1}{2}\sigma^2\int_t^s (m_1 c_s(u) + m_2 d(s,T,u))^2 du\right),$$
$$E\left[e^{-\int_t^s q_u(m_1 c_s(u)+m_2 d(s,T,u))dY_u}\right] = \exp\left(\mu\int_t^s \left[e^{-q_u(m_1 c_s(u)+m_2 d(s,T,u))} - 1\right]du\right). \tag{3.39}$$

Therefore, combining (3.39) and (3.31), we obtain

$$
\begin{aligned}
E_t\left[e^I\right] &= e^{I_2}E_t\left[e^{I_1}\right] \\
&\overset{\text{(i)}}{=} \exp\left(-\left(1+b_1^B+b_1^A\right)\int_t^s f(t,u)du - b_1^A Ks + \left(1+b_1^B+b_1^A\right)Kt - \left(1+b_1^B\right)KT\right) \\
&\quad\cdot \exp\left[\left(1+b_1^B+b_1^A\right)Kc_s(t) + \left(1+b_1^B\right)(K-r_0)d(s,T,0) - f_1(t,s)\right] \\
&\quad\cdot E_t\left[e^{-\int_t^s \sigma[(1+b_1^B+b_1^A)c_s(u)+(1+b_1^B)d(s,T,u)]dW_u}\right]\cdot E_t\left[e^{-\int_t^s q_u[(1+b_1^B+b_1^A)c_s(u)+(1+b_1^B)d(s,T,u)]dY_u}\right] \\
&\overset{\text{(ii)}}{=} \exp\left(-\left(1+b_1^B+b_1^A\right)\int_t^s f(t,u)du - b_1^A Ks + \left(1+b_1^B+b_1^A\right)Kt - \left(1+b_1^B\right)KT\right) \\
&\quad\cdot \exp\left[\left(1+b_1^B+b_1^A\right)Kc_s(t) + \left(1+b_1^B\right)(K-r_0)d(s,T,0) - f_1(t,s)\right] \\
&\quad\cdot \exp\left[\frac{1}{2}\sigma^2\int_t^s \left[\left(1+b_1^B+b_1^A\right)c_s(u) + \left(1+b_1^B\right)d(s,T,u)\right]^2 du\right] \\
&\quad\cdot \exp\left[\mu\int_t^s \left[e^{-q_u((1+b_1^B+b_1^A)c_s(u)+(1+b_1^B)d(s,T,u))} - 1\right]du\right] \\
&\quad\cdot \exp\left[\int_s^T \left[\frac{1}{2}\sigma^2\left(1+b_1^B\right)^2 c_T^2(u) + \mu\left(e^{-(1+b_1^B)q_u c_T(u)} - 1\right)\right]du\right],
\end{aligned}
\tag{3.40}
$$

where (i) follows from the independence of W_u and Y_u and the property of conditional expectation and (ii) holds from the Markov property and the independent increment property of W_u and Y_u. Finally, we substitute $E_t[e^I]$ into (3.29) and obtain (3.23). The proof is completed. □

3.2. CDS's Pricing

Firm C holds a bond issued by the reference firm A with the maturity date T_1. To decrease the possible loss, firm C buys protection with the maturity date T_2 ($T_2 \leq T_1$) from firm B on condition that firm C gives the payments to firm B at a fixed swap rate in time while firm B promises to make up firm C for the loss caused by the default of firm A at a certain rate. Each party has the obligation to make payments until its own default. The source of credit risk may be from three parties: the issuer of bond, the buyer of CDS and the seller of CDS.

In the following, we consider a simple situation which only contains the risk from reference firm A and firm B. At the same time, to make the calculation convenient, we suppose the recovery rate of the bond issued by firm A is zero and the face value is 1 dollar. In the event of firm A's default, firm B compensates firm C for 1 dollar if he does not default, otherwise 0 dollar. There are four cases for the defaults of firm A and firm B.

Case 1. The defaults of firm A and firm B are mutually independent conditional on the risk-free interest rate.

Case 2. Firm A is the primary party whose default only depends on the risk-free interest rate (the only economy state variable) and the firm B is the secondary party whose default depends on the risk-free interest rate and the default state of firm A.

Case 3. Firm B is the primary party and the firm A is the secondary party.

Case 4. The defaults of firm A, and firm B are mutually contagious (looping default).

Now, we make use of the results in previous sections to price the CDS in Case 2. We assume firm A is the primary party and the firm B is the secondary party. Denoted the swap rate by a constant c and interest rate by r_t, let the default times of firm A and B be τ^A with the intensity λ^A and τ^B with the intensity λ^B, respectively.

Theorem 3.6. *Suppose the risk-free interest rate r_t satisfies* (3.1) *and the intensities λ^A and λ^B satisfy* (3.6) *and* (3.7)*, respectively. Then, the swap rate c has the following expression:*

$$c = \frac{V^B(0,T_2) - e^{-(b_0^B + b_0^A)T_2} g\left(1 + b_1^B + b_1^A, 0, T_2\right)}{\int_0^{T_2} g(1,0,s)ds}, \tag{3.41}$$

where $g(\cdot,\cdot,\cdot)$ and $V^B(0,T_2)$ are given by Theorems 3.4 and 3.5, respectively.

Proof. Firstly, the time-0 market value of buyer C's payments to seller B is

$$E\left[\int_0^{T_2} ce^{-\int_0^s r_u du} ds\right] = c\int_0^{T_2} E\left[e^{-R_{0,s}}\right] ds, \tag{3.42}$$

where $R_{0,s}$ is defined as Theorem 3.4.

Secondly, the time-0 market value of firm B's promised payoff in case of firm A's default is

$$E\left[1_{\{\tau^A \le T_2\}} e^{-\int_0^{T_2} r_u du} 1_{\{\tau^B > T_2\}}\right]. \tag{3.43}$$

Thus, in accordance with the arbitrage-free principle, we obtain

$$c = \frac{E\left[1_{\{\tau^A \le T_2\}} e^{-\int_0^{T_2} r_u du} 1_{\{\tau^B > T_2\}}\right]}{\int_0^{T_2} E\left[e^{-R_{0,s}}\right] ds}. \tag{3.44}$$

Further, we can use the properties of conditional expectation to simplify (3.44) as follows:

$$\begin{aligned} c &= \frac{E\left[E\left[1_{\{\tau^A\le T_2\}} \mid \mathcal{F}^r_{T^*}\vee\mathcal{H}^B_{T^*}\right]e^{-\int_0^{T_2}r_u du}1_{\{\tau^B>T_2\}}\right]}{\int_0^{T_2}E\left[e^{-R_{0,s}}\right]ds} \\ &= \frac{E\left[1_{\{\tau^B>T_2\}}e^{-\int_0^{T_2}r_u du}\right]-E\left[1_{\{\tau^A>T_2\}}e^{-\int_0^{T_2}r_u du}1_{\{\tau^B>T_2\}}\right]}{\int_0^{T_2}E\left[e^{-R_{0,s}}\right]ds} \\ &= \frac{V^B(0,T_2)-E\left[1_{\{\tau^A>T_2\}}e^{-\int_0^{T_2}(r_u+\lambda^B_u)du}\right]}{\int_0^{T_2}E\left[e^{-R_{0,s}}\right]ds}, \end{aligned} \tag{3.45}$$

where the last one is obtained by (3.4). Note that $V^B(0,T_2)$ can be obtained by (3.23) and $E[e^{-R_{0,s}}]=g(1,0,s)$ by Theorem 3.4. We substitute (3.7) into the above expectation term

$$\begin{aligned} &E\left[1_{\{\tau^A>T_2\}}\exp\left(-\int_0^{T_2}\left(r_u+\lambda^B_u\right)du\right)\right] \\ &\quad= E\left[1_{\{\tau^A>T_2\}}\exp\left(-\int_0^{T_2}\left(b^B_0+\left(1+b^B_1\right)r_u+b1_{\{\tau^A\le u\}}\right)du\right)\right] \\ &\quad= E\left[1_{\{\tau^A>T_2\}}\exp\left(-b^B_0T_2-\left(1+b^B_1\right)\int_0^{T_2}r_u du-b\left(T_2-\tau^A\right)1_{\{\tau^A\le T_2\}}\right)\right] \\ &\quad= E\left[1_{\{\tau^A>T_2\}}e^{-b^B_0T_2-(1+b^B_1)R_{0,T_2}}\right] \\ &\quad\overset{\text{(i)}}{=} E\left[E\left[1_{\{\tau^A>T_2\}}\mid\mathcal{F}^r_{T^*}\right]e^{-b^B_0T_2-(1+b^B_1)R_{0,T_2}}\right] \\ &\quad\overset{\text{(ii)}}{=} E\left[e^{-b^B_0T_2-(1+b^B_1)R_{0,T_2}-\int_0^{T_2}\lambda^A_u du}\right] \\ &\quad\overset{\text{(iii)}}{=} e^{-(b^B_0+b^A_0)T_2}E\left[e^{-(1+b^B_1+b^A_1)\int_0^{T_2}r_u du}\right] \\ &\quad\overset{\text{(iv)}}{=} e^{-(b^B_0+b^A_0)T_2}g\left(1+b^B_1+b^A_1,0,T_2\right), \end{aligned} \tag{3.46}$$

where (i) involves the property of conditional expectation, (ii) follows from (2.6), and (iv) follows from Theorem 3.4. Now, substituting these results into (3.45) , we show (3.41) holds. The proof is complete. □

Remark 3.7. The model in Case 1 can be considered a special case of primary-secondary model and the price of CDS can be derived by the similar method. The pricing of CDS in Case 4 will be discussed in another paper. In Case 3, if λ^A_t and λ^B_t satisfy the below relations:

$$\begin{aligned} \lambda^A_t &= b^A_0+b^A_1r_t+b1_{\{\tau^B\le t\}}, \\ \lambda^B_t &= b^B_0+b^B_1r_t, \end{aligned} \tag{3.47}$$

where b_0^A, b_1^A, b_0^B, b_1^B, and b are positive constants, then the swap rate

$$c = \frac{g\left(1 + b_1^B, 0, T_2\right)e^{-b_0^B T_2} - e^{-(b_0^B + b_0^A)T_2} g\left(1 + b_1^B + b_1^A, 0, T_2\right)}{\int_0^{T_2} g(1, 0, s)ds}, \tag{3.48}$$

where $g(\cdot, \cdot, \cdot)$ are given by Theorem 3.4. The deriving process is similar to Theorem 3.6, so we omit it.

Remark 3.8. In our models, to make the expressions comparatively simple, we all assume that the recovery rates are zero. When the relevant recovery rates are nonzero constant, the pricing formulas are still easily obtained from Lemma 3.2 because we can get $p(t, T) = g(1, t, T)$ from Theorem 3.4. We omit the process.

4. Conclusion

This paper gives the pricing formulas of defaultable bonds and CDSs. In our model, we consider the case that the default intensity is correlated with the risk-free interest rate following jump-diffusion process and the counterparty's default, which is more realistic. We involve the jump risk of risk-free interest rate in the pricing, generalizing the contagious model in Jarrow and Yu [12].

In fact, we only consider the comparatively simple situation. We can further study the more general model. For example, we consider the case that the relevant recovery rates are stochastic and the interest rate satisfies more general jump-diffusion process. Moreover, the model in this paper is actually one-factor model with one state variable, while we can discuss multifactor models in which there are several state variables. In a word, the contagious model of credit security with counterparty risk is very necessary to be further discussed in the future.

Acknowledgments

The authors gratefully acknowledge the support from the National Basic Research Program of China (973 Program no. 2007CB814903) and thank the reviewers for their valued comments.

References

[1] R. C. Merton, "On the pricing of corporate debt: the risk structure of interest rates," *Journal of Finance*, vol. 29, pp. 449–470, 1974.
[2] F. Black and J. C. Cox, "Valuing corporate securities: some effects of bond indenture provisions," *Journal of Finance*, vol. 31, pp. 351–367, 1976.
[3] F. A. Longstaff and E. S. Schwartz, "A simple approach to valuing risky fixed and floating rate debt," *Journal of Finance*, vol. 50, pp. 789–819, 1995.
[4] C. S. Zhou, *A Jump-Diffusion Approach to Modeling Credit Risk and Valuing Defaultable Securities*, Federal Reserve Board, Washington, DC, USA, 1997.
[5] A. K. Steve and F. A. Amir, "Using Brownian bridge for fast simulation of jump-diffusion processes and barrier options," *Journal of Derivatives*, pp. 43–54, 2002.
[6] D. Zhang and R. V. N. Melnik, "First passage time for multivariate jump-diffusion processes in finance and other areas of applications," *Applied Stochastic Models in Business and Industry*, vol. 25, no. 5, pp. 565–582, 2009.

[7] S. G. Kou and H. Wang, “First passage times of a jump diffusion process,” *Advances in Applied Probability*, vol. 35, no. 2, pp. 504–531, 2003.
[8] S. G. Kou, G. Petrella, and H. Wang, “Pricing path-dependent options with jump risk via Laplace transforms,” *Kyoto Economic Review*, vol. 74, pp. 1–23, 2005.
[9] R. A. Jarrow, D. Lando, and S. Turnbull, “A Markov model for the term structure of credit risk spreads,” Working paper, Conell University, 1994.
[10] D. Duffie and K. J. Singleton, “Modeling term structures of defaultable bonds,” Working paper, Stanford University Business School, 1995.
[11] M. Davis and V. Lo, “Infectious defaults,” *Quantitative Finance*, pp. 382–387, 2001.
[12] R. A. Jarrow and F. Yu, “Counterparty risk and the pricing of defaultable securities,” *Journal of Finance*, vol. 56, no. 5, pp. 1765–1799, 2001.
[13] S. Y. Leung and Y. K. Kwork, “Credit default swap valuation with counterparty risk,” *Kyoto Economic Review*, vol. 74, no. 1, pp. 25–45, 2005.
[14] Y.-F. Bai, X.-H. Hu, and Z.-X. Ye, “A model for dependent default with hypberbolic attenuation effect and valuation of credit default swap,” *Applied Mathematics and Mechanics. English Edition*, vol. 28, no. 12, pp. 1643–1649, 2007.
[15] F. Yu, “Correlated defaults in intensity-based models,” *Mathematical Finance. An International Journal of Mathematics, Statistics and Financial Economics*, vol. 17, no. 2, pp. 155–173, 2007.
[16] K. S. Leung and Y. K. Kwok, “Counterparty risk for credit default swaps: Markov chain interacting intensities model with stochastic intensity,” *Asia-Pacific Financial Markets*, vol. 16, no. 3, pp. 169–181, 2009.
[17] H. S. Park, “The survival probability of mortality intensity with jump-diffusion,” *Journal of the Korean Statistical Society*, vol. 37, no. 4, pp. 355–363, 2008.