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Research Article

Monotone Positive Solutions for an Elastic Beam Equation with Nonlinear Boundary Conditions

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This paper is concerned with the existence of monotone positive solutions for an elastic beam equation with nonlinear boundary conditions. By applying monotone iteration method, we not only obtain the existence of monotone positive solutions but also establish iterative schemes for approximating the solutions. It is worth mentioning that these iterative schemes start off with zero function or quadratic function, which is very useful and feasible for computational purpose. An example is also included to illustrate the main results obtained.

1. Introduction

It is well known that beam is one of the basic structures in architecture. The deformations of an elastic beam in equilibrium state can be described by the following equation of deflection curve:

$$\frac{d^2}{dt^2}\left(EI_z\frac{d^2u}{dt^2}\right) = q(t), \tag{1.1}$$

where E is Yang's modulus constant, I_z is moment of inertia with respect to z axes, and $q(t)$ is loading at t. If the loading of beam considered is in relation to deflection and rate of change of deflection, we need to study a more general equation:

$$u^{(4)}(t) = f\left(t, u(t), u'(t)\right). \tag{1.2}$$

According to different forms of supporting, various boundary value problems (BVPs for short) should be considered.

Owing to its importance in engineering, physics, and material mechanics, fourth-order BVPs have attracted much attention from many authors; see, for example [1–15] and the references therein. However, almost all of the papers we mentioned focused their attention on the null boundary conditions. When the boundary conditions are nonzero or nonlinear, fourth-order equations can model beams resting on elastic bearings located in their extremities. Up to now, a little work has been done for fourth-order BVPs with nonlinear boundary conditions. It is worth mentioning that, in 2009, Alves et al. [16] studied some fourth-order BVPs with nonlinear boundary conditions, which models an elastic beam whose left end is fixed and right end is attached to a bearing device or sliding clamped. Their main tool was monotone iterative method. For more on monotone iterative techniques, one can refer to [17–20] and the references therein.

Motivated greatly by the above-mentioned excellent works, in this paper we consider the existence and iteration of monotone positive solutions for the following fourth-order BVP with nonlinear boundary conditions:

$$\begin{gathered} u^{(4)}(t) = f\left(t, u(t), u'(t)\right), \quad t \in (0,1), \\ u(0) = u''(0) = 0, \\ u'(1) = 0, \qquad u'''(1) = g(u(1)), \end{gathered} \tag{1.3}$$

which models an elastic beam whose left end is simply supported and right end is sliding clamped, given by the function g. By applying monotone iterative method, we not only obtain the existence of monotone positive solutions for the BVP (1.3) but also establish iterative schemes for approximating the solutions. These iterative schemes start off with zero function or quadratic function, which is very useful and feasible for computational purpose. Our main tool is the following theorem [21].

Theorem 1.1. *Let K be a normal cone of a Banach space E and $v_0 \le w_0$. Suppose that*

(a_1) *$T : [v_0, w_0] \to E$ is completely continuous,*

(a_2) *T is monotone increasing on $[v_0, w_0]$,*

(a_3) *v_0 is a lower solution of T, that is, $v_0 \le Tv_0$,*

(a_4) *w_0 is an upper solution of T, that is, $Tw_0 \le w_0$.*

Then the iterative sequences

$$v_n = Tv_{n-1}, \qquad w_n = Tw_{n-1} \quad (n = 1, 2, 3, \ldots) \tag{1.4}$$

satisfy

$$v_0 \le v_1 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0 \tag{1.5}$$

and converge to, respectively, v and $w \in [v_0, w_0]$, which are fixed points of T.

Throughout this paper, we always assume that the following conditions are satisfied:

(A_1) $f \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$;

(A_2) $g \in C([0,+\infty), (-\infty,0])$.

2. Preliminary

In order to obtain the main results of this paper, we first present several fundamental lemmas in this section.

Lemma 2.1. *Let γ be a constant and $y \in C[0,1]$. Then the BVP*

$$\begin{gathered} u^{(4)}(t) = y(t), \quad t \in (0,1), \\ u(0) = u''(0) = 0, \\ u'(1) = 0, \qquad u'''(1) = \gamma \end{gathered} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds - \gamma\phi(t), \quad t \in [0,1], \tag{2.2}$$

where

$$\begin{aligned} G(t,s) &= \frac{1}{6}\begin{cases} s\left(6t - 3t^2 - s^2\right), & 0 \le s \le t \le 1, \\ t\left(6s - 3s^2 - t^2\right), & 0 \le t \le s \le 1, \end{cases} \\ \phi(t) &= \frac{1}{2}t - \frac{1}{6}t^3, \quad t \in [0,1]. \end{aligned} \tag{2.3}$$

Proof. Let u be a solution of the BVP (2.1). Then we may suppose that

$$u(t) = \int_0^1 G(t,s)y(s)ds + At^3 + Bt^2 + Ct + D, \quad t \in [0,1]. \tag{2.4}$$

By the boundary conditions in (2.1), we have

$$A = \frac{\gamma}{6}, \qquad B = 0, \qquad C = -\frac{\gamma}{2}, \qquad D = 0. \tag{2.5}$$

Therefore, the BVP (2.1) has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds - \gamma\phi(t), \quad t \in [0,1]. \tag{2.6}$$

□

Lemma 2.2. *For any* $(t,s) \in [0,1] \times [0,1]$, *one has*

$$0 \leq \frac{\partial G(t,s)}{\partial t} \leq (1-t)s, \qquad \frac{1}{3}t^2 s \leq G(t,s) \leq \frac{1}{2}\left(2t-t^2\right)s. \tag{2.7}$$

Proof. For any fixed $s \in [0,1]$, it is easy to know that

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1, \\ (1-t)s - \dfrac{1}{2}(s-t)^2, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.8}$$

which shows that

$$0 \leq \frac{\partial G(t,s)}{\partial t} \leq (1-t)s \quad \text{for } (t,s) \in [0,1] \times [0,1], \tag{2.9}$$

and so

$$G(t,s) = \int_0^t \frac{\partial G(\tau,s)}{\partial \tau} d\tau \leq \int_0^t s(1-\tau)d\tau = \frac{1}{2}\left(2t-t^2\right)s \quad \text{for } (t,s) \in [0,1] \times [0,1]. \tag{2.10}$$

On the other hand, it follows from the expression of $G(t,s)$ that

$$\begin{aligned} G(t,s) &= \frac{1}{6}s\left(6t-3t^2-s^2\right) \geq \frac{1}{6}s\left(6t-4t^2\right) \geq \frac{1}{3}t^2 s, \quad 0 \leq s \leq t \leq 1, \\ G(t,s) &= \frac{1}{6}t\left(6s-3s^2-t^2\right) \geq \frac{1}{6}t\left(6s-4s^2\right) \geq \frac{1}{3}t^2 s, \quad 0 \leq t \leq s \leq 1. \end{aligned} \tag{2.11}$$

□

Lemma 2.3. *For any* $t \in [0,1]$, *one has*

$$0 \leq \phi(t) \leq \frac{1}{2}\left(2t-t^2\right), \qquad 0 \leq \phi'(t) \leq 1-t. \tag{2.12}$$

Proof. It is obvious. □

3. Main Results

Theorem 3.1. *Assume that* $f(t,0,0) \not\equiv 0$ *for* $t \in [0,1]$ *and there exists a constant* $a > 0$ *such that the following conditions are satisfied:*

(H_1) $f(t,u_1,v_1) \leq f(t,u_2,v_2) \leq a,\ 0 \leq t \leq 1,\ 0 \leq u_1 \leq u_2 \leq a,\ 0 \leq v_1 \leq v_2 \leq a;$

(H_2) $(-a/2) \leq g(z_2) \leq g(z_1),\ 0 \leq z_1 \leq z_2 \leq a.$

Then the BVP (1.3) *has monotone positive solutions.*

Proof. Let $E = C^1[0,1]$ be equipped with the norm

$$\|u\| = \max\left\{\max_{t\in[0,1]}|u(t)|,\ \max_{t\in[0,1]}\left|u'(t)\right|\right\},$$
$$K = \left\{u \in E:\ u(t) \geq 0,\ u'(t) \geq 0 \text{ for } t \in [0,1]\right\}. \tag{3.1}$$

Then K is a normal cone in Banach space E. Note that this induces an order relation $\leq$ in E by defining $u \leq v$ if and only if $v - u \in K$. If we define an operator $T : K \to E$ by

$$(Tu)(t) = \int_0^1 G(t,s)f\left(s,u(s),u'(s)\right)ds - g(u(1))\phi(\mathrm{t}), \quad t \in [0,1], \tag{3.2}$$

then

$$(Tu)'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t} f\left(s,u(s),u'(s)\right)ds - g(u(1))\phi'(t), \quad t \in [0,1], \tag{3.3}$$

which together with (A_1), (A_2), and Lemmas 2.2 and 2.3 implies that $T : K \to K$. Obviously, fixed points of T are monotone nonnegative solutions of the BVP (1.3).

Let $v_0(t) \equiv 0$ and $w_0(t) = a(2t - t^2)/2$, $t \in [0,1]$. First, it is easy to verify that $T : [v_0, w_0] \to K$ is completely continuous by an application of Arzela-Ascoli theorem. Now, we divide our proof into the following steps.

Step 1. We assert that T is monotone increasing on $[v_0, w_0]$.

Suppose that $u,\ v \in [v_0, w_0]$ and $u \leq v$. Then $0 \leq u(t) \leq v(t) \leq a$ and $0 \leq u'(t) \leq v'(t) \leq a$ for $t \in [0,1]$. By (H_1) and (H_2), we have

$$\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)f\left(s,u(s),u'(s)\right)ds - g(u(1))\phi(t) \\
&\leq \int_0^1 G(t,s)f\left(s,v(s),v'(s)\right)ds - g(v(1))\phi(t) \\
&= (Tv)(t), \quad t \in [0,1], \\
(Tu)'(t) &= \int_0^1 \frac{\partial G(t,s)}{\partial t} f\left(s,u(s),u'(s)\right)ds - g(u(1))\phi'(t) \\
&\leq \int_0^1 \frac{\partial G(t,s)}{\partial t} f\left(s,v(s),v'(s)\right)ds - g(v(1))\phi'(t) \\
&= (Tv)'(t), \quad t \in [0,1],
\end{aligned} \tag{3.4}$$

which shows that $Tu \leq Tv$.

Step 2. We prove that v_0 is a lower solution of T.

For any $t \in [0,1]$, we know that

$$(Tv_0)(t) = \int_0^1 G(t,s) f(s,0,0)\,ds - g(0)\phi(t) \geq 0 = v_0(t),$$
$$(Tv_0)'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,0,0)\,ds - g(0)\phi'(t) \geq 0 = v_0'(t), \tag{3.5}$$

which implies that $v_0 \leq Tv_0$.

Step 3. We show that w_0 is an upper solution of T.

It follows from Lemmas 2.2 and 2.3, (H_1), and (H_2) that

$$\begin{aligned}
(Tw_0)(t) &= \int_0^1 G(t,s) f\left(s, w_0(s), w_0'(s)\right) ds - g(w_0(1))\phi(t) \\
&\leq \frac{2t-t^2}{2} \int_0^1 s f\left(s, w_0(s), w_0'(s)\right) ds + \frac{a\left(2t-t^2\right)}{4} \\
&\leq \frac{a\left(2t-t^2\right)}{4} + \frac{a\left(2t-t^2\right)}{4} \\
&= w_0(t), \quad t \in [0,1], \\
(Tw_0)'(t) &= \int_0^1 \frac{\partial G(t,s)}{\partial t} f\left(s, w_0(s), w_0'(s)\right) ds - g(w_0(1))\phi'(t) \\
&\leq (1-t) \int_0^1 s f\left(s, w_0(s), w_0'(s)\right) ds + \frac{a(1-t)}{2} \\
&\leq \frac{a(1-t)}{2} + \frac{a(1-t)}{2} \\
&= w_0'(t), \quad t \in [0,1],
\end{aligned} \tag{3.6}$$

which indicates that $Tw_0 \leq w_0$.

Step 4. We claim that the BVP (1.3) has monotone positive solutions.

In fact, if we construct sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ as follows:

$$v_n = Tv_{n-1}, \qquad w_n = Tw_{n-1}, \quad n = 1,2,3,\ldots, \tag{3.7}$$

then it follows from Theorem 1.1 that

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0, \tag{3.8}$$

and $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ converge to, respectively, v and $w \in [v_0, w_0]$, which are monotone solutions of the BVP (1.3). Moreover, for any $t \in (0,1]$, by Lemmas 2.2 and 2.3, we know that

$$\begin{aligned}(Tv_0)(t) &= \int_0^1 G(t,s) f(s,0,0) ds - g(0)\phi(t) \\ &\geq \frac{t^2}{3}\int_0^1 s f(s,0,0) ds \\ &> 0,\end{aligned} \tag{3.9}$$

and so

$$0 < (Tv_0)(t) \leq (Tv)(t) = v(t) \leq w(t), \quad t \in (0,1], \tag{3.10}$$

which shows that v and w are positive solutions of the BVP (1.3). □

4. An Example

Example 4.1. Consider the BVP

$$\begin{gathered} u^{(4)}(t) = \frac{1}{4}u(t) + \frac{1}{16}(1+t)\left(u'(t)\right)^2 + 1, \quad t \in (0,1), \\ u(0) = u''(0) = 0, \\ u'(1) = 0, \qquad u'''(1) = -\frac{1}{4}(u(1))^2. \end{gathered} \tag{4.1}$$

If we let $f(t,u,v) = (1/4)u + 1/16(1+t)v^2 + 1$ for $(t,u,v) \in [0,1] \times [0,+\infty) \times [0,+\infty)$ and $g(u) = -(1/4)u^2$ for $u \in [0,+\infty)$, then all the hypotheses of Theorem 3.1 are fulfilled with $a = 2$. It follows from Theorem 3.1 that the BVP (4.1) has monotone positive solutions v and w. Moreover, the two iterative schemes are

$$\begin{aligned} v_0(t) &\equiv 0, \quad t \in [0,1], \\ v_{n+1}(t) &= \frac{1}{6}\int_0^t s\left(6t - 3t^2 - s^2\right)\left(\frac{1}{4}v_n(s) + \frac{1}{16}(1+s)\left(v_n'(s)\right)^2 + 1\right) ds \\ &\quad + \frac{1}{6}\int_t^1 t\left(6s - 3s^2 - t^2\right)\left(\frac{1}{4}v_n(s) + \frac{1}{16}(1+s)\left(v_n'(s)\right)^2 + 1\right) ds \\ &\quad + \frac{1}{4}(v_n(1))^2\left(\frac{1}{2}t - \frac{1}{6}t^3\right), \quad t \in [0,1], \ n = 0,1,2,\ldots, \end{aligned}$$

$$
\begin{aligned}
w_0(t) &= 2t - t^2, \quad t \in [0,1], \\
w_{n+1}(t) &= \frac{1}{6}\int_0^t s\left(6t - 3t^2 - s^2\right)\left(\frac{1}{4}w_n(s) + \frac{1}{16}(1+s)\left(w_n'(s)\right)^2 + 1\right)ds \\
&\quad + \frac{1}{6}\int_t^1 t\left(6s - 3s^2 - t^2\right)\left(\frac{1}{4}w_n(s) + \frac{1}{16}(1+s)\left(w_n'(s)\right)^2 + 1\right)ds \\
&\quad + \frac{1}{4}(w_n(1))^2\left(\frac{1}{2}t - \frac{1}{6}t^3\right), \quad t \in [0,1],\ n = 0,1,2,\ldots.
\end{aligned}
\tag{4.2}
$$

The first, second, and third terms of the two schemes are as follows:

$$
\begin{aligned}
v_0(t) &\equiv 0, \\
v_1(t) &= \frac{1}{3}t - \frac{1}{6}t^3 + \frac{1}{24}t^4, \\
v_2(t) &= \frac{205151}{580608}t - \frac{169109}{967680}t^3 + \frac{145}{3456}t^4 + \frac{13}{17280}t^5 - \frac{1}{17280}t^6 - \frac{1}{15120}t^7 + \frac{19}{967680}t^8 \\
&\quad + \frac{1}{580608}t^9 - \frac{1}{580608}t^{10} + \frac{1}{4561920}t^{11}, \\
w_0(t) &= 2t - t^2, \\
w_1(t) &= \frac{263}{480}t - \frac{73}{288}t^3 + \frac{5}{96}t^4 + \frac{1}{480}t^5 - \frac{1}{720}t^6 + \frac{1}{3360}t^7, \\
w_2(t) &= \frac{21844130402150536441991}{58425017821892349788160}t - \frac{5343730809041282235713}{29212508910946174894080\,0}t^3 + \frac{3755569}{88473600}t^4 \\
&\quad + \frac{574129}{442368000}t^5 - \frac{19199}{132710400}t^6 - \frac{12433}{103219200}t^7 + \frac{47291}{1238630400}t^8 + \frac{8953}{1592524800}t^9 \\
&\quad - \frac{17377}{4644864000}t^{10} + \frac{19373}{51093504000}t^{11} + \frac{161}{2919628800}t^{12} - \frac{269}{12651724800}t^{13} \\
&\quad + \frac{31}{14760345600}t^{14} + \frac{1}{6709248000}t^{15} - \frac{1}{23003136000}t^{16} + \frac{1}{210567168000}t^{17}.
\end{aligned}
\tag{4.3}
$$

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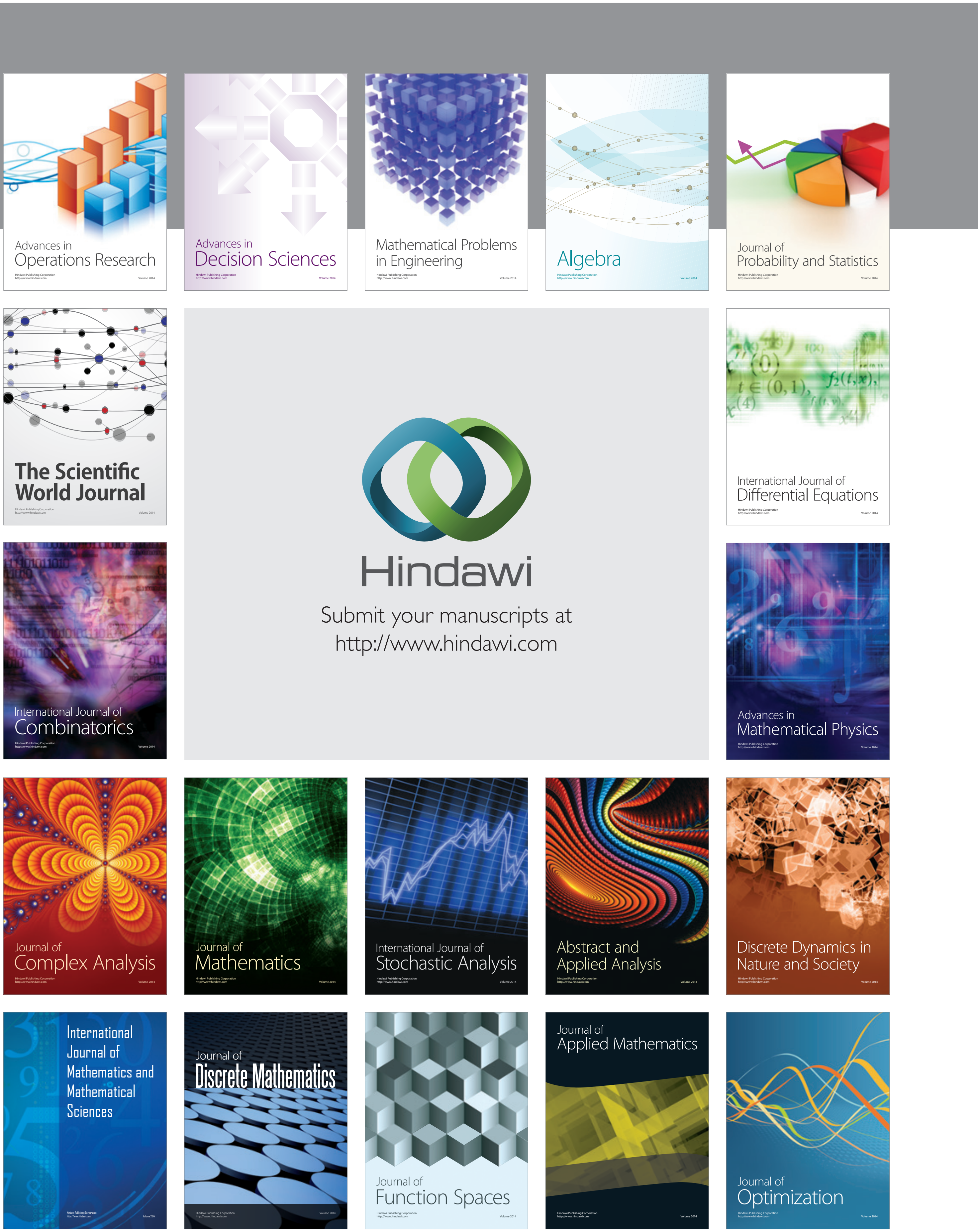
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