Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 591256, 14 pages
doi:10.1155/2012/591256

Research Article

An Analog of the Adjugate Matrix for the Outer Inverse $A_{T,S}^{(2)}$

Xiaoji Liu,[1,2] Guangyan Zhu,[1] Guangping Zhou,[1] and Yaoming Yu[3]

[1] *School of Science, Guangxi University for Nationalities, Nanning 530006, China*
[2] *Guangxi Key Laboratory of Hybrid Computational and IC Design Analysis, Nanning 530006, China*
[3] *School of Mathematical Sciences, Monash University, Caulfield East, VIC 3800, Australia*

Correspondence should be addressed to Xiaoji Liu, liuxiaoji.2003@yahoo.com.cn

Received 5 August 2011; Revised 24 November 2011; Accepted 7 December 2011

Academic Editor: Kui Fu Chen

We investigate the determinantal representation by exploiting the limiting expression for the generalized inverse $A_{T,S}^{(2)}$. We show the equivalent relationship between the existence and limiting expression of $A_{T,S}^{(2)}$ and some limiting processes of matrices and deduce the new determinantal representations of $A_{T,S}^{(2)}$, based on some analog of the classical adjoint matrix. Using the analog of the classical adjoint matrix, we present Cramer rules for the restricted matrix equation $AXB = D$, $\mathcal{R}(X) \subset T$, $\mathcal{N}(X) \supset \widetilde{S}$.

1. Introduction

Throughout this paper $\mathbb{C}^{m\times n}$ denotes the set of $m \times n$ matrices over the complex number field $\mathbb{C}$, and $\mathbb{C}_r^{m\times n}$ denotes its subset in which every matrix has rank r. I stands for the identity matrix of appropriate order (dimension).

Let $A \in \mathbb{C}^{m\times n}$, and let M and N be Hermitian positive definite matrices of orders m and n, respectively. Consider the following equations:

$$AXA = A, \tag{1}$$

$$XAX = A, \tag{2}$$

$$(AX)^* = AX, \tag{3}$$

$$(MAX)^* = MAX, \tag{3M}$$

$$(XA)^* = XA, \tag{4}$$

$$(NXA)^* = NXA. \tag{4N}$$

X is called a $\{2\}$- (or outer) inverse of A if it satisfies (2) and denoted by $A^{(2)}$. X is called the Moore-Penrose inverse of A if it satisfies (1), (2), (3), and (5) and denoted by $A^{\dagger}$. X is called the weighted Moore-Penrose inverse of A (with respect to M, N) if it satisfies (1), (2), (4), and (6) and denoted by A^{+}_{MN} (see, e.g., [1, 2]).

Let $A \in \mathbb{C}^{n\times n}$. Then a matrix X satisfying

$$A^k XA = A^k, \tag{1^k}$$

$$XAX = X, \tag{2^*}$$

$$AX = XA, \tag{5}$$

where k is some positive integer, is called the Drazin inverse of A and denoted by A_d. The smallest positive integer k such that X and A satisfy (7), (8), and (9), then it is called the Drazin index and denoted by $k = \mathrm{Ind}(A)$. It is clear that $\mathrm{Ind}(A)$ is the smallest positive integer k satisfying $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$ (see [3]). If $k = 1$, then X is called the group inverse of A and denoted by A_g. As is well known, A_g exists if and only if $\mathrm{rank}\, A = \mathrm{rank}\, A^2$. The generalized inverses, and in particular Moore-Penrose, group and Drazin inverses, have also been studied in the context of semigroups, rings of Banach and C^* algebras (see [4–8]).

In addition, if a matrix X satisfies (1) and (5), then it is called a $\{1,5\}$-inverse of A and is denoted by $A^{(1,5)}$.

Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$. Then the matrix $X \in \mathbb{C}^{m\times n}$ satisfying

$$(AW)^{k+1} XW = (AW)^k, \tag{1^kW}$$

$$XWAWX = X, \tag{2W}$$

$$AWX = XWA, \tag{5W}$$

where k is some nonnegative integer, is called the W-weighted Drazin inverse of A, and is denoted by $X = A_{d,W}$ (see [9]). It is obvious that when $m = n$ and $W = I_n$, X is called the Drazin inverse of A.

Lemma 1.1 (see [1, Theorem 2.14]). *Let $A \in \mathbb{C}^{m\times n}_r$, and let T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, with $\dim T = \dim S^{\perp} = t \le r$. Then A has a $\{2\}$-inverse X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if*

$$AT \oplus S = \mathbb{C}^m \tag{1.1}$$

in which case X is unique and denoted by $A^{(2)}_{T,S}$.

If $A^{(2)}_{T,S}$ exists and there exists a matrix G such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$, then $GAA^{(2)}_{T,S} = G$ and $A^{(2)}_{T,S}AG = G$.

It is well known that several important generalized inverses, such as the Moore-Penrose inverse $A^{\dagger}$, the weighted Moore-Penrose inverse $A^{+}_{M,N}$, the Drazin inverse A_d, and the group inverse A_g, are outer inverses $A^{(2)}_{T,S}$ for some specific choice of T and S, are all the generalized inverse $A^{(2)}_{T,S}$, $\{2\}$- (or outer) inverse of A with the prescribed range T and null space S (see [2, 10] in the context of complex matrices and [11] in the context of semigroups).

Determinantal representation of the generalized inverse $A^{(2)}_{T,S}$ was studied in [12, 13]. We will investigate further such representation by exploiting the limiting expression for $A^{(2)}_{T,S}$. The paper is organized as follows. In Section 2, we investigate the equivalent relationship between the existence of $A^{(2)}_{T,S}$ and the limiting process of matrices $\lim_{\lambda\to 0}G(AG+\lambda I)^{-1}$ or $\lim_{\lambda\to 0}(GA+\lambda I)^{-1}G$ and deduce the new determinantal representations of $A^{(2)}_{T,S}$, based on some analog of the classical adjoint matrix, by exploiting limiting expression. In Section 3, using the analog of the classical adjoint matrix in Section 2, we present Cramer rules for the restricted matrix equation $AXB = D$, $\mathcal{R}(X) \subset T$, $\mathcal{N}(X) \supset \widetilde{S}$. In Section 4, we give an example for solving the solution of the restricted matrix equation by using our expression. We introduce the following notations.

For $1 \le k \le n$, the symbol $Q_{k,n}$ denotes the set $\{\alpha : \alpha = (\alpha_1,\ldots,\alpha_k), 1 \le \alpha_1 < \cdots < \alpha_k \le n$, where α_i, $i = 1,\ldots,k$, are integers$\}$. And $Q_{k,n}\{j\} := \{\beta : \beta \in Q_{k,n}, j \in \beta\}$, where $j \in \{1,\ldots,n\}$.

Let $A = (a_{ij}) \in \mathbb{C}^{m\times n}$. The symbols $a_{.j}$ and $a_{i.}$ stand for the jth column and the ith row of A, respectively. In the same way, denote by $a^{*}_{.j}$ and $a^{*}_{i.}$ the jth column and the ith row of Hermitian adjoint matrix A^{*}. The symbol $A_{.j}(b)$ (or $A_{j.}(b)$) denotes the matrix obtained from A by replacing its jth column (or row) with some vector b (or b^T). We write the range of A by $\mathcal{R}(A) = \{Ax : x \in \mathbb{C}^n\}$ and the null space of A by $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$. Let $B \in \mathbb{C}^{p\times q}$. We define the range of a pair of A and B as $\mathcal{R}(A,B) = \{AWB : W \in \mathbb{C}^{n\times p}\}$.

Let $\alpha \in Q_{k,m}$ and $\beta \in Q_{k,n}$, where $1 \le k \le \min\{m,n\}$. Then $|A^{\alpha}_{\beta}|$ denotes a minor of A determined by the row indexed by α and the columns indexed by β. When $m = n$, the cofactor of a_{ij} in A is denoted by $\partial|A|/\partial a_{ij}$.

2. Analogs of the Adjugate Matrix for $A^{(2)}_{T,S}$

We start with the following theorem which reveals the intrinsic relation between the existence of $A^{(2)}_{T,S}$ and of $\lim_{\lambda\to 0}G(AG+\lambda I)^{-1}$ or $\lim_{\lambda\to 0}(GA+\lambda I)^{-1}G$. Here $\lambda \to 0$ means $\lambda \to 0$ through any neighborhood of 0 in $\mathbb{C}$ which excludes the nonzero eigenvalues of a square matrix. In [14], Wei pointed out that the existence of $A^{(2)}_{T,S}$ implies the existence of $\lim_{\lambda\to 0}G(AG+\lambda I)^{-1}$ or $\lim_{\lambda\to 0}(GA+\lambda I)^{-1}G$. The following result will show that the converse is true under some condition.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times n}_r$, and let T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, with* $\dim T = \dim S^{\perp} = t \le r$. *Let $G \in \mathbb{C}^{n\times m}_r$ with $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$. Then the following statements are equivalent:*

(i) $A^{(2)}_{T,S}$ *exists;*

(ii) $\lim_{\lambda\to 0}G(AG+\lambda I)^{-1}$ *exists and* $\operatorname{rank}(AG) = \operatorname{rank}(G)$;

(iii) $\lim_{\lambda\to 0}(GA+\lambda I)^{-1}G$ *exists and* $\operatorname{rank}(GA) = \operatorname{rank}(G)$.

In this case,

$$A_{T,S}^{(2)} = \lim_{\lambda\to 0}(GA+\lambda I)^{-1}G = \lim_{\lambda\to 0}G(AG+\lambda I)^{-1}. \tag{2.1}$$

Proof. (i)⇔(ii) Assume that $A_{T,S}^{(2)}$ exists. By [14, Theorem 2.4], $\lim_{\lambda\to 0}G(AG+\lambda I)^{-1}$ exists. Since $G = A_{T,S}^{(2)}AG$, $\operatorname{rank}(AG) = \operatorname{rank}(G)$.

Conversely, assume that $\lim_{\lambda\to 0}G(AG+\lambda I)^{-1}$ exists and $\operatorname{rank}(AG) = \operatorname{rank}(G)$. So

$$\lim_{\lambda\to 0}(AG+\lambda I)^{-1}AG = \lim_{\lambda\to 0}AG(AG+\lambda I)^{-1} \tag{2.2}$$

exists. By [15, Theorem], $(AG)_g$ exists. So $(AG)^{(1,5)}$ exists, and then, by [13, Theorem 2], $A_{T,S}^{(2)}$ exists.

Similarly, we can show that (i)⇔(iii). Equation (2.1) comes from [14, equation (2.16)]. □

Lemma 2.2. *Let* $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ *and* $G = (g_{ij}) \in \mathbb{C}_t^{n\times m}$*. Then* $\operatorname{rank}(GA)_{.i}(g_{.j}) \leq t$*, where* $1 \leq i \leq n$*,* $1 \leq j \leq m$*, and* $\operatorname{rank}(AG)_{i.}(g_{j.}) \leq t$*, where* $1 \leq i \leq m$*,* $1 \leq j \leq n$*.*

Proof. Let $P_{ik}(a)$ be an $n\times n$ matrix with a in the (i,k) entry, 1 in all diagonal entries, and 0 in others. It is an elementary matrix and

$$
\begin{aligned}
(GA)_{.i}(g_{.j})\prod_{k\neq i}P_{ik}(-a_{jk}) &= \begin{array}{c}\begin{pmatrix} \sum_{k\neq j} g_{1k}a_{k1} & \cdots & g_{1j} & \cdots & \sum_{k\neq j} g_{1k}a_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k\neq j} g_{nk}a_{k1} & \cdots & g_{nj} & \cdots & \sum_{k\neq j} g_{nk}a_{kn}\end{pmatrix} \\ i\text{th} \end{array} \\
&= \begin{pmatrix} g_{11} & \cdots & g_{1j} & \cdots & g_{1m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{i1} & \cdots & g_{ij} & \cdots & g_{im} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n1} & \cdots & g_{nj} & \cdots & g_{nm} \end{pmatrix} \begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & 0 & \cdots & a_{mm} \end{pmatrix} & j\text{th}. \\ i\text{th} & \end{array}
\end{aligned} \tag{2.3}
$$

It follows from the invertibility of $P_{ik}(a), i\neq k$, that $\operatorname{rank}(GA)_{.i}(g_{.j}) \leq t$.

Analogously, the inequation $\operatorname{rank}(AG)_{i.}(g_{j.}) \leq t$ can be proved. So the proof is complete. □

Recall that if $f_A(\lambda) = \det(\lambda I + A) = \lambda^n + d_1\lambda^{n-1}\cdots + d_{n-1}\lambda + d_n$ is the characteristic polynomial of an $n\times n$ matrix—A over $\mathbb{C}$, then d_i is the sum of all $i\times i$ principal minors of A, where $i = 1,\ldots,n$ (see, e.g., [16]).

Theorem 2.3. *Let A, T, S, and G be the same as in Theorem 2.1. Write $G = (g_{ij})$. Suppose that the generalized inverse $A_{T,S}^{(2)}$ of A exists. Then $A_{T,S}^{(2)}$ can be represented as follows:*

$$A_{T,S}^{(2)} = \left(\frac{x_{ij}}{d_t(GA)}\right)_{n\times m}, \tag{2.4}$$

where

$$x_{ij} = \sum_{\beta\in Q_{t,n}\{i\}} \left|\left((GA)_{.i}(g_{.j})\right)_{\beta}^{\beta}\right|, \qquad d_t(GA) = \sum_{\beta\in Q_{t,n}} \left|(GA)_{\beta}^{\beta}\right|, \tag{2.5}$$

or

$$A_{T,S}^{(2)} = \left(\frac{y_{ij}}{d_t(AG)}\right)_{n\times m}, \tag{2.6}$$

where

$$y_{ij} = \sum_{\alpha\in Q_{t,m}\{j\}} \left|\left((AG)_{j.}(g_{i.})\right)_{\alpha}^{\alpha}\right|, \qquad d_t(AG) = \sum_{\alpha\in Q_{t,m}} \left|(AG)_{\alpha}^{\alpha}\right|. \tag{2.7}$$

Proof. We will only show the representation (2.5) since the proof of (2.7) is similar. If $-\lambda$ is not the eigenvalue of GA, then the matrix $\lambda I + GA$ is invertible, and

$$(\lambda I + GA)^{-1} = \frac{1}{\det(\lambda I + GA)} \begin{pmatrix} X_{11} & X_{21} & \cdots & X_{n1} \\ X_{12} & X_{22} & \cdots & X_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ X_{1n} & X_{2n} & \cdots & X_{nn} \end{pmatrix}, \tag{2.8}$$

where $X_{ij},\ i, j = 1, \ldots, n$, are cofactors of $\lambda I + GA$. It is easy to see that

$$\sum_{l=1}^{n} X_{il} g_{lj} = \det\left(\lambda I + GA\right)_{.i}\left(g_{.j}\right). \tag{2.9}$$

So, by (2.1),

$$A_{T,S}^{(2)} = \lim_{\lambda\to 0} \begin{pmatrix} \dfrac{\det\left(\lambda I + GA\right)_{.1}\left(g_{.1}\right)}{\det(\lambda I + GA)} & \cdots & \dfrac{\det\left(\lambda I + GA\right)_{.1}\left(g_{.m}\right)}{\det(\lambda I + GA)} \\ \vdots & \vdots & \vdots \\ \dfrac{\det\left(\lambda I + GA\right)_{.n}\left(g_{.1}\right)}{\det(\lambda I + GA)} & \cdots & \dfrac{\det\left(\lambda I + GA\right)_{.n}\left(g_{.m}\right)}{\det(\lambda I + GA)} \end{pmatrix}. \tag{2.10}$$

We have the characteristic polynomial of GA

$$f_{GA}(\lambda) = \det(\lambda I + GA) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \cdots + d_n, \tag{2.11}$$

where d_i $(1 \le i \le n)$ is a sum of $i \times i$ principal minors of GA. Since $\operatorname{rank}(GA) \le \operatorname{rank}(G) = t$, $d_n = d_{n-1} = \cdots = d_{t+1} = 0$ and

$$\det(\lambda I + GA) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \cdots + d_t\lambda^{n-t}. \tag{2.12}$$

Expanding $\det(\lambda I + GA)_{.i}(g_{.j})$, we have

$$\det(\lambda I + GA)_{.i}(g_{.j}) = x_1^{(ij)}\lambda^{n-1} + x_2^{(ij)}\lambda^{n-2} + \cdots + x_n^{(ij)}, \tag{2.13}$$

where $x_k^{(ij)} = \sum_{\beta\in Q_{k,n}\{i\}} |((GA)_{.i}(g_{.j}))_\beta^\beta|$, $1 \le k \le n$, for $1 \le i \le n$ and $1 \le j \le m$.

By Lemma 2.2, $\operatorname{rank}(GA)_{.i}(g_{.j}) \le t$ and so $|((GA)_{.i}(g_{.j}))_\beta^\beta| = 0$, $k > t$ and $\beta \in Q_{k,n}\{i\}$, for all i, j. Therefore, $x_k^{(ij)} = 0$, $k \le n$, for all i, j. Consequently,

$$\det(\lambda I + GA)_{.i}(g_{.j}) = x_1^{(ij)}\lambda^{n-1} + x_2^{(ij)}\lambda^{n-2} + \cdots + x_t^{(ij)}\lambda^{n-t}. \tag{2.14}$$

Substituting (2.12) and (2.14) into (2.10) yields

$$\begin{aligned} A_{T,S}^{(2)} &= \lim_{\lambda\to 0}\begin{pmatrix} \dfrac{x_1^{(11)}\lambda^{n-1} + \cdots + x_t^{(11)}\lambda^{n-t}}{\lambda^n + d_1\lambda^{n-1} + \cdots + d_t\lambda^{n-t}} & \cdots & \dfrac{x_1^{(1m)}\lambda^{n-1} + \cdots + x_t^{(1m)}\lambda^{n-t}}{\lambda^n + d_1\lambda^{n-1} + \cdots + d_t\lambda^{n-t}} \\ \vdots & \vdots & \vdots \\ \dfrac{x_1^{(n1)}\lambda^{n-1} + \cdots + x_t^{(n1)}\lambda^{n-t}}{\lambda^n + d_1\lambda^{n-1} + \cdots + d_t\lambda^{n-t}} & \cdots & \dfrac{x_1^{(nm)}\lambda^{n-1} + \cdots + x_t^{(nm)}\lambda^{n-t}}{\lambda^n + d_1\lambda^{n-1} + \cdots + d_t\lambda^{n-t}} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{x_t^{(11)}}{d_t} & \cdots & \dfrac{x_t^{(1m)}}{d_t} \\ \vdots & \vdots & \vdots \\ \dfrac{x_t^{(n1)}}{d_t} & \cdots & \dfrac{x_t^{(nm)}}{d_t} \end{pmatrix}. \end{aligned} \tag{2.15}$$

Substituting x_{ij} for $x_t^{(ij)}$ in the above equation, we reach (2.5). □

Remark 2.4. The proofs of Lemma 2.2 and Theorem 2.3 are based on the general techniques and methods obtained previously by [17], respectively.

Remark 2.5. (i) By using (2.5), we can obtain (2.17) in [12, Theorem 2.3]. In fact, $u = d_t(GA)$ and, by the Binet-Cauchy formula,

$$\begin{aligned} x_{ij} &= \sum_{\beta\in Q_{t,n}\{i\}} \left|((GA)_{.i}(g_{.j}))_\beta^\beta\right| = \sum_{\beta\in Q_{t,n}\{i\}} \sum_k g_{kj} \frac{\partial\left|(GA)_\beta^\beta\right|}{\partial s_{ki}} = \sum_{\beta\in Q_{t,n},\alpha\in Q_{t,m}} \sum_k g_{kj} \frac{\partial\left|G_\beta^\alpha\right|}{\partial g_{kj}} \frac{\partial\left|A_\alpha^\beta\right|}{\partial a_{ji}} \\ &= \sum_{\alpha\in Q_{t,m},\beta\in Q_{t,n}} \det\left(G_\beta^\alpha\right) \frac{\partial\left|A_\alpha^\beta\right|}{\partial a_{ji}}, \end{aligned} \tag{2.16}$$

where $s_{kj} = (GA)_{kj}$. Note that $\partial|A_{\alpha}^{\beta}|/\partial a_{ij} = 0$ if $i \notin \alpha$ or $j \notin \beta$. In addition, using the symbols in [13], we can rewrite (2.5) as [13, equation (13)] over $\mathbb{C}$.

(ii) This method is especially efficient when GA or AG is given (comparing with [12, Theorem 2]).

Observing the particular case from Theorem 2.3, $G = (g_{ij}) = N^{-1}A^{*}M$, where M and N are Hermitian positive definite matrices, we obtain the following corollary in which the symbols $g_{.j} := (g)_{.j}$ and $g_{i.} := (g)_{i.}$.

Corollary 2.6. *Let $A \in \mathbb{C}_r^{m\times n}$ and $G = N^{-1}A^{*}M$, where M and N are Hermitian positive definite matrices of order m and n, respectively, Then*

$$A_{MN}^{\dagger} = \left(\frac{x_{ij}}{d_r(GA)}\right)_{n\times m}, \tag{2.17}$$

where

$$x_{ij} = \sum_{\beta\in Q_{r,n}\{i\}} \left|\left((GA)_{.i}(g_{.j})\right)_{\beta}^{\beta}\right|, \qquad d_r(GA) = \sum_{\beta\in Q_{r,n}} \left|(GA)_{\beta}^{\beta}\right|, \tag{2.18}$$

or

$$A_{MN}^{\dagger} = \left(\frac{y_{ij}}{d_r(AG)}\right)_{n\times m}, \tag{2.19}$$

where

$$y_{ij} = \sum_{\alpha\in Q_{r,m}\{j\}} \left|\left((AG)_{j.}(g_{i.})\right)_{\alpha}^{\alpha}\right|, \qquad d_r(AG) = \sum_{\alpha\in Q_{r,m}} \left|(AG)_{\alpha}^{\alpha}\right|. \tag{2.20}$$

If M and N are identity matrices, then we can obtain the following result.

Corollary 2.7 (see [17, Theorem 2.2]). *The Moore-Penrose inverse $A^{\dagger}$ of $A = (a_{ij}) \in \mathbb{C}_r^{m\times n}$ can be represented as follows:*

$$A^{\dagger} = \left(\frac{x_{ij}}{d_r(A^{*}A)}\right)_{n\times m}, \tag{2.21}$$

where

$$x_{ij} = \sum_{\beta\in Q_{r,n}\{i\}} \left|\left((A^{*}A)_{.i}\left(a_{.j}^{*}\right)\right)_{\beta}^{\beta}\right|, \qquad d_r(A^{*}A) = \sum_{\beta\in Q_{r,n}} \left|(A^{*}A)_{\beta}^{\beta}\right|, \tag{2.22}$$

or

$$A^{\dagger} = \left(\frac{y_{ij}}{d_r(AA^{*})}\right)_{n\times m}, \tag{2.23}$$

where

$$y_{ij} = \sum_{\alpha \in Q_{r,m}\{j\}} \left| \left((AA^*)_{j.}(a^*_{i.}) \right)^{\alpha}_{\alpha} \right|, \qquad d_r(AA^*) = \sum_{\alpha \in Q_{r,m}} \left| (AA^*)^{\alpha}_{\alpha} \right|. \tag{2.24}$$

Note that $A_{d,W} = (WAW)^{(2)}_{\mathcal{R}((AW)^k A),\, \mathcal{N}((AW)^k A)}$. Therefore, when $G = (AW)^k A$ in Theorem 2.3, we have the following corollary.

Corollary 2.8. *Let* $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, *and* $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. *If* $\mathrm{rank}\,(AW)^k = t$, $\mathrm{rank}\,(WA)^k = r$, *and* $(AW)^k A = (c_{ij})_{m\times n}$, *then*

$$A_{d,W} = \left(\frac{x_{ij}}{d_t\left((AW)^{k+2}\right)} \right)_{m\times n}, \tag{2.25}$$

where

$$x_{ij} = \sum_{\beta \in Q_{t,m}\{i\}} \left| \left(\left((AW)^{k+2} \right)_{.i} (c_{.j}) \right)^{\beta}_{\beta} \right|, \qquad d_t\left((WA)^{k+2}\right) = \sum_{\beta \in Q_{t,m}} \left| \left((AW)^{k+2} \right)^{\beta}_{\beta} \right|, \tag{2.26}$$

or

$$A_{d,W} = \left(\frac{y_{ij}}{d_r\left((WA)^{k+2}\right)} \right)_{m\times n}, \tag{2.27}$$

where

$$y_{ij} = \sum_{\alpha \in Q_{r,n}\{j\}} \left| \left(\left((WA)^{k+2} \right)_{j.} (c_{i.}) \right)^{\alpha}_{\alpha} \right|, \qquad d_r\left((WA)^{k+2}\right) = \sum_{\alpha \in Q_{r,n}} \left| \left((WA)^{k+2} \right)^{\alpha}_{\alpha} \right|. \tag{2.28}$$

When $G = A^k$ with $k = \mathrm{Ind}(A)$ in Theorem 2.3, we have the following corollary.

Corollary 2.9 (see [17, Theorem 3.3]). *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}A = k$ *and* $\mathrm{rank}\,A^k = r$, *and* $A^k = (a^{(k)}_{ij})_{n\times n}$. *Then*

$$A_d = \left(\frac{x_{ij}}{d_r(A^{k+1})} \right)_{n\times n}, \tag{2.29}$$

where

$$d_{ij} = \sum_{\beta \in Q_{r,n}\{i\}} \left| \left(\left(A^{k+1}_{.i} \right) \left(a^{(k)}_{.j} \right) \right)^{\beta}_{\beta} \right|, \qquad d_r\left(A^{k+1}\right) = \sum_{\beta \in Q_{r,n}} \left| \left(A^{k+1} \right)^{\beta}_{\beta} \right|. \tag{2.30}$$

Finally, we turn our attention to the two projectors $A_{T,S}^{(2)}A$ and $AA_{T,S}^{(2)}$. The limiting expressions for $A_{T,S}^{(2)}$ in (2.1) bring us the following:

$$\begin{aligned}A_{T,S}^{(2)}A &= \lim_{\lambda\to 0}(GA+\lambda I)^{-1}GA,\\ AA_{T,S}^{(2)} &= \lim_{\lambda\to 0}AG(AG+\lambda I)^{-1}.\end{aligned} \tag{2.31}$$

Corollary 2.10. *Let A, T, S, and G be the same as in Theorem 2.1. Write $GA = (s_{ij})$ and $AG = (h_{ij})$. Suppose that $A_{T,S}^{(2)}$ exists. Then $A_{T,S}^{(2)}A$ of $AA_{T,S}^{(2)}$ can be represented as follows:*

$$A_{T,S}^{(2)}A = \left(\frac{x_{ij}}{d_t(GA)}\right)_{n\times n}, \tag{2.32}$$

where

$$\begin{gathered}x_{ij} = \sum_{\beta\in Q_{t,n}\{i\}}\left|\left((GA)_{.i}(s_{.j})\right)_{\beta}^{\beta}\right|, \qquad d_t(GA) = \sum_{\beta\in Q_{t,n}}\left|(GA)_{\beta}^{\beta}\right|,\\ AA_{T,S}^{(2)} = \left(\frac{y_{ij}}{d_t(AG)}\right)_{m\times m},\end{gathered} \tag{2.33}$$

where

$$y_{ij} = \sum_{\alpha\in Q_{t,m}\{j\}}\left|\left((AG)_{j.}(h_{i.})\right)_{\alpha}^{\alpha}\right|, \qquad d_t(AG) = \sum_{\alpha\in Q_{t,m}}\left|(AG)_{\alpha}^{\alpha}\right|. \tag{2.34}$$

3. Cramer Rules for the Solution of the Restricted Matrix Equation

The restricted matrix equation problem is mainly to find solution of a matrix equation or a system of matrix equations in a set of matrices which satisfy some constraint conditions. Such problems play an important role in applications in structural design, system identification, principal component analysis, exploration, remote sensing, biology, electricity, molecular spectroscopy, automatics control theory, vibration theory, finite elements, circuit theory, linear optimal, and so on. For example, the finite-element static model correction problem can be transformed to solve some constraint condition solution and its best approximation of the matrix equation $AX = B$. The undamped finite-element dynamic model correction problem can be attributed to solve some constraint condition solution and its best approximation of the matrix equation $A^TXA = B$. These motivate the gradual development of theory in respect of the solution to the restricted matrix equation in recent years (see [18–27]).

In this section, we consider the restricted matrix equation

$$AXB = D, \qquad \mathcal{R}(X)\subset T, \qquad \mathcal{N}(X)\supset \widetilde{S}, \tag{3.1}$$

where $A \in \mathbb{C}_r^{m\times n}$, $B \in \mathbb{C}_{\tilde{r}}^{p\times q}$, $D \in \mathbb{C}^{m\times q}$, $T \subset \mathbb{C}^n$, $S \subset \mathbb{C}^m$, $\tilde{T} \subset \mathbb{C}^q$ and $\tilde{S} \subset \mathbb{C}^p$, satisfy

$$\dim(T) = \dim\left(S^{\perp}\right) = t \le r, \qquad \dim\left(\tilde{T}\right) = \dim\left(\tilde{S}^{\perp}\right) = \tilde{t} \le \tilde{r}. \tag{3.2}$$

Assume that there exist matrices $G \in \mathbb{C}^{n\times m}$ and $\tilde{G} \in \mathbb{C}^{q\times p}$ satisfying

$$\mathcal{R}(G) = T, \qquad \mathcal{N}(G) = S, \qquad \mathcal{R}\left(\tilde{G}\right) = \tilde{T}, \qquad \mathcal{N}\left(\tilde{G}\right) = \tilde{S}. \tag{3.3}$$

If $A_{T,S}^{(2)}$ and $B_{\tilde{T},\tilde{S}}^{(2)}$ exist and $D \in \mathcal{R}(AG, \tilde{G}B)$, then the restricted matrix equation (3.1) has the unique solution

$$X = A_{T,S}^{(2)} D B_{\tilde{T},\tilde{S}}^{(2)} \tag{3.4}$$

(see [2, Theorem 3.3.3] for the proof).

In particular, when D is a vector b and $B = \tilde{G} = I_1$, the restricted matrix equation (3.1) becomes the restricted linear equation

$$Ax = b, \quad x \in T. \tag{3.5}$$

If $b \in A\mathcal{R}(G)$, then $x = A_{T,S}^{(2)} b$ is the unique solution of the restricted linear equation (3.5) (see also [10, Theorem 2.1]).

Theorem 3.1. *Given $A, B, D = (d_{ij}), G = (g_{ij}), \tilde{G} = (\tilde{g}_{ij}), T, S, \tilde{T}$, and $\tilde{S}$ as above. If $A_{T,S}^{(2)}$ and $B_{\tilde{T},\tilde{S}}^{(2)}$ exist and $D \in \mathcal{R}(AG, \tilde{G}B)$, then $X = A_{T,S}^{(2)} D B_{\tilde{T},\tilde{S}}^{(2)}$ is the unique solution of the restricted matrix equation* (3.1) *and it can be represented as*

$$x_{ij} = \frac{\sum_{k=1}^{m} \sum_{\beta\in Q_{t,n}\{i\},\alpha\in Q_{\tilde{t},p}\{j\}} \left|\left((GA)_{.i}(g_{.k})\right)_{\beta}^{\beta}\right| \left|\left(\left(B\tilde{G}\right)_{j.}\left(\tilde{f}_{k.}\right)\right)_{\alpha}^{\alpha}\right|}{d_t(GA)\ d_{\tilde{t}}\left(B\tilde{G}\right)} \tag{3.6}$$

or

$$x_{ij} = \frac{\sum_{k=1}^{q} \sum_{\beta\in Q_{t,n}\{i\},\alpha\in Q_{\tilde{t},p}\{j\}} \left|\left((GA)_{.i}(f_{.k})\right)_{\beta}^{\beta}\right| \left|\left(\left(B\tilde{G}\right)_{j.}(\tilde{g}_{k.})\right)_{\alpha}^{\alpha}\right|}{d_t(GA)\ d_{\tilde{t}}\left(B\tilde{G}\right)}, \tag{3.7}$$

where $\tilde{f}_{k.} = d_{k.}\tilde{G}$ and $f_{.k} = Gd_{.k}$, $i = 1, \ldots, n$, and $j = 1, \ldots, p$.

Proof. By the argument above, we have $X = A^{(2)}_{T,S} D B^{(2)}_{\tilde{T},\tilde{S}}$ is the unique solution of the restricted matrix equation (3.1). Setting $Y = DB^{(2)}_{\tilde{T},\tilde{S}}$ and using (2.7), we get that

$$\begin{aligned} y_{kj} &= \sum_{h=1}^{q} d_{kh} \left(B^{(2)}_{\tilde{T},\tilde{S}}\right)_{hj} = \sum_{h=1}^{q} d_{kh} \frac{\sum_{\alpha \in Q_{\tilde{t},p}\{j\}} \left| \left(\left(B\widetilde{G}\right)_{j.} (\tilde{g}_{h.}) \right)^{\alpha}_{\alpha} \right|}{d_{\tilde{t}}\left(B\widetilde{G}\right)} \\ &= \frac{\sum_{\alpha \in Q_{\tilde{t},p}\{j\}} \left| \left(\left(B\widetilde{G}\right)_{j.} \left(\sum_{h=1}^{q} d_{kh}\tilde{g}_{h.}\right) \right)^{\alpha}_{\alpha} \right|}{d_{\tilde{t}}\left(B\widetilde{G}\right)} = \frac{\sum_{\alpha \in Q_{\tilde{t},p}\{j\}} \left| \left(\left(B\widetilde{G}\right)_{j.} \left(\tilde{f}_{k.}\right) \right)^{\alpha}_{\alpha} \right|}{d_{\tilde{t}}\left(B\widetilde{G}\right)}, \end{aligned} \tag{3.8}$$

where $\tilde{f}_{k.} = d_{k.}\widetilde{G}$. Since $X = A^{(2)}_{T,S} Y$, by (2.5),

$$x_{ij} = \sum_{k=1}^{m} \left(A^{(2)}_{T,S}\right)_{ik} y_{kj} = \sum_{k=1}^{m} \frac{\sum_{\beta \in Q_{t,n}\{i\}} \left| \left((GA)_{.i} (g_{.k})\right)^{\beta}_{\beta} \right|}{d_t(GA)} \frac{\sum_{\alpha \in Q_{\tilde{t},p}\{j\}} \left| \left(\left(B\widetilde{G}\right)_{j.} \left(\tilde{f}_{k.}\right) \right)^{\alpha}_{\alpha} \right|}{d_{\tilde{t}}\left(B\widetilde{G}\right)}. \tag{3.9}$$

Hence, we have (3.6).

We can obtain (3.7) in the same way. □

In particular, when D is a vector b and $B = \widetilde{G} = I_1$ in the above theorem, we have the following result from (3.7).

Theorem 3.2. *Given A, G, T, and S as above. If $b \in A\mathcal{R}(G)$, then $x = A^{(2)}_{T,S} b$ is the unique solution of the restricted linear equation $Ax = b$, $x \in T$, and it can be represented as*

$$x_i = \frac{\sum_{\beta \in Q_{t,n}\{i\}} \left| \left((GA)_{.i} (f)\right)^{\beta}_{\beta} \right|}{d_t(GA)}, \quad j = 1, \ldots, n, \tag{3.10}$$

where $f = Gb$.

Remark 3.3. Using the symbols in [13], we can rewrite (3.10) as [13, equation (27)].

4. Example

Let

$$A = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 & -1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \tag{4.1}$$

Obviously, rank $A = 3$, $\dim AT = \dim T = 2$, and

$$T = \mathcal{R}(G) \subset \mathbb{C}^4, \qquad S = \mathcal{N}(G) \subset \mathbb{C}^5, \qquad \widetilde{T} = \mathcal{R}\left(\widetilde{G}\right) \subset \mathbb{C}^2, \qquad \widetilde{S} = \mathcal{N}\left(\widetilde{G}\right) \subset \mathbb{C}^3. \tag{4.2}$$

It is easy to verify that $AT \oplus S = \mathbb{C}^5$ and $B\widetilde{T} \oplus \widetilde{S} = \mathbb{C}^3$. Thus, $A^{(2)}_{T,S}$ and $B^{(2)}_{\widetilde{T},\widetilde{S}}$ exist by Lemma 1.1.

Now consider the restricted matrix equation

$$AXB = D, \qquad \mathcal{R}(X) \subset T, \qquad \mathcal{N}(X) \supset \widetilde{S}. \tag{4.3}$$

Clearly,

$$AG = \begin{pmatrix} 1 & 3 & 4 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{G}B = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \tag{4.4}$$

and it is easy to verify that $\mathcal{R}(D) \subset \mathcal{R}(AG)$ and $\mathcal{N}(D) \supset \mathcal{N}(\widetilde{G}B)$ hold.

Note that $\mathcal{R}(D) \subset \mathcal{R}(AG)$ and $\mathcal{N}(D) \supset \mathcal{N}(\widetilde{G}B)$ if and only if $D \in \mathcal{R}(AG, \widetilde{G}B)$. So, by Theorem 3.1, the unique solution of (4.3) exists.

Table 1

y_{ik}	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$k = 1$	4	−2	0	0
$k = 2$	4	0	0	0

Table 2

z_{kj}	$j = 1$	$j = 2$	$j = 3$
$k = 1$	0	−2	0
$k = 2$	−2	4	0

Computing

$$GA = \begin{pmatrix} 1 & 0 & 9 & 5 \\ 0 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B\widetilde{G} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

$$d_2(GA) = \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 9 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 5 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 4 & 4 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 4,$$

$$d_2\left(B\widetilde{G}\right) = \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} = -2, \tag{4.5}$$

$$\widetilde{f} = \begin{pmatrix} 1 & -1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and setting $y_{ik} = \sum_{\beta \in Q_{t,n}\{i\}} |((GA)_{.i}(\widetilde{f}_{.k}))_{\beta}^{\beta}|$, we have Table 1.

Similarly, setting $z_{kj} = \sum_{\alpha \in Q_{\widetilde{t},p}\{j\}} |((B\widetilde{G})_{j.}(\widetilde{g}_{k.}))_{\alpha}^{\alpha}|$, we have Table 2.
So, by (3.7), we have

$$X = \left(x_{ij}\right) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -\dfrac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

Acknowledgments

The authors would like to thank the referees for their helpful comments and suggestions. This work was supported by the National Natural Science Foundation of China 11061005, the Ministry of Education Science and Technology Key Project under Grant 210164, and Grants HCIC201103 of Guangxi Key Laboratory of Hybrid Computational and IC Design Analysis Open Fund.

References

[1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverse Theory and Application*, Springer-Verlag, New York, NY, USA, 2nd edition, 2003.
[2] G. Wang, Y. Wei, and S. Qiao, *Generalized Inverse: Theory and Computations*, Science Press, Beijing, China, 2004.
[3] M. P. Drazin, "Pseudo-inverses in associative rings and semigroups," *The American Mathematical Monthly*, vol. 65, pp. 506–514, 1958.
[4] P. Patricio and R. Puystjens, "Drazin-moore-penrose invertibility in rings," *Linear Algebra and its Applications*, vol. 389, pp. 159–173, 2004.
[5] D. Huang, "Group inverses and Drazin inverses over Banach algebras," *Integral Equations and Operator Theory*, vol. 17, no. 1, pp. 54–67, 1993.
[6] R. Harte and M. Mbekhta, "On generalized inverses in C*-algebras," *Studia Mathematica*, vol. 103, no. 1, pp. 71–77, 1992.
[7] P. S. Stanimirović and D. S. Cvetković-Ilić, "Successive matrix squaring algorithm for computing outer inverses," *Applied Mathematics and Computation*, vol. 203, no. 1, pp. 19–29, 2008.
[8] M. Miladinović, S. Miljković, and P. Stanimirović, "Modified SMS method for computing outer inverses of Toeplitz matrices," *Applied Mathematics and Computation*, vol. 218, no. 7, pp. 3131–3143, 2011.
[9] R. E. Cline and T. N. E. Greville, "A Drazin inverse for rectangular matrices," *Linear Algebra and its Applications*, vol. 29, pp. 54–62, 1980.
[10] Y. L. Chen, "A Cramer rule for solution of the general restricted linear equation," *Linear and Multilinear Algebra*, vol. 34, no. 2, pp. 177–186, 1993.
[11] X. Mary, "On generalized inverses and Green's relations," *Linear Algebra and its Applications*, vol. 434, no. 8, pp. 1836–1844, 2011.
[12] Y. Yu and G. Wang, "On the generalized inverse $A_{T,S}^{(2)}$ over integer domains," *The Australian Journal of Mathematical Analysis and Applications*, vol. 4, no. 1, article 16, pp. 1–20, 2007.
[13] Y. Yu and Y. Wei, "Determinantal representation of the generalized inverse $A_{T,S}^{(2)}$ over integral domains and its applications," *Linear and Multilinear Algebra*, vol. 57, no. 6, pp. 547–559, 2009.
[14] Y. Wei, "A characterization and representation of the generalized inverse $A_{T,S}^{(2)}$ and its applications," *Linear Algebra and its Applications*, vol. 280, no. 2-3, pp. 87–96, 1998.
[15] A. Ben-Israel, "On matrices of index zero or one," *Society for Industrial and Applied Mathematics Journal on Applied Mathematics*, vol. 17, pp. 1118–1121, 1969.
[16] R. A. Horn and C. R. Johnson, *The Matrix analysis*, Cambridge University Press, New York, NY, USA, 1985.
[17] I. I. Kyrchei, "Analogs of the adjoint matrix for generalized inverses and corresponding Cramer rules," *Linear and Multilinear Algebra*, vol. 56, no. 4, pp. 453–469, 2008.
[18] M. Baruch, "Optimal correction of mass and stiffness matrices usingmeasured modes," *American Institute of Aeronautics and Astronautics Journal*, vol. 20, no. 11, pp. 1623–1626, 1982.
[19] A. Bjierhammer, "Rectangular reciprocal matrices with special reference to geodetic calculations," *Kungliga tekniska hogskolan Handl Stackholm*, vol. 45, pp. 1–86, 1951.
[20] M. Bixon and J. Jortner, "Intramolecular radiationless transitions," *Journal of Chemical and Physics*, vol. 48, pp. 715–726, 1968.
[21] J. W. Gadzuk, "Localized vibrational modes in Fermi liquids. General theory," *Physical Review B*, vol. 24, no. 4, pp. 1651–1663, 1981.
[22] L. Datta and S. D. Morgera, "On the reducibility of centrosymmetric matrices applications in engineering problems," *Circuits, Systems, and Signal Processing*, vol. 8, no. 1, pp. 71–96, 1989.
[23] Y. M. Ram and G. G. L. Gladwell, "Contructing a finite elemnt model of a vibratory rod from eigendata," *Journal of Sound and Vibration*, vol. 169, no. 2, pp. 229–237, 1994.
[24] A. Jameson and E. Kreindler, "Inverse problem of linear optimal control," *Society for Industrial and Applied Mathematics Journal on Control and Optimization*, vol. 11, pp. 1–19, 1973.
[25] G. W. Stagg and A. H. El-Abiad, *Computer Methodes in Power System Analysis*, MeGrawHill, New York, NY, USA, 1968.
[26] H. C. Chen, "The SAS domain decomposition method for structural anlaysis," CSRD Technical Report 754, Center for Supercomputing Research and Development. University of Illinois, Urbana, Ill, USA, 1988.
[27] F. S. Wei, "Analytical dynamic model improvement using vibration test data," *American Institute of Aeronautics and Astronautics Journal*, vol. 28, no. 1, pp. 175–177, 1990.