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Research Article

A New Jacobian-Like Method for the Polyhedral Cone-Constrained Eigenvalue Problem

Guo Sun

School of Management Science, Qufu Normal University, Shandong, Rizhao 276800, China

Correspondence should be addressed to Guo Sun, qrsunguo@163.com

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The eigenvalue problem over a polyhedral cone is studied in this paper. Based on the F-B NCP function, we reformulate this problem as a system of equations and propose a Jacobian-like method. The global convergence and local quadratic convergence of the proposed method are established under suitable assumptions. Preliminary numerical experiments for a special polyhedral cone are reported in this paper to show the validity of the proposed method.

1. Introduction

In this paper, we consider the cone-constrained eigenvalue problem of finding a nonzero vector $x \in R^n$ and $\lambda \in R$ such that

$$\mathcal{K} \ni x \perp (Ax - \lambda Bx) \in \mathcal{K}^+, \tag{1.1}$$

with $\mathcal{K}$ being a closed convex cone, $\mathcal{K}^+$ being its dual in the Euclidean space, and (A, B) being a pair of order n matrices such that B is symmetric and positive definite. If (x, λ) is a solution of the problem, λ is called a $\mathcal{K}$-eigenvalue of (A, B) and x is called a $\mathcal{K}$-eigenvector of (A, B) corresponding to λ.

For this problem, we consider the polyhedral cone $\mathcal{K}$ in R^n as

$$\mathcal{K} = \{v \in R^n \mid Uv \geqslant 0\}, \tag{1.2}$$

where $U \in R^{s\times n}$. It is easy to verify that its dual cone $\mathcal{K}^+$ assumes the following form:

$$\mathcal{K}^+ = \left\{u \in R^n \mid u = U^\top \xi,\ \xi \in R^s_+\right\}, \tag{1.3}$$

where R^s_+ stands for the nonnegative orthant of R^s.

Throughout this paper, the cone-constrained eigenvalue problem (1.1) with $\mathcal{K}$ being of the form (1.2) is referred, for simplicity, as the polyhedral cone-constrained eigenvalue problem in this paper.

When $\mathcal{K} = R^n_+$, the cone-constrained eigenvalue problem reduces to the Pareto eigenvalue problem [1], which has the following form:

$$x \geq 0, \quad Ax - \lambda Bx \geq 0, \quad \langle x, Ax - \lambda Bx\rangle = 0. \tag{1.4}$$

If A is symmetric and B is symmetric and positive definite in addition, the problem is reduced to a generalized Rayleigh quotient problem [2].

The polyhedral cone-constrained eigenvalue problem is an important problem arising from the study of static equilibrium states of finite-dimensional mechanical systems with unilateral frictional contact [3]. Due to wide applications of the cone-constrained eigenvalue problem, the theoretical analysis and numerical solution methods are established [4–6]. Recently, Adly and Seeger have transformed the cone-constrained eigenvalue problem into a system of equations [1]; therefore, it is possible to apply the semismooth and smooth Newton method to solve the problem. Motivated by the work in [7, 8] for nonlinear complementarity problems, we design a smoothing Jacobian-like Method to solve the polyhedral cone-constrained eigenvalue problem. Compared with the smoothing method given in [7], the smoothing parameter is adjusted in a relaxed manner. The given numerical experiments show the efficiency of the proposed method.

The remainder of this paper is organized as follows. In Section 2, the reformulation of the polyhedral cone-constrained eigenvalue problem is presented, some basic well-known conclusions are reviewed, and the nonsingularity condition of the Jacobian matrix of the system of equations is established. In Section 3, we develop a new Jacobian-like method for the polyhedral cone-constrained eigenvalue problem, and the global convergence and local quadratic convergence of the proposed algorithm are discussed. Numerical experiments are presented in Section 4. Finally, the conclusions are given in Section 5.

To finish this section, we give some notations used in this paper. For a continuously differentiable mapping $G : R^n \to R^m$, let $G'(x) \in R^{m\times n}$ denote the Jacobian of G at a point $x \in R^n$, and let $\nabla G(x)$ denote its transposition. For a locally Lipschitz continuous function $G : R^n \to R^m$, its generalized Jacobian in the sense of Clarke [9] is defined as

$$\partial G(x) = \operatorname{conv}\left\{V \in R^{m\times n} \mid V = \lim_{x_k \to x} G'(x_k),\ G(x) \text{ is differentiable at } x_k\right\}, \tag{1.5}$$

where $\operatorname{conv}(A)$ is the convex hull of a set A. Proposition 2.6.2 (e) in [9] provides an overestimation of the generalized Jacobian of mapping G as follows:

$$\partial G(x)^\top \subseteq \partial G_1(x) \times \cdots \times \partial G_m(x), \tag{1.6}$$

where the right-hand side denotes the set of matrices in $R^{n\times m}$ whose i th column is the generalized gradient of the ith component function G_i. Traditionally, the right-hand side is denoted as

$$\partial_C G(x)^\top = \partial G_1(x) \times \cdots \times \partial G_m(x) \tag{1.7}$$

and $\partial_C G(x)$ is called the C-subdifferential of G at x [10].

The inner product of vectors $x, y \in R^n$ is denoted by $x^\top y$. We denote by $\|x\|$ the Euclidean norm of x and denote by $\|\cdot\|_B$ the ellipsoidal norm with respect to symmetric and positive definite matrix B. Similarly, $\|A\|$ denotes the spectral norm of a matrix $A \in R^{n\times n}$, which is the induced matrix norm of the Euclidean vector norm.

If $A \in R^{n\times n}$ is any given matrix and $\Xi \subseteq R^{n\times n}$ is a nonempty set of matrices, we denote the distance from A to Ξ by $\mathrm{dist}(A, \Xi) = \inf_{B\in\Xi}\|A - B\|$.

For a vector $a \in R^n$, $D_a = \mathrm{diag}(a)$ denotes the diagonal matrix whose ith diagonal element is a_i. R^n_+ (R^n_{++}) stands for the nonnegative (positive) orthant of R^n.

2. Preliminaries

First, using a method similar to the one of [11], we transform the polyhedral cone-constrained eigenvalue problem as a system of equations via the well-known Fischer-Burmeister (F-B) function [12] from R^2 to R:

$$\phi_{\mathrm{FB}}(a,b) = \sqrt{a^2 + b^2} - a - b \quad \text{for } a,b \in R. \tag{2.1}$$

For the polyhedral cone-constrained eigenvalue problem, if (x, λ) is a solution of the problem, then (tx, λ) is also a solution of the problem for any $t > 0$. Hence, to solve the problem, we only need to find a solution with x being a unit vector in the B-norm. In this sense, (x, λ) is a solution of the problem if and only if the following statements hold:

$$\begin{gathered}
x \in \mathcal{K} = \{v \in R^n \mid Uv \geq 0\}, \\
Ax - \lambda Bx \in \mathcal{K}^+ = \left\{u \in R^n \mid u = U^\top \xi,\ \xi \in R^s_+\right\}, \\
x^\top (Ax - \lambda Bx) = 0, \\
\frac{1}{2}\|x\|_B^2 - 1 = 0.
\end{gathered} \tag{2.2}$$

By simple reformulation, the above form is equivalent to the following system:

$$\begin{aligned} &Ux \geq 0, \\ &\xi \geq 0, \\ &\xi^{\top} U x = 0, \\ &Ax - \lambda Bx = U^{\top}\xi, \\ &\frac{1}{2}\|x\|_B^2 - 1 = 0. \end{aligned} \tag{2.3}$$

Using the F-B merit function, we can express the problem (1.1) as the following system of

$$\begin{aligned} &\Phi(Ux, \xi) = 0, \\ &Ax - \lambda Bx - U^{\top}\xi = 0, \\ &\frac{1}{2}\|x\|_B^2 - 1 = 0, \end{aligned} \tag{2.4}$$

where

$$\Phi(Ux, \xi) = \begin{pmatrix} \phi_{FB}((Ux)_1, \xi_1) \\ \vdots \\ \phi_{FB}((Ux)_s, \xi_s) \end{pmatrix}. \tag{2.5}$$

Denote $z = (x, \xi, \lambda)$ and a vector-valued function $H : R^{n+s+1} \to R^{n+s+1}$ as follows:

$$H(z) = H(x, \xi, \lambda) = \begin{pmatrix} \Phi(Ux, \xi) \\ Ax - \lambda Bx - U^{\top}\xi \\ \frac{1}{2}x^{\top}Bx - 1 \end{pmatrix}. \tag{2.6}$$

Then the polyhedral cone-constrained eigenvalue problem is equivalent to the equation $H(z) = 0$.

Due to the nonsmoothness of the equation above, we approximate it by the following function [13]:

$$\phi_{\mathrm{FB}}^{\mu}(a, b) = \sqrt{a^2 + b^2 + \mu^2} - a - b \quad \text{for } \mu \in R_{++},\ a, b \in R. \tag{2.7}$$

Correspondingly, the nonsmooth functions $\Phi(Ux,\xi)$ and $H(z)$ can be approximated via the following smooth functions $\Phi_\mu(Ux,\xi)$ and $H_\mu(z)$, respectively,

$$\Phi_\mu(Ux,\xi) = \begin{pmatrix} \phi^\mu_{\mathrm{FB}}((Ux)_1,\xi_1) \\ \vdots \\ \phi^\mu_{\mathrm{FB}}((Ux)_s,\xi_s) \end{pmatrix}, \tag{2.8}$$

$$H_\mu(z) = \begin{pmatrix} \Phi_\mu(Ux,\xi) \\ Ax - \lambda Bx - U^\top \xi \\ \dfrac{1}{2}x^\top Bx - 1 \end{pmatrix}. \tag{2.9}$$

Obviously, $H_\mu(z) \to H(z)$ as $\mu \to 0^+$. Thus, the original problem is equivalent to the following equation:

$$H_\mu(z) = 0. \tag{2.10}$$

In the following, we will summarize some properties of function $H_\mu(z)$, which will be used in the subsequent analysis. The first result can be proved similarly to Proposition 2.3 in [8].

Proposition 2.1. *The function Φ_μ defined by* (2.8) *satisfies the inequality*

$$\left\|\Phi_{\mu_1}(Ux,\xi) - \Phi_{\mu_2}(Ux,\xi)\right\| \le \kappa\left\|\mu_1 - \mu_2\right\| \tag{2.11}$$

for all $x \in R^n$, $\xi \in R^s$, and μ_1, $\mu_2 > 0$, where $\kappa = \sqrt{s}$. In particular,

$$\left\|\Phi_\mu(Ux,\xi) - \Phi(Ux,\xi)\right\| \le \kappa\mu \tag{2.12}$$

for all $x \in R^n$, $\xi \in R^s$ and $\mu > 0$.

As an immediate consequence of Proposition 2.1, we obtain the following corollary.

Corollary 2.2. *The function H_μ defined by* (2.9) *satisfies the inequality*

$$\left\|H_{\mu_1}(z) - H_{\mu_2}(z)\right\| = \left\|\begin{pmatrix} \Phi_{\mu_1}(Ux,\xi) - \Phi_{\mu_2}(Ux,\xi) \\ 0 \\ 0 \end{pmatrix}\right\| \le \kappa\left|\mu_1 - \mu_2\right| \tag{2.13}$$

for all $z = (x,\xi,\lambda) \in R^{n+s+1}$ and μ_1, $\mu_2 > 0$, where $\kappa = \sqrt{s}$. In particular,

$$\left\|H_\mu(z) - H(z)\right\| \le \kappa\mu \tag{2.14}$$

for all $z = (x,\xi,\lambda) \in R^{n+s+1}$ and $\mu > 0$.

By direct calculation, one has the following conclusion.

Proposition 2.3. *For an arbitrary $z \in R^{n+s+1}$, it holds that*

$$H'_\mu(z) = \begin{pmatrix} \overline{D}_a U & \overline{D}_b & 0 \\ A - \lambda B & -U^\top & -Bx \\ (Bx)^\top & 0 & 0 \end{pmatrix}, \tag{2.15}$$

where $\overline{D}_a = \operatorname{diag}(\overline{a}_1, \overline{a}_2, \dots, \overline{a}_s)$, and $\overline{D}_b = \operatorname{diag}(\overline{b}_1, \overline{b}_2, \dots, \overline{b}_s)$ such that

$$\overline{a}_i = \frac{(Ux)_i}{\sqrt{(Ux)_i^2 + {\xi_i}^2 + \mu^2}} - 1, \qquad \overline{b}_i = \frac{\xi_i}{\sqrt{(Ux)_i^2 + {\xi_i}^2 + \mu^2}} - 1, \tag{2.16}$$

for $\mu > 0$.

Obviously,

$$-2 < \overline{a}_i < 0, \quad -2 < \overline{b}_i < 0, \quad \forall i = 1, 2, \dots, s. \tag{2.17}$$

Hence, both $\overline{D}_a$ and $\overline{D}_b$ are negative definite diagonal matrices.

Next, we will characterize differential properties of the nonsmooth mapping $H(z)$ defined in (2.6). For the mapping $H(z)$, we can make an overestimate of its generalized Jacobian in the sense of Clarke [9] by the definition of the C-subdifferential in the following Proposition.

Proposition 2.4. *For an arbitrary $z \in R^{n+s+1}$, it holds that*

$$\partial_C H(z) = \begin{pmatrix} D_a U & D_b & 0 \\ A - \lambda B & -U^\top & -Bx \\ (Bx)^\top & 0 & 0 \end{pmatrix}, \tag{2.18}$$

where $D_a = \operatorname{diag}(a_1, a_2, \dots, a_s)$, and $D_b = \operatorname{diag}(b_1, b_2, \dots, b_s)$ such that

$$a_i = \frac{(Ux)_i}{\sqrt{(Ux)_i^2 + \xi_i^2}} - 1, \quad b_i = \frac{\xi_i}{\sqrt{(Ux)_i^2 + {\xi_i}^2}} - 1 \tag{2.19}$$

if $(Ux)_i^2 + {\xi_i}^2 \neq 0$, and $a_i = \tau_i - 1$, $b_i = \eta_i - 1$ for any $(\tau_i, \eta_i) \in R^2$ satisfying $\tau_i^2 + \eta_i^2 \leq 1$ for the case that $(Ux)_i^2 + \xi_i^2 = 0$.

In order to guarantee local fast convergence of our proposed algorithm, we need to consider the compatibility of the Jacobian matrix $H'_\mu(z)$ in the following result.

Lemma 2.5. *For any $z \in R^{n+s+1}$, it holds that*

$$\lim_{\mu \downarrow 0} \operatorname{dist}\left(H'_\mu(z), \partial_C H(z)\right) = 0. \tag{2.20}$$

Proof. Since the difference only lies in the first s columns between $\nabla H_\mu(z)$ and $\partial_C H(z)^\top$, it suffices to consider the distance between the first s columns of $\nabla H_\mu(z)$ and $\partial_C H(z)^\top$.

Define

$$\beta(z) = \{i \mid [Ux]_i = \xi_i = 0,\ i = 1, 2, \ldots, s\}. \tag{2.21}$$

Let $\nabla H_{\mu,i}(z)$ be the ith column of matrix $\nabla H_\mu(z)$ and $U_i^\top$ the ith column of matrix $U^\top$. Then

$$\lim_{\mu \downarrow 0} \nabla H_{\mu,i}(z) = \begin{cases} \begin{pmatrix} U_i^\top a_i \\ e_i b_i \\ 0 \end{pmatrix}, & i \in \{1, 2, \ldots, s\} \setminus \beta(z), \\ \begin{pmatrix} -U_i^\top \\ -e_i \\ 0 \end{pmatrix}, & i \in \beta(z), \end{cases} \tag{2.22}$$

where $e_i \in R^s$ is the ith unit vector.

The assertion follows from Proposition 2.4 with $(\tau_i, \eta_i) = (0, 0)$ for $i \in \beta(z)$. □

To establish global convergence of our algorithm, it is necessary to study the condition under which $H'_\mu(z)$ is nonsingular for any $\mu > 0$.

Theorem 2.6. *For $z = (x, \xi, \lambda) \in R^n \times R^s \times R$, if x is a nonzero vector, and $(A - \lambda B)$ is a positive definite matrix, then $H'_\mu(z)$ is nonsingular for any $\mu > 0$.*

Proof. Noticing that the nonsingularity of the matrix $\nabla H'_\mu(z)$ defined in Proposition 2.3 is equivalent to showing that the only solution of the following system:

$$\begin{pmatrix} \overline{D}_a U & \overline{D}_b & 0 \\ A - \lambda B & -U^\top & -Bx \\ (Bx)^\top & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0 \tag{2.23}$$

is the zero vector. This system can be rewritten as

$$\overline{D}_a U p + \overline{D}_b q = 0, \tag{2.24}$$

$$(A - \lambda B)p - U^\top q - Bxr = 0, \tag{2.25}$$

$$(Bx)^\top p = 0. \tag{2.26}$$

Recalling that $\overline{D}_b$ defined in Proposition 2.3 is a negative definite matrix, we have and $\overline{D}_b$ is nonsingular. From (2.24), it holds that

$$q = -\overline{D}_b^{-1}\overline{D}_b U p. \tag{2.27}$$

By (2.25) and (2.27), we obtain that

$$\left[(A - \lambda B) + U^\top \overline{D}_b^{-1}\overline{D}_a U\right] p - Bxr = 0. \tag{2.28}$$

Premultiplying the equation above by $p^\top$ yields that

$$p^\top \left[(A - \lambda B) + U^\top \overline{D}_b^{-1}\overline{D}_a U\right] p - p^\top Bxr = 0. \tag{2.29}$$

Using (2.26), then

$$p^T \left[(A - \lambda B) + U^\top \overline{D}_b^{-1}\overline{D}_a U\right] p = 0. \tag{2.30}$$

From the fact that both $\overline{D}_a$ and $\overline{D}_b$ are negative definite matrices, it is not difficult to prove that the matrix $U^\top \overline{D}_b^{-1}\overline{D}_a U$ is positive semidefinite. Since $(A-\lambda B)$ is a positive definite matrix, $(A - \lambda B) + U^\top \overline{D}_b^{-1}\overline{D}_a U$ must be a positive definite one. This implies, by (2.27) and (2.30),

$$p = 0, \qquad q = 0. \tag{2.31}$$

Now, it follows from (2.25) that

$$Bxr = 0. \tag{2.32}$$

Pre-multiplying (2.25) by $x^\top$ yields

$$x^\top Bxr = 0. \tag{2.33}$$

From the positive definiteness of B and the fact that x is nonzero, we assert that $r = 0$. Hence, $H'_\mu(z)$ is nonsingular and the desired result follows. □

3. Algorithm and Convergence

Based on the analysis in Section 2, the polyhedral cone-constrained problem is equivalently transformed into the system of smooth equations (2.10). Motivated by the work in [7, 8] for nonlinear complementarity problem, we propose a new smoothing Jacobian-like method for the polyhedral cone-constrained eigenvalue problem. For convenience of description, we denote

$$\psi(z) = \frac{1}{2}H(z)^{\top}H(z),$$
$$\psi_{\mu}(z) = \frac{1}{2}H_{\mu}(z)^{\top}H_{\mu}(z). \tag{3.1}$$

At each iteration of the new proposed method, the searching direction Δz is computed by the Newton equation of the mapping $\psi_{\mu}(z)$. Meanwhile, μ is viewed as a smoothing parameter, which needs to be updated according to certain rule. Compared with the smoothing method given in [7], μ is adjusted in a relaxed manner. The following is the formal description of our designed method.

Algorithm 3.1. Step 0. Take tolerance $\epsilon \geq 0$, choose constants $z_0 \in R^{n+s+1}$, $\eta \in (0,1)$, $\alpha \in (0,1/2)$, $\sigma \in (0, 1-\alpha)$, $\gamma > 0$, $\kappa = \sqrt{s}$, $\mu_0 = (\alpha/2\kappa)\|H(z_0)\|$. Set $k = 0$.

Step 1. Solve the linear system

$$H'_{\mu_k}(z_k)\Delta z_k = -H(z_k) \tag{3.2}$$

to obtain $\Delta z_k \in R^{s+n+1}$ and find the smallest $m_k \in \{0,1,2,\ldots\}$ such that

$$\psi_{\mu_k}\left(z_k + \eta^{m_k}\Delta z_k\right) - \psi_{\mu_k}(z_k) \leq -2\sigma\eta^{m_k}\psi(z_k). \tag{3.3}$$

Let $t_k = \eta^{m_k}$, $z_{k+1} = z_k + t_k\Delta z_k$.

Step 2. If $H(z_{k+1}) \leq \epsilon$, stop; otherwise, if

$$\|H(z_{k+1})\| \leq \alpha\|H(z_k)\| + \frac{\kappa}{\alpha}\mu_k, \tag{3.4}$$

then take μ_{k+1} such that

$$0 < \mu_{k+1} \leq \min\left\{\frac{\mu_k}{2}, \frac{\alpha}{2\kappa}\|H(z_{k+1})\|\right\},$$
$$\operatorname{dist}\left(H'_{\mu}(z_{k+1}), \partial_C H(z_{k+1})\right) \leq \gamma\|H(z_{k+1})\|, \tag{3.5}$$

and in other case, let $\mu_{k+1} = \mu_k$.

Step 3. Let $k = k+1$ and go to *Step* 1.

Remark 3.2. (1) The positive sequence $\{\mu_k\}$ is nonincreasing and satisfies

$$\kappa\mu_k \leq \alpha\|H(z_k)\|, \quad \forall k = 0,1,2,\ldots \tag{3.6}$$

(2) Parameter μ_k remains unchanged except if condition (3.4) is met.

The following result shows that Algorithm 3.1 is well defined.

Lemma 3.3. *Under the conditions in Theorem 2.6, Algorithm 3.1 is well defined.*

Proof. Obviously, for the sequence $\{\mu_k\}$ generated by Algorithm 3.1, it holds that $\mu_k > 0$. Hence, the linear system (3.2) is solvable by Theorem 2.6. To show the assertion, it suffices to show that for every k, the line search step terminates after a finite number of steps.

Since $\nabla\psi_{\mu_k}(z_k) = \nabla H_{\mu_k}(z_k)H_{\mu_k}(z_k)$, one has

$$\begin{aligned}
\psi_{\mu_k}(z_k + t\Delta z_k) - \psi_{\mu_k}(z_k) &= t\nabla\psi_{\mu_k}^{\top}(z_k)\Delta z_k + o(t) \\
&= tH_{\mu_k}^{\top}(z_k)H_{\mu_k}'(z_k)\Delta z_k + o(t) \\
&= -tH_{\mu_k}^{\top}(z_k)H(z_k) + o(t) \\
&= -tH^{\top}(z_k)H(z_k) + t\Big(H(z_k) - H_{\mu_k}(z_k)\Big)^{\top}H(z_k) + o(t) \\
&\le -t\|H(z_k)\|^2 + \kappa\mu_k t\|H(z_k)\| + o(t) \\
&\le -(1-\alpha)t\|H(z_k)\|^2 + o(t) \\
&\le -2\sigma\psi(z_k) + o(t),
\end{aligned} \tag{3.7}$$

where the second inequality follows from Remark 3.2 (1), and the last inequality uses the fact that $(1-\alpha) > \sigma$. This shows that the inequality (3.3) holds for all m sufficiently large. Hence the desired result follows. □

To establish the convergence of Algorithm 3.1, we need a technical lemma.

Lemma 3.4. *Let sequence $\{z_k\}$ be generated by Algorithm 3.1, then $\{z_k\}$ remains in the level set $L_0 = \{z \in R^{s+n+1} \mid \|H(z)\| \le ((4\alpha+1)/(2\alpha-1))\|H(z_0)\|\}$.*

Proof. For convenience, denote $\Lambda = \{0\} \cup \{k \mid \|H(z_k)\| \le \alpha\|H(z_{k-1})\| + (\kappa/\alpha)\mu_{k-1}\}$.

Assume Λ consists of $k_0 = 0 < k_1 < k_2 < \cdots$. For any $k \in N = \{0, 1, 2, 3, \ldots\}$, let k_i be the largest number in Λ such that $k_i \le k < k_{i+1}$. Then one has $\mu_k = \mu_{k_i}$ by Remark 3.2 (2).

Together with the line search rule (3.2), one further has

$$\|H_{\mu_k}(z_k)\| = \left\|H_{\mu_{k_i}}(z_k)\right\| \le \left\|H_{\mu_{k_i}}(z_{k-1})\right\| \le \cdots \le \left\|H_{\mu_{k_i}}(z_{k_i})\right\|. \tag{3.8}$$

Combining this with Corollary 2.2 yields

$$\begin{aligned}
\|H(z_k)\| &\le \left\|H(z_k) - H_{\mu_k}(z_k)\right\| + \left\|H_{\mu_k}(z_k)\right\| \\
&= \left\|H(z_k) - H_{\mu_k}(z_k)\right\| + \left\|H_{\mu_{k_i}}(z_k)\right\| \\
&\le \kappa\mu_k + \left\|H_{\mu_{k_i}}(z_{k_i})\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \kappa\mu_{k_i} + \left\| H_{\mu_{k_i}}(z_{k_i}) - H(z_{k_i}) \right\| + \|H(z_{k_i})\| \\
&\leq 2\kappa\mu_{k_i} + \|H(z_{k_i})\|.
\end{aligned}
\tag{3.9}
$$

On the other hand, it follows from condition (3.4) and Corollary 2.2 that

$$
\begin{aligned}
\|H(z_{k_{i+1}})\| &\leq \alpha\|H(z_{k_{i+1}-1})\| + \kappa\alpha^{-1}\mu_{k_{i+1}-1} \\
&\leq \alpha\left\| H_{\mu_{k_i}}(z_{k_{i+1}-1}) \right\| + \alpha\left\| H_{\mu_{k_i}}(z_{k_{i+1}-1}) - H(z_{k_i}) \right\| + \kappa\alpha^{-1}\mu_{k_i} \\
&\leq \alpha\left\| H_{\mu_{k_i}}(z_{k_i}) \right\| + \kappa\left(\alpha + \alpha^{-1}\right)\mu_{k_i} \\
&\leq \alpha\|H(z_{k_i})\| + \kappa\left(2\alpha + \alpha^{-1}\right)\mu_{k_i}.
\end{aligned}
\tag{3.10}
$$

Obviously, from the update rule of μ_k in (3.5), we obtain

$$
\mu_{k_i} \leq \frac{1}{2}\mu_{k_i-1} = \frac{1}{2}\mu_{k_{i-1}} \leq \cdots \leq \frac{1}{2^i}\mu_0 = \frac{1}{2^i}\frac{\alpha}{2\kappa}\|H(z_0)\|.
\tag{3.11}
$$

Using (3.10) and (3.11), one further has

$$
\begin{aligned}
\|H(z_{k_{i+1}})\| &\leq \|\alpha H(z_{k_i})\| + \kappa\left(2\alpha + \alpha^{-1}\right)\mu_{k_i} \\
&\leq \alpha\left(\alpha\|H(z_{k_{i-1}})\| + \kappa\left(2\alpha + \alpha^{-1}\right)\mu_{k_{i-1}}\right) + \kappa\left(2\alpha + \alpha^{-1}\right)\mu_{k_i} \\
&\;\;\vdots \\
&\leq \alpha^{i+1}\|H(z_0)\| + \alpha^i\kappa\left(2\alpha + \alpha^{-1}\right)\mu_{k_0} + \alpha^{i-1}\kappa\left(2\alpha + \alpha^{-1}\right)\mu_{k_1} + \cdots + \kappa\left(2\alpha + \alpha^{-1}\right)\mu_{k_i} \\
&\leq \alpha^{i+1}\|H(z_0)\| + \alpha^i\kappa\left(2\alpha + \alpha^{-1}\right)\mu_{k_0}\left[1 + \frac{1}{2\alpha} + \left(\frac{1}{2\alpha}\right)^2 + \cdots + \left(\frac{1}{2\alpha}\right)^i\right] \\
&\leq \alpha^{i+1}\|H(z_0)\| + \alpha^i\kappa\left(2\alpha + \alpha^{-1}\right)\mu_{k_0}\left(\frac{1}{1-(1/2\alpha)}\right) \\
&= \left(\frac{2\alpha^2 + 2\alpha}{2\alpha - 1}\right)\alpha^{i+1}\|H(z_0)\|,
\end{aligned}
\tag{3.12}
$$

where the last equality uses the fact that $\mu_{k_0} = \mu_0 = (\alpha/2\kappa)\|H(z_0)\|$.

Combining (3.9) with (3.11) and (3.12) yields

$$\begin{aligned}\|H(z_k)\| &\le 2\kappa\frac{1}{2^i}\mu_0 + \left(\frac{2\alpha^2+2\alpha}{2\alpha-1}\right)\alpha^i\|H(z_0)\| \\ &= 2\kappa\frac{1}{2^i}\frac{\alpha}{2\kappa}\|H(z_0)\| + \left(\frac{2\alpha^2+2\alpha}{2\alpha-1}\right)\alpha^i\|H(z_0)\| \\ &\le \left(\frac{4\alpha+1}{2\alpha-1}\right)\alpha^i\|H(z_0)\| \\ &\le \left(\frac{4\alpha+1}{2\alpha-1}\right)\|H(z_0)\|.\end{aligned} \tag{3.13}$$

This shows that $\{z_k\} \subset L_0$. □

Now, we are at the position to state our main result in this section.

Theorem 3.5. *Suppose the level set L_0 defined in Lemma 3.3 is bounded. Let $\{z_k = (x_k, \xi_k, \lambda_k)\}$ be a sequence generated by Algorithm 3.1, and let $z^* = (x^*, \xi^*, \lambda^*)$ be an accumulation point of $\{z_k\}$. Under the conditions of Theorem 2.6, then $\|H(z^*)\| = 0$, that is, (x^*, λ^*) is a solution of the polyhedral cone-constrained eigenvalue problem.*

Proof. If Λ is an infinite set, we conclude from (3.13) that

$$\|H(z^*)\| = \lim_{k\to\infty}\|H(z_k)\| \le \lim_{i\to\infty}\left(\frac{4\alpha+1}{2\alpha-1}\right)\alpha^i\|H(z_0)\| = 0, \tag{3.14}$$

and the desired results follow.

In the following, we will show that the index set Λ is an infinite set by reductio ad absurdum. Suppose that Λ is a finite set. Let $\widehat{k}$ be the largest number in Λ, then for all $k-1 \ge \overline{k}$, one has $\mu_{k-1} = \mu_{\overline{k}}$ and

$$\|H(z_k)\| > \alpha\|H(z_{k-1})\| + \frac{\kappa}{\alpha}\mu_{k-1} \ge \mu_{\overline{k}} > 0. \tag{3.15}$$

On the other hand, from the line search condition (3.3), one has

$$2\sigma t_k\psi(z_k) \le \varphi_{\mu_{\overline{k}}}(z_k) - \varphi_{\mu_{\overline{k}}}(z_{k+1}). \tag{3.16}$$

Summarizing these inequalities yields that $\lim_{k\to\infty}t_k\psi(z_k) = 0$.

Since the sequence $\{\psi(z_k)\}$ is bounded away from zero by (3.15), we claim that $t_k \to 0$ as $k \to \infty$. Hence the full step size is never accepted for all k sufficiently large, and inequality (3.3) does not hold for $\overline{t}_k = \eta^{m_k-1}$. That is, for sufficiently large k, it holds that

$$-2\sigma\psi(z_k) < \frac{\varphi_{\mu_{\overline{k}}}(z_k + \eta^{m_k-1}\Delta z_k) - \varphi_{\mu_{\overline{k}}}(z_k)}{\eta^{m_k-1}}. \tag{3.17}$$

Since the level set L_0 is bounded and matrix $H'_\mu(z)$ is nonsingular by Theorem 2.6, then the sequence $\{\Delta z_k\}$ generated by the linear equation (3.2) is also bounded. Without loss of generality, we may assume that $\{z_k\}$ and $\{\Delta z_k\}$ converge to vectors $\overline{z}$ and $\Delta\overline{z}$, respectively. Taking limits in both sides of inequality above yields that

$$\begin{aligned} -2\sigma\psi(\overline{z}) \le \nabla\psi_{\mu_{\overline{k}}}^{\top}(\overline{z})\Delta\overline{z} &= H_{\mu_{\overline{k}}}(\overline{z})^{\top}H'_{\mu_{\overline{k}}}(\overline{z})\Delta\overline{z} \\ &= -H_{\mu_{\overline{k}}}(\overline{z})^{\top}H(\overline{z}) \\ &= -\|H(\overline{z})\|^2 + \Big(H(\overline{z}) - H_{\mu_{\overline{k}}}(\overline{z})\Big)^{\top}H(\overline{z}) \\ &\le -2(1-\alpha)\psi(\overline{z}), \end{aligned} \tag{3.18}$$

where the last inequality follows from Remark 3.2 (1).

Using the fact that the sequence $\{\psi(z_k)\}$ is bounded away from zero by (3.15) again, one has $\sigma \ge 1-\alpha$, which is a contradiction with the choice of the parameter σ. Thus Λ is an infinite set and the desired result is proved. □

In order to analyze the local superlinear convergence of the Algorithm 3.1, we need the concepts of semismooth and strongly semismooth [14, 15].

For a locally Lipschitizian mapping $G(x) : R^n \to R^m$, it is said to be semismooth at $x \in R^n$ if the limit

$$\lim_{\substack{V\in\partial G(x+th') \\ h'\to h, t\downarrow 0}} Vh' \tag{3.19}$$

exists for any $h \in R^n$ [15].

$G(x) : R^n \to R^m$ is called strongly semismooth at $x \in R^n$ if $G(x)$ is semismooth at x, and for any $V \in \partial G(x+h)$ with $h > 0$ being sufficiently small, it holds that

$$G(x+h) - G(x) - Vh = \mathrm{O}\Big(\|h\|^2\Big). \tag{3.20}$$

Obviously, the vector-value function H defined in (2.6) is semismooth and strongly semismooth according to Lemma 2.4(a) in [16]. In the following, we can obtain the following convergence rate result in a similar way to the proof of Theorem 3 in [7].

Theorem 3.6. *Suppose the conditions in Theorem 2.6 hold, and assume the level set L_0 defined in Lemma 3.3 is bounded. $z^* = (x^*, \xi^*, \lambda^*)$ is an accumulation of the sequence $\{z_k\}$ generated by Algorithm 3.1. If every element in $\partial_c H(z^*)$ is nonsingular, then $\{z_k\}$ converges quadratically to z^*.*

Table 1: Numerical result of Example 4.1.

SP				Iter	Pareto	Pareto eigenvector x			HV
λ	x				eigenvalue	x_1	x_2	x_3	
4.4737	−0.03	0.8305	0.5563	3	$\lambda_1 = 4.133975$	0	0.9659	0.2588	6.51×10^{-14}
5.6473	−0.15	0.6575	−0.7384	5	$\lambda_2 = 4.602084$	0.1814	0.9792	0.0907	2.7873×10^{-20}
4.9735	0.3011	0.8987	0.0112	2	$\lambda_3 = 5$	0.3162	0.9487	0	3.269×10^{-20}
4.179	−0.1735	0.3621	0.9158	4	$\lambda_4 = 5.866025$	0	0.2588	0.9659	5.142×10^{-16}
4.5108	−0.075	−0.83	0.5536	5	$\lambda_5 = 6$	0	0	1	6.537×10^{-18}
7.7682	0.7014	0.689	0.1823	3	$\lambda_6 = 7$	0.7071	0.7017	0	4.9206×10^{-16}
6.672	0.98	−0.077	−0.1846	3	$\lambda_7 = 8$	1	0	0	4.0538×10^{-18}
8.9328	0.5986	0.5013	0.2996	3	$\lambda_8 = 9.397916$	0.7875	0.4741	0.3937	2.7843×10^{-22}
10.1813	0.7681	0.038	0.6391	4	$\lambda_9 = 10$	0.8944	0	0.4472	1.6339×10^{-18}

4. Numerical Experiments

In this section, we give preliminary numerical experiments for a special polyhedral cone-constrained eigenvalue problem, that is, the Pareto eigenvalue problem. We implement the algorithm in MATLAB 7.0.

In our numerical experiments, we consider the eigenvalue complementarity problem with $B = I$. As for the parameters used in our numerical experiments, we take $\eta = 0.5$, $\alpha = 0.25$, $\sigma = 0.3$, $\gamma = 20$, and let the algorithm terminates when $\|\nabla H(z)\| \leq 10^{-6}$.

To generate the initial point for these examples, we first generate a random vector $\varsigma \in \mathbb{R}^n$ with uniform distribution on $[-1, 1]^n$, and then set

$$x^0 = \frac{\varsigma}{\langle \mathbf{1}_n, \varsigma \rangle}, \quad \lambda^0 = \frac{\langle x^0, Ax^0 \rangle}{\langle x^0, Bx^0 \rangle}, \quad y^0 = Ax^0 - \lambda^0 Bx^0, \\ \mu^0 = \frac{\alpha}{2\kappa} \|H(z_0)\| \tag{4.1}$$

to generate the initial point $z_0 = (\mu^0, x^0, y^0, \lambda^0)$.

Example 4.1. Consider the pareto eigenvalue problem [1] for which

$$A = \begin{bmatrix} 8 & -1 & 4 \\ 3 & 4 & \frac{1}{2} \\ 2 & -\frac{1}{2} & 6 \end{bmatrix}. \tag{4.2}$$

This problem has 9 Pareto eigenvalues [5], which can all be detected using our algorithm via distinct initial points as seen from Table 1. In this table, SP denotes the starting point of z^0, Iter denotes the number of iterations, and HV denotes the final value of $H(z_k)$ when the algorithm terminates.

Example 4.2. Consider the Pareto eigenvalue problem with A being randomly generated from $[-1, 1]$ such that

$$\begin{bmatrix} -0.02418 & -0.3211 & 0.8033 & -0.5086 & -0.1005 & -0.5833 & 0.3562 & 0.8881 & 0.692 & -0.2711 \\ 0.783 & -0.981 & -0.3515 & 0.7952 & 0.2862 & -0.4964 & 0.0175 & 0.8242 & -0.6552 & 0.9514 \\ 0.5246 & -0.3324 & 0.1454 & 0.2082 & -0.5095 & -0.207 & -0.4462 & 0.63 & -0.926 & 0.4338 \\ 0.3106 & -0.5063 & -0.63 & -0.7205 & -0.064 & -0.0387 & 0.1577 & 0.3793 & -0.3749 & 0.2866 \\ 0.943 & 0.9881 & 0.4758 & -0.6774 & -0.0051 & 0.0186 & 0.6457 & -0.3827 & 0.6346 & 0.5651 \\ -0.6577 & 0.8111 & -0.7765 & 0.7663 & -0.85 & 0.25 & 0.883 & 0.1164 & -0.5307 & -0.067 \\ -0.7281 & -0.3001 & -0.2325 & 0.9572 & 0.5332 & 0.2511 & -0.1114 & 0.2735 & 0.1749 & -0.5354 \\ 0.5104 & -0.4438 & 0.7247 & 0.2026 & -0.9093 & 0.9823 & -0.1535 & 0.5382 & 0.8484 & -0.3642 \\ -0.3736 & -0.8142 & 0.7142 & -0.2764 & -0.6698 & -0.2816 & 0.9924 & -0.892 & -0.34 & 0.5776 \\ -0.0735 & -0.5189 & 0.6387 & -0.5371 & 0.5544 & -0.448 & 0.2282 & -0.7703 & -0.5889 & 0.2689 \end{bmatrix}. \tag{4.3}$$

For this matrix, if we take the initial point

$$\begin{aligned} z_0 = {} & (0.03572, 0.3626, -0.4312, 0.1223, 0.4, -0.4128, \\ & -0.1927, 0.1593, 0.0775, -0.3911, 0.3409, 0.1971, 1.3022, 1.2591, 0.58, \\ & -0.5704, 0.1238, -0.1847, 0.3483, 0.6675, 0.4573, -0.7037)^T \end{aligned} \tag{4.4}$$

and apply the JL method, we can obtain one Pareto eigenvalue 0.0592 and a corresponding eigenvector of $(0, 0, 0.1204, 0.315, 0, 0, 0.8402, 0.227, 0, 0.3591)^T$ after 6 iteration steps within computer 1.016 seconds on a PIV 2.0 GHz personal computer.

5. Conclusions

In this paper, we first reformulate the the polyhedral cone-constrained eigenvalue problem as a system of equations based on the F-B function and then establish a smooth version by parameterizing the F-B function. A smoothing Jacobian-like method is developed to solve the problem. Some preliminary numerical results reported in the paper suggest efficiency of the proposed algorithm.

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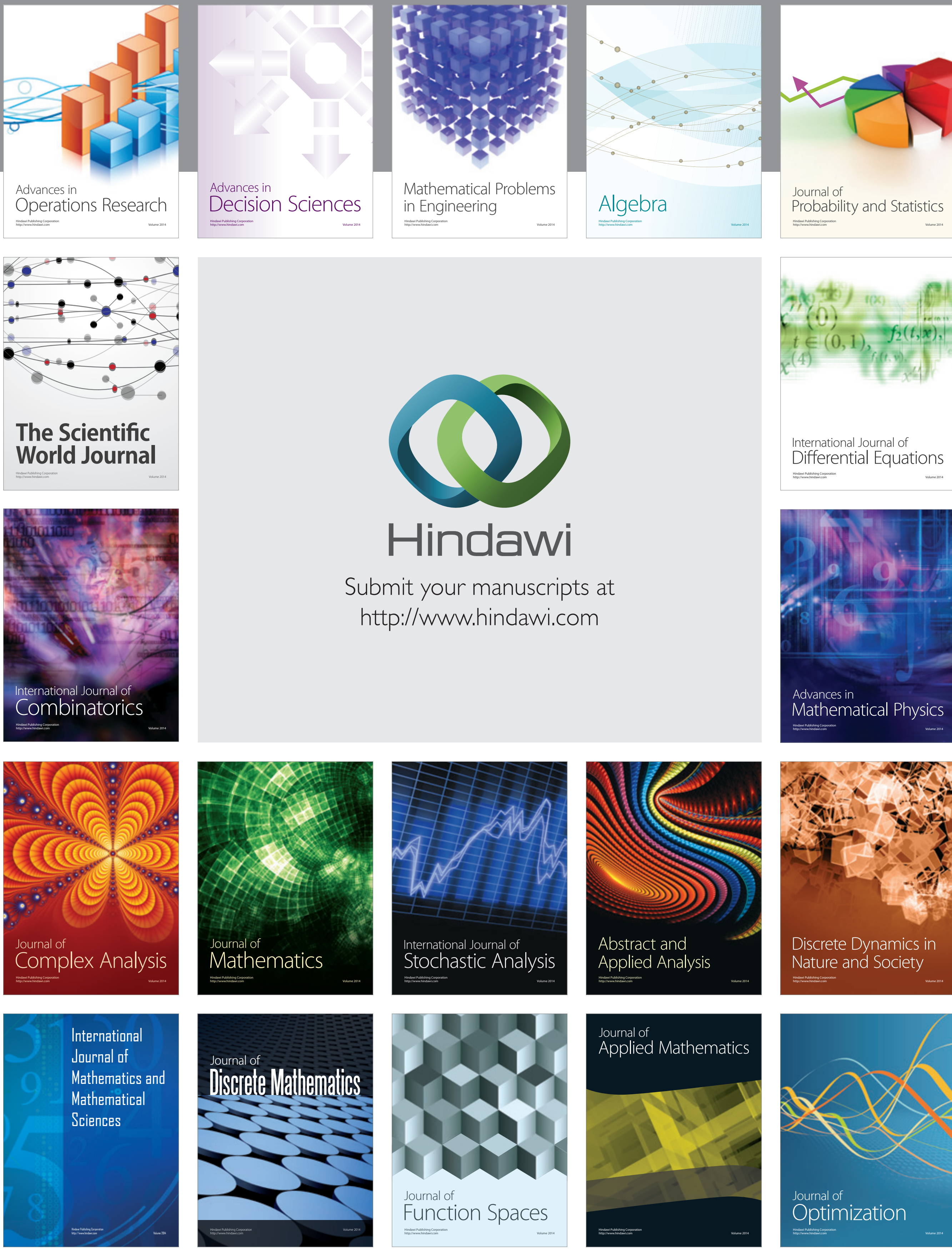
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