

Research Article

An Extension of the Legendre-Galerkin Method for Solving Sixth-Order Differential Equations with Variable Polynomial Coefficients

A. H. Bhrawy,^{1,2} A. S. Alofi,¹ and S. I. El-Soubhy³

¹ Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

² Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt

³ Department of Mathematics, Faculty of Science, Taibah University, Madinah 20012, Saudi Arabia

Correspondence should be addressed to A. H. Bhrawy, alibhrawy@yahoo.co.uk

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We extend the application of Legendre-Galerkin algorithms for sixth-order elliptic problems with constant coefficients to sixth-order elliptic equations with variable polynomial coefficients. The complexities of the algorithm are $O(N)$ operations for a one-dimensional domain with $(N - 5)$ unknowns. An efficient and accurate direct solution for algorithms based on the Legendre-Galerkin approximations developed for the two-dimensional sixth-order elliptic equations with variable coefficients relies upon a tensor product process. The proposed Legendre-Galerkin method for solving variable coefficients problem is more efficient than pseudospectral method. Numerical examples are considered aiming to demonstrate the validity and applicability of the proposed techniques.

1. Introduction

Spectral methods are preferable in numerical solutions of ordinary and partial differential equations due to its high-order accuracy whenever it works [1, 2]. Recently, renewed interest in the Galerkin technique has been prompted by the decisive work of Shen [3], where new Legendre polynomial bases for which the matrices systems are sparse are introduced. We introduce a generalization of Shen's basis to numerically solve the sixth-order differential equations with variable polynomial coefficients.

Sixth-order boundary-value problems arise in astrophysics; the narrow convecting layers bounded by stable layers, which are believed to surround A-type stars, may be modeled by sixth-order boundary-value problems [4, 5]. Further discussion of the sixth-order boundary-value problems is given in [6]. The literature of numerical analysis contains little work on the solution of the sixth-order boundary-value problems [4, 5, 7, 8]. Theorems that

list conditions for the existence and uniqueness of solutions of such problems are thoroughly discussed in [9], but no numerical methods are contained therein.

From the numerical point of view, Shen [3], Doha and Bhrawy [10–12], and Doha et al. [13] have constructed efficient spectral-Galerkin algorithms using compact combinations of orthogonal polynomials for solving elliptic equations of the second and fourth order with constant coefficients in various situations. Recently, the authors in [14, 15] and [16] have developed efficient Jacobi dual-Petrov-Galerkin and Jacobi-Gauss collocation methods for solving some odd-order differential equations. Moreover, the Bernstein polynomials have been applied for the numerical solution of high even-order differential equations (see, [17, 18]).

For sixth-order differential equations, Twizell and Boutayeb [5] developed finite-difference methods of order two, four, six, and eight for solving such problems. Siddiqi and Twizell [7] used sixth-degree splines, where spline values at the mid knots of the interpolation interval and the corresponding values of the even order derivatives were related through consistency relations. A sixth-degree B-spline functions is used to construct an approximate solution for sixth-order boundary-value problems (see [19]). Moreover, Septic spline solutions of sixth-order boundary value problems are introduced in [20]. El-Gamel et al. [8] proposed Sinc-Galerkin method for the solutions of sixth-order boundary-value problems. In fact, the decomposition and modified domain decomposition methods to investigate solution of the sixth-order boundary-value problems are introduced in [21]. Recently, Bhrawy [22] developed a spectral Legendre-Galerkin method for solving sixth-order boundary-value problems with constant coefficients. In this work, we introduce an efficient direct solution algorithm to generalize the work in [3, 22].

The main aim of this paper is to extend the application of Legendre-Galerkin method (LGM) to solve sixth-order elliptic differential equations with variable coefficients by using the expansion coefficients of the moments of the Legendre polynomials and their high-order derivatives. We present appropriate basis functions for the Legendre-Galerkin method applied to these equations. This leads to discrete systems with sparse matrices that can be efficiently inverted. The complexities of the algorithm is $O(N)$ operations for a one-dimensional domain with $(N-5)$ unknowns. The direct solution algorithms developed for the homogeneous problem in two-dimensions with constant and variable coefficients rely upon a tensor product process. Numerical results indicating the high accuracy and effectiveness of these algorithms are presented.

This paper is organized as follows. In the next section, we discuss an algorithm for solving the one-dimensional sixth-order elliptic equations with variable polynomial coefficients. In Section 3, we extend our results of Sections 2 to the two-dimensional sixth-order equations with variable polynomial coefficients. In Section 4, we present two numerical examples to exhibit the accuracy and efficiency of the proposed numerical algorithms. Also a conclusion is given in Section 5.

2. One-Dimensional Sixth-Order Equations with Polynomial Coefficients

We first introduce some basic notation which will be used in the sequel. We denote by $L_n(x)$ the n th degree Legendre polynomial, and we set

$$S_N = \text{span}\{L_0(x), L_1(x), \dots, L_N(x)\}, \quad W_N = \left\{v \in S_N : v^{(j)}(\pm 1) = 0, j = 0, 1, 2\right\}, \quad (2.1)$$

where $v^{(j)}(x)$ denotes j th-order differentiation of $v(x)$ with respect to x .

We recall that the $\{L_n(x)\}$ satisfy the orthogonality relation

$$\int_{-1}^1 L_m(x)L_n(x)dx = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases} \quad h_n = \frac{2}{2n+1}, \quad \forall m, n \geq 0. \quad (2.2)$$

We recall also that $L_n(x)$ is a polynomial of degree n and therefore $L_n^{(q)}(x) \in S_{n-q}$. The following relation (the q th derivative of $L_k(x)$) will be needed for our main results (see Doha [23])

$$D^q L_k(x) = \sum_{\substack{i=0 \\ (k+i) \text{ even}}}^{k-q} C_q(k,i)L_i(x), \quad (2.3)$$

where

$$C_q(k,i) = \frac{2^{q-1}(2i+1)\Gamma((1/2)(q+k-i))\Gamma((1/2)(q+k+i+1))}{\Gamma(q)\Gamma((1/2)(2-q+k-i))\Gamma((1/2)(3-q+k+i))}. \quad (2.4)$$

Some other useful relations are

$$L_n(\pm 1) = (\pm 1)^n, \quad L_n^{(q)}(\pm 1) = (\pm 1)^{n+q} \frac{(n+q)!}{2^q(n-q)!q!}. \quad (2.5)$$

In this section, we are interested in using the Legendre-Galerkin method to solve the variable polynomial coefficients sixth-order differential equation in the form:

$$-\chi_3(x)u^{(6)}(x) + \sum_{i=0}^2 (-1)^i \alpha_i \chi_i(x)u^{(2i)}(x) = f(x), \quad x \in [-1, 1], \quad (2.6)$$

subject to

$$u^{(j)}(\pm 1) = 0, \quad j = 0, 1, 2, \quad (2.7)$$

where $\alpha_i, i = 0, 1, 2$ are constants and $\chi_i(x), i = 0, 1, 2, 3$ are given polynomials. Moreover, $f(x)$ is a given source function. Without loss of generality, we suppose that $\chi_3(x) = x^\mu, \chi_1(x) = x^\nu$, and $\chi_0(x) = x^\sigma$ where μ, ν , and σ are positive integers.

2.1. Basis of Functions

The problem of approximating solutions of ordinary or partial differential equations by Galerkin approximation involves the projection onto the span of some appropriate set of basis functions, typically arising as the eigenfunctions of a singular Sturm-Liouville problem. The members of the basis may satisfy automatically the boundary conditions imposed on the problem. As suggested in [3, 10–12], one should choose compact combinations of orthogonal

polynomials as basis functions to minimize the bandwidth and condition number of the resulting system. As a general rule, for one-dimensional sixth-order differential equations with six boundary conditions, one can choose the basis functions of expansion $\phi_k(x)$ of the form

$$\phi_k(x) = \sum_{j=0}^3 \eta_j(k) L_{k+2j}(x); \quad \eta_0(k) = 1, \quad k = 0, 1, \dots, N-6. \quad (2.8)$$

We will choose the coefficients $\{\eta_j(k)\}$ such that $\phi_k(x)$ verifies the boundary conditions (2.7). Making use of (2.5) and (2.8), hence $\{\eta_j(k)\}$ can be uniquely determined to obtain

$$\eta_j(k) = (-1)^j \frac{3(2k+4j+1)(k+3/2)_3}{(3-j)!j!(k+j+1/2)_4}, \quad j = 1, 2, 3, \quad (2.9)$$

where $(a)_b = \Gamma(a+b)/\Gamma(a)$. Now, substitution of (2.9) into (2.8) yields

$$\phi_k(x) = L_k(x) + \sum_{j=1}^3 (-1)^j \frac{3(2k+4j+1)(k+3/2)_3}{(3-j)!j!(k+j+1/2)_4} L_{k+2j}(x), \quad k = 0, 1, 2, \dots, N-6. \quad (2.10)$$

It is obvious that $\{\phi_k(x)\}$ are linearly independent. Therefore by dimension argument we have

$$W_N = \text{span}\{\phi_k(x) : k = 0, 1, 2, \dots, N-6\}. \quad (2.11)$$

2.2. Treatment of Variable Polynomial Coefficients

A more general situation which often arises in the numerical solution of differential equations with polynomial coefficients by using the Legendre Galerkin method is the evaluation of the expansion coefficients of the moments of high-order derivatives of infinitely differentiable functions. The formula of Legendre coefficients of the moments of one single Legendre polynomials of any degree is

$$x^m L_j(x) = \sum_{n=0}^{2m} \Theta_{m,n}(j) L_{j+m-n}(x), \quad \forall m, j \geq 0, \quad (2.12)$$

with $L_{-r}(x) = 0$, $r \geq 1$, where

$$\Theta_{m,n}(j) = \frac{(-1)^n 2^{j+m-n} m! (2j+2m-2n+1) \Gamma(j+1)}{\Gamma(j+m-n+1)} \\ \times \sum_{k=\max(0, j-n)}^{\min(j+m-n, j)} \binom{j+m-n}{k} \frac{\Gamma(j+k+1)}{2^k (n+k-j)! \Gamma(3j+2m-2n-k+2)}$$

$$\begin{aligned} & \times \sum_{\ell=0}^{j-k} \frac{(-1)^\ell \Gamma(2j+m-n-k-\ell+1) \Gamma(j+m+\ell-n+1)}{\ell!(j-k-\ell)! \Gamma(j-\ell+1) \Gamma(k+\ell+1)} \\ & \times {}_2F_1(j-k-n, j+m+\ell-n+1; 3j+2m-2n-k+2; 2). \end{aligned} \tag{2.13}$$

For more details about the above formula, the reader is referred to Doha [23]. This formula can be used to facilitate greatly the setting up of the algebraic systems to be obtained by applying the LGM for solving differential equations with polynomial coefficients of any order. The following lemma is very important and needed in what follows.

Lemma 2.1. *We have, for arbitrary constants $\ell, j \geq 0$,*

$$\begin{aligned} x^\ell \phi_j^{(q)}(x) &= \sum_{i=0}^3 \left[\sum_{n=0}^{2\ell} \Theta_{\ell,n}(j+2i) L_{j+2i+\ell-n}(x) \right] \eta_i(j), \\ x^\ell \phi_j^{(q)}(x) &= \sum_{i=0}^3 \left[\sum_{s=0}^{j+2i-q} \left[\sum_{n=0}^{2\ell} \Theta_{\ell,n}(s) L_{s+\ell-n}(x) \right] C_q(j+2i, s) \right] \eta_i(j), \end{aligned} \tag{2.14}$$

where $\phi_j(x)$, $\eta_i(j)$, and $\Theta_{\ell,n}(j)$ are as defined in (2.8), (2.9), and (2.13), respectively.

Proof. Immediately obtained from relations (2.3), (2.8), and (2.12), the Legendre-Galerkin approximation to (2.6)-(2.7) is, to find $u_N \in W_N$ such that

$$\begin{aligned} \left(-x^\mu u_N^{(6)}(x), v(x) \right) + \sum_{i=1}^2 \alpha_i \left((-1)^i x^\nu u_N^{(2i)}(x), v(x) \right) + \alpha_0 (x^\sigma u_N(x), v(x)) &= (f(x), v(x))_N, \\ & \forall v \in W_N, \end{aligned} \tag{2.15}$$

where $(u, v) = \int_{-1}^1 u v dx$ is the scalar product in $L^2(-1, 1)$ and $(\cdot, \cdot)_N$ is the inner product associated with the Legendre-Gauss-Lobatto quadrature. It is clear that if we take $\phi_k(x)$ as defined in (2.8) and $v(x) = \phi_k(x)$, then we find that (2.15) is equivalent to

$$\begin{aligned} \left(-x^\mu u_N^{(6)}(x), \phi_k(x) \right) + \sum_{i=1}^2 \alpha_i \left((-1)^i x^\nu u_N^{(2i)}(x), \phi_k(x) \right) + \alpha_0 (x^\sigma u_N(x), \phi_k(x)) &= (f(x), \phi_k(x))_N, \\ & k = 0, 1, \dots, N-6. \end{aligned} \tag{2.16}$$

Hence, by setting

$$\begin{aligned} f_k &= (f, \phi_k(x))_N, \quad \mathbf{f} = (f_0, f_1, \dots, f_{N-6})^T, \\ u_N(x) &= \sum_{n=0}^{N-6} a_n \phi_n(x), \quad \mathbf{a} = (a_0, a_1, \dots, a_{N-6})^T, \\ Z_i^\ell &= \left(z_{kj}^{i,\ell} \right)_{0 \leq k, j \leq N-6}; \quad 0 \leq i \leq 3, \ell \text{ is a positive integer,} \end{aligned} \tag{2.17}$$

where

$$z_{kj}^{r,\ell} = \left((-1)^r x^\ell \phi_j^{(2r)}(x), \phi_k(x) \right), \quad r = 0, 1, 2, 3, \quad (2.18)$$

then the matrix system associated with (2.16) becomes

$$\left(Z_3^\mu + \sum_{i=1}^2 \alpha_i Z_i^\nu + \alpha_0 Z_0^\sigma \right) \mathbf{a} = \mathbf{f}, \quad (2.19)$$

where the nonzero elements of the matrices Z_3^μ , Z_2^ν , Z_1^ν , and Z_0^σ are given explicitly in the following theorem. \square

Theorem 2.2. *If we take $\phi_k(x)$ as defined in (2.8), and if we denote $z_{kj}^{3,\mu} = (-x^\mu \phi_j^{(6)}(x), \phi_k(x))$, $z_{kj}^{r,\nu} = ((-1)^r x^\nu \phi_j^{(2r)}(x), \phi_k(x))$, $r = 1, 2$ and $z_{kj}^{0,\sigma} = (x^\sigma \phi_j(x), \phi_k(x))$ then the nonzero elements $(z_{kj}^{3,\mu})$, $(z_{kj}^{r,\nu})$, $(z_{kj}^{0,\sigma})$ for $0 \leq k, j \leq N - 6$ are given by*

$$\begin{aligned} z_{kj}^{3,\mu} &= - \sum_{\ell=0}^3 \left[\sum_{i=0}^3 \left(\sum_{s=0}^{2\mu} (C_6(j+2i, k+2\ell-\mu+s) \Theta_{\mu,s}(k+2\ell-\mu+s)) \eta_i(j) \right) \eta_\ell(k) h_{k+2\ell} \right], \\ & \quad j = k + 2p - \mu, p = 0, 1, \dots, \mu, \\ z_{kj}^{r,\nu} &= (-1)^r \sum_{\ell=0}^3 \left[\sum_{i=0}^3 \left(\sum_{s=0}^{2\nu} (C_{2r}(j+2i, k+2\ell-\nu+s) \Theta_{\nu,s}(k+2\ell-\nu+s)) \eta_i(j) \right) \eta_\ell(k) h_{k+2\ell} \right], \\ & \quad j = k + 2p - \nu + 2r - 6, p = 0, 1, \dots, \nu + 6 - 2r, r = 1, 2, \\ z_{kj}^{0,\sigma} &= \sum_{\ell=0}^3 \left[\sum_{i=0}^3 (\Theta_{\sigma,j+2i+\sigma-(k+2\ell)}(j+2i) \eta_i(j)) \eta_\ell(k) h_{k+2\ell} \right], \quad j = k + 2p - \sigma - 6, \\ & \quad p = 0, 1, \dots, \sigma + 6. \end{aligned} \quad (2.20)$$

Proof. The proof of this theorem is rather lengthy, but it is not difficult once Lemma 2.1 is applied. \square

From Theorem 2.2, we see that Z_3^μ is a band matrix with an upper bandwidth of μ , lower bandwidth of μ , and an overall bandwidth $2\mu + 1$. The sparse matrices Z_2^ν , Z_1^ν , and Z_0^σ have bandwidths of $2\nu + 5$, $2\nu + 9$, and $2\sigma + 13$, respectively.

In general, the expense of calculating an LU factorization of an $N \times N$ dense matrix A is $O(N^3)$ operations, and the expense of solving $Ax = b$, provided that the factorization is known, is $O(N^2)$. However, in the case that a banded A has bandwidth of r , we need just $O(r^2N)$ operations to factorize and $O(rN)$ operations to solve a linear system. In the case of $\alpha_i = 0$, $i = 0, 1, 2$, the square matrix Z_3^μ has bandwidth of $2\mu + 1$. We need just $O((2\mu + 1)^2N)$ operations to factorize and $O((2\mu + 1)N)$ operations to solve the linear system (2.19). If

$\mu \ll N$ this represents a very substantial saving. Notice also that the system (2.19) reduces to a diagonal system for $\mu = 0$ and $\alpha_i = 0, i = 0, 1, 2$.

2.3. Constant Coefficients

In the special case, ($\mu = \nu = \sigma = 0$, i.e., the sixth-order differential equation with constant coefficients), the corresponding matrix system becomes $(Z_3^0 + \sum_{i=1}^3 \alpha_i Z_i^0) \mathbf{a} = \mathbf{f}$, where $Z_i^0 = (z_{kj}^{i,0})_{0 \leq k, j \leq N-6; i = 0, 1, 2, 3}$.

Corollary 2.3. *If $\mu = \nu = \sigma = 0$ then the nonzero elements $(z_{kj}^{3,0}), (z_{kj}^{2,0}), (z_{kj}^{1,0}), (z_{kj}^{0,0})$ for $0 \leq k, j \leq N - 6$ are given as follows:*

$$z_{kk}^{3,0} = 2^6 \left(k + \frac{3}{2}\right)_2 \left(k + \frac{3}{2}\right)_3, \quad (2.21)$$

$$z_{kj}^{2,0} = \sum_{\ell=0}^3 \left[\sum_{i=0}^3 \eta_i(j) C_4(j + 2i, k + 2\ell) \right] \eta_\ell(k) h_{k+2\ell}, \quad j = k + 2p - 2, \quad p = 0, 1, 2,$$

$$z_{kj}^{1,0} = - \sum_{\ell=0}^3 \left[\sum_{i=0}^3 \eta_i(j) C_2(j + 2i, k + 2\ell) \right] \eta_\ell(k) h_{k+2\ell}, \quad j = k + 2p - 4, \quad p = 0, 1, 2, 3, 4,$$

$$z_{k+2p,k}^{0,0} = z_{k,k+2p}^{0,0} = \sum_{i=0}^{3-p} (-1)^{2i+p} \binom{3}{i} \binom{3}{i+p} \frac{(2k + 4p + 4i + 1)(k + 3/2)_3 (k + 2p + 3/2)_3}{2(k + p + i + 1/2)_4 (k + 2p + i + 1/2)_4},$$

$$p = 0, 1, 2, 3. \quad (2.22)$$

Note that the results of Corollary 2.3 can be obtained immediately as a special case from Theorem 2.2. For more details see [22].

It is worthy to note here that if $\alpha_i = 0, i = 0, 1, 2$, then the nonzero elements of the matrix Z_3^0 are given by (2.21) and the solution of the linear system is given explicitly by $\mathbf{a}_k = (f, \phi_k)_N / z_{kk}^{3,0}$.

Obviously Z_2^0, Z_1^0 , and Z_0^0 are symmetric positive definite matrices. Furthermore, Z_3^0 is a diagonal matrix, Z_2^0 can be split into two tridiagonal submatrices, Z_1^0 can be split into two pentadiagonal submatrices, and A_0 can be split into two sparse submatrices with bandwidth of 7. Therefore, the system can be efficiently solved. More precisely for $k + j$ odd, $z_{kj}^{r,0} = 0, r = 0, 1, 2, 3$. Hence system $(Z_3^0 + \sum_{i=1}^3 \alpha_i Z_i^0) \mathbf{a} = \mathbf{f}$ of order $N - 5$ can be decoupled into two separate systems of order $(N/2 - 2)$ and $(N/2 - 3)$, respectively. In this way one needs to solve two systems of order n instead of one of order $2n$, which leads to substantial savings. Moreover, in the case of $\alpha_i \neq 0, i = 0, 1, 2$, we can form explicitly the LU factorization, that is, $Z_3^0 + \sum_{i=1}^3 \alpha_i Z_i^0 = LU$. The special structure of L and U allows us to obtain the solution in $O(N)$ operations.

Remark 2.4. If the boundary conditions are nonhomogeneous, one can split the solution $u(x)$ into the sum of a low-degree polynomial which satisfies the nonhomogeneous boundary conditions plus a sum over the basis functions $\phi(x)$ that satisfy the equivalent homogeneous boundary conditions.

3. Two-Dimensional Sixth-Order Equations with Polynomial Coefficients

In this section, we extend the results of Section 2 to deal with the two-dimensional sixth-order differential equations with variable polynomial coefficients:

$$-X_3(x)Y_3(y)(\Delta)^3 u(x, y) + \sum_{r=1}^2 \alpha_r X_r(x)Y_r(y)(-\Delta)^r u(x, y) + \alpha_0 X_0(x)Y_0(y)u(x, y) = f(x, y),$$

in Ω ,
(3.1)

subject to the boundary conditions

$$\begin{aligned} \frac{\partial^i}{\partial x^i} u(\pm 1, y) &= 0, \quad i = 0, 1, 2, \\ \frac{\partial^i}{\partial y^i} u(x, \pm 1) &= 0, \quad i = 0, 1, 2, \end{aligned}$$

(3.2)

where $\Omega = (-1, 1) \times (-1, 1)$, the differential operator Δ is the well-known Laplacian defined by $\Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, $(\partial^0/\partial x^0)u(\pm 1, y) = u(\pm 1, y)$, $(\partial^0/\partial x^0)u(x, \pm 1) = u(x, \pm 1)$ and $f(x, y)$ is a given source function. Moreover, $X_i(x)$ and $Y_i(y)$, $i = 0, 1, 2, 3$ are given polynomials. Without loss of generality, we suppose that $X_3(x) = x^\mu$, $Y_3(y) = y^\nu$, $X_i(x) = x^\rho$, $Y_i(y) = y^\sigma$, $i = 1, 2$, $X_0(x) = x^\delta$, and $Y_0(y) = y^\varepsilon$ where $\mu, \nu, \rho, \sigma, \delta$, and ε are positive integers.

The Legendre-Galerkin approximation to (3.1)-(3.2) is, to find $u_N \in W_N^2$ such that

$$\left(x^\mu y^\nu (-\Delta)^3 u_N, v \right) + \sum_{r=1}^2 \alpha_r \left(x^\rho y^\sigma (-\Delta)^r u_N, v \right) + \alpha_0 \left(x^\delta y^\varepsilon u_N, v \right) = (f, v)_N \quad \forall v \in W_N^2.$$

(3.3)

It is clear that if we take $\phi_k(x)$ as defined in (2.8), then

$$W_N^2 = \text{span}\{\phi_i(x)\phi_j(y), i, j = 0, 1, \dots, N-6\}.$$

(3.4)

We denote

$$\begin{aligned} u_N &= \sum_{k=0}^{N-6} \sum_{j=0}^{N-6} u_{kj} \phi_k(x) \phi_j(y), \quad f_{kj} = (f, \phi_k(x)\phi_j(y))_N, \\ U &= (u_{kj}), \quad F = (f_{kj}), \quad k, j = 0, 1, \dots, N-6. \end{aligned}$$

(3.5)

Taking $v(x, y) = \phi_\ell(x)\phi_m(y)$ in (3.3) for $\ell, m = 0, 1, \dots, N-6$, then one can observe that (3.3) is equivalent to the following equation:

$$\begin{aligned} \sum_{\ell, m=0}^{N-6} \left\{ \left\{ z_{i\ell}^{3,\mu} u_{\ell m} z_{jm}^{0,\nu} + 3z_{i\ell}^{2,\mu} u_{\ell m} z_{jm}^{1,\nu} + 3z_{i\ell}^{1,\mu} u_{\ell m} z_{jm}^{2,\nu} + z_{i\ell}^{0,\mu} u_{\ell m} z_{jm}^{3,\nu} \right\} \right. \\ \left. + \alpha_2 \left\{ z_{i\ell}^{2,\rho} u_{\ell m} z_{jm}^{0,\sigma} + 2z_{i\ell}^{1,\rho} u_{\ell m} z_{jm}^{1,\sigma} + z_{i\ell}^{0,\rho} u_{\ell m} z_{jm}^{2,\sigma} \right\} \right\} \end{aligned}$$

$$\begin{aligned}
& + \alpha_1 \left\{ z_{i\ell}^{1,\rho} u_{\ell m} z_{jm}^{0,\sigma} + z_{i\ell}^{0,\rho} u_{\ell m} z_{jm}^{1,\sigma} \right\} \\
& + \alpha_0 \left\{ z_{i\ell}^{0,\delta} u_{\ell m} z_{jm}^{0,\varepsilon} \right\} = f_{ij}, \quad i, j = 0, 1, \dots, N-6,
\end{aligned} \tag{3.6}$$

which may be written in the matrix form

$$\begin{aligned}
& Z_3^\mu U Z_0^\nu + 3Z_2^\mu U Z_1^\nu + 3Z_1^\mu U Z_2^\nu + Z_0^\mu U Z_3^\nu + \alpha_2 \left(Z_2^\rho U Z_0^\sigma + 2Z_1^\rho U Z_1^\sigma + Z_0^\rho U Z_2^\sigma \right) \\
& + \alpha_1 \left(Z_1^\rho U Z_0^\sigma + Z_0^\rho U Z_1^\sigma \right) + \alpha_0 Z_0^\delta U Z_0^\varepsilon = F,
\end{aligned} \tag{3.7}$$

where $\{Z_i^q$ for $i = 0, 1, 2, 3$ and q is a positive integer} are the matrices defined in Theorem 2.2.

The direct solution algorithm here developed for the sixth-order elliptic differential equation in two dimensions relies upon a tensor product process, which is defined as follows. Let P and R be two matrices of size $n \times n$ and $m \times m$, respectively. Their tensor product

$$P \otimes R = \begin{pmatrix} P_{11}R & \cdots & P_{1n}R \\ \vdots & \ddots & \vdots \\ P_{n1}R & \cdots & P_{nn}R \end{pmatrix}, \tag{3.8}$$

is a matrix of size $mn \times mn$.

We can also rewrite (3.7) in the following form using the Kronecker matrix algebra (See, Graham [24]):

$$\begin{aligned}
L\mathbf{v} \equiv & \left[Z_3^\mu \otimes Z_0^\nu + 3Z_2^\mu \otimes Z_1^\nu + 3Z_1^\mu \otimes Z_2^\nu + Z_0^\mu \otimes Z_3^\nu + \alpha_2 \left(Z_2^\rho \otimes Z_0^\sigma + 2Z_1^\rho \otimes Z_1^\sigma + Z_0^\rho \otimes Z_2^\sigma \right) \right. \\
& \left. + \alpha_1 \left(Z_1^\rho \otimes Z_0^\sigma + Z_0^\rho \otimes Z_1^\sigma \right) + \alpha_0 Z_0^\delta \otimes Z_0^\varepsilon \right] \mathbf{v} = \mathbf{f},
\end{aligned} \tag{3.9}$$

where \mathbf{f} and \mathbf{v} are F and U written in a column vector, that is,

$$\begin{aligned}
\mathbf{f} &= (f_{00}, f_{10}, \dots, f_{N-6,0}; f_{01}, f_{11}, \dots, f_{N-6,1}; \dots; f_{0,N-6}, \dots, f_{N-6,N-6})^T, \\
\mathbf{v} &= (u_{00}, u_{10}, \dots, u_{N-6,0}; u_{01}, u_{11}, \dots, u_{N-6,1}; \dots; u_{0,N-6}, \dots, u_{N-6,N-6})^T,
\end{aligned} \tag{3.10}$$

and \otimes denotes the tensor product of matrices, that is, $Z_3^\mu \otimes Z_0^\nu = (Z_3^\mu z_{ij}^{0,\nu})_{i,j=0,1,\dots,N-6}$. In brief, the solution of (3.1) subject to (3.2) can be summarized in Algorithm 1.

4. Numerical Results

We report on two numerical examples by using the algorithms presented in the previous sections. It is worthy to mention that the pure spectral-Galerkin technique is rarely used in

Compute the matrices $\{Z_j^u, Z_j^v\}$ for $j = 0, 1, 2, 3$, $\{Z_j^p, Z_j^q\}$ for $j = 0, 1, 2$ and $\{Z_0^\delta, Z_0^\epsilon\}$ from Theorem 2.2.
 Find the tensor product of the matrices in the previous step.
 Compute F and write it in a column vector \mathbf{f} .
 Obtain a column vector \mathbf{v} by solving (3.9).

Algorithm 1

practice, since for a general right-hand side function f we are not able to compute exactly its representation by Legendre polynomials. In fact, the so-called pseudospectral method is used to treat the right-hand side; that is, we replace f by $I_N f$ (polynomial interpolation over the set of Gauss-Lobatto points); see for instance; Funaro [25].

Example 4.1. Here we present some numerical results for a one-dimensional sixth-order equation with polynomial coefficients. We only consider the case $\chi_3(x) = (x^4 + x + 1)$, $\chi_2(x) = x^2 + 2x + 3$, $\chi_1(x) = x^2 + 1$ and $\chi_0(x) = 1$; that is, we consider the

$$-(x^4 + 1)u^{(6)}(x) + \gamma_2(x^2 + 2x + 3)u^{(4)}(x) - \gamma_1(x^2 + 1)u''(x) + \gamma_0 u(x) = f(x), \quad (4.1)$$

subject to

$$u(\pm 1) = u'(\pm 1) = u''(\pm 1) = 0, \quad x \in I, \quad (4.2)$$

where $f(x)$ is chosen such that the exact solution of (4.1) is $u(x) = (1 - x^2)\sin^2(2\pi x)$.

For LGM, we have $u_N(x) = \sum_{k=0}^{N-6} a_k \phi_k(x)$, the vectors of unknowns \mathbf{a} is the solution of the following system

$$\left((Z_3^4 + Z_3^0) + \gamma_2(Z_2^2 + 2Z_2^1 + 3Z_2^0) + \gamma_1(Z_1^2 + Z_1^0) + \gamma_0 Z_0^0 \right) \mathbf{a} = \mathbf{f}, \quad (4.3)$$

where the nonzero elements of the matrices $Z_3^4, Z_3^0, Z_2^2, Z_2^1, Z_2^0, Z_1^2, Z_1^0$ and Z_0^0 can be evaluated explicitly from Theorem 2.2. Table 1 lists the maximum pointwise error of $u - u_N$, using the LGM with various choices of $\gamma_0, \gamma_1, \gamma_2$ and N . Numerical results of this problem show that the Legendre Galerkin method converge exponentially.

In order to examine the algorithm proposed in Section 3, we will consider a problem for a two-dimensional sixth-order elliptic differential equation with constant coefficients.

Example 4.2. Consider the two-dimensional sixth-order equation

$$-\Delta^3 u + \gamma_2 \Delta^2 u - \gamma_1 \Delta u + \gamma_0 u = f(x, y), \quad (4.4)$$

Table 1: Maximum pointwise error of $u - u_N$ for $N = 20, 30, 40$.

N	γ_0	γ_1	γ_2	LGM	γ_0	γ_1	γ_2	LGM
20				$8.66 \cdot 10^{-4}$				$9.22 \cdot 10^{-4}$
30	30	1	1	$1.67 \cdot 10^{-9}$	1	3	5	$1.67 \cdot 10^{-9}$
40				$6.10 \cdot 10^{-16}$				$6.10 \cdot 10^{-16}$

Table 2: Maximum pointwise error of $u - u_N$ for $N = 20, 30, 40$.

N	γ_0	γ_1	γ_2	LGM	γ_0	γ_1	γ_2	LGM
20				$1.18 \cdot 10^{-3}$				$1.18 \cdot 10^{-3}$
30	3	1	1	$2.39 \cdot 10^{-9}$	0	0	0	$1.18 \cdot 10^{-9}$
40				$2.11 \cdot 10^{-14}$				$2.08 \cdot 10^{-14}$

subject to the boundary conditions

$$\begin{aligned}
 u(\pm 1, y) = u(x, \pm 1) = 0, \quad \frac{\partial u}{\partial x}(\pm 1, y) = \frac{\partial u}{\partial y}(x, \pm 1) = 0, \\
 \frac{\partial^2 u}{\partial x^2}(\pm 1, y) = \frac{\partial^2 u}{\partial y^2}(x, \pm 1) = 0,
 \end{aligned} \tag{4.5}$$

where $f(x, y)$ is chosen such that the exact solution of (4.4)-(4.5) is

$$u(x, y) = (1 - x^2)(1 - y^2)\sin^2(2\pi x)\sin^2(2\pi y). \tag{4.6}$$

In Table 2, we list the maximum pointwise error of $u - u_N$ by the LGM with two choices of $\gamma_0, \gamma_1, \gamma_2$ and various choices of N . The results indicate that the spectral accuracy is achieved and that the effect of roundoff errors is very limited.

5. Concluding Remarks

We have presented stable and efficient spectral Galerkin method using Legendre polynomials as basis functions for sixth-order elliptic differential equations in one and two dimensions. We concentrated on applying our algorithms to solve variable polynomials coefficients differential equations by using the expansion coefficients of the moments of the Legendre polynomials and their high-order derivatives. Numerical results are presented which exhibit the high accuracy of the proposed algorithms.

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