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Research Article

A Reduced-Order TS Fuzzy Observer Scheme with Application to Actuator Faults Reconstruction

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This paper focuses on the principle for designing reduced-order fuzzy-observer-based actuator fault reconstruction for a class of nonlinear systems. The problem addressed can be indicated as an approach for a kind of reduced-order fuzzy observer design with special gain matrix structure that depends on a given matching condition specification. Using the Lyapunov theory, the stability conditions are obtained and expressed in terms of linear matrix inequalities, and the conditions for asymptotic estimation of actuator faults are derived. Simulation results illustrate the observer design procedure and demonstrate the actuator fault reconstruction effectiveness and performance.

1. Introduction

Automated diagnosis has been one of the most fruitful applications in sophisticated control systems, with potential significance for domains in which systems diagnosis must proceed, while the system is operative and testing opportunities are limited by operational considerations. A real problem is usually to fix the system with faults so that it can continue its mission for some time with some limitations in functionality. Consequently, diagnosis is a part of a larger problem known as Fault Detection, Identification and Reconfiguration (FDIR). The classical principles include observer-based methods, parity space methods, and parameter identification based methods, which have been thoroughly studied (see, e.g., [1, 2] and the references therein).

Observer design is an actual research topic, important in the observer-based fault estimation, and in the fault detection and isolation [3–5]. The nonlinear system theory, exploiting Lipschitz condition, is emerged as an approach capable of use in the state estimation design for nonlinear systems [6], although Lipschitz condition is a restrictive limitation and many classes of systems may not be satisfied. Application of this principle in state estimator design results only in a sufficient condition for the asymptotic stability of estimation error, and, in fact, there is no straightforward method for selecting the observer gain to satisfy such conditions [7]. Because of strong restrictions, an observer structure with adaptively adjusted parameters is proposed in [8], where Lipchitz constant can be unknown. Concerning fault detection, for example, in [9–11] there are proposed sliding-mode observers. Since they are conditioned by matching conditions, these approaches are not sufficient to ensure safe operation in all applications.

Recently, fault estimation and reconstruction are preferred as an option to fault detection, where, instead of generating residuals, observer-based methods are used to reconstruct sensor and actuator fault signals in nonlinear systems. These practices primarily use adaptive and unknown input observer structures (see e.g., [12–14]), ensuring disturbance rejection and robustness properties of fault estimation.

An alternative approach is the Takagi-Sugeno (TS) fuzzy approximation of the nonlinear system model equations. Since the TS fuzzy method provides the suitable model for a certain class of nonlinear dynamic systems [15], the well-known nonlinear observers are based on TS fuzzy system model. Using TS fuzzy model, a nonlinear system is represented by the fuzzy rules. Each rule utilizes the local system dynamics by a linear model, and the nonlinear system is represented by a collection of fuzzy rules. In this sense, the TS fuzzy model can be viewed as an expansion of piecewise linear partition for the nonlinear system. Since such description allows the utilization of system state representation, model order reduction and error approximation problems have to be solved using the projection methods [16], generally given in the form of linear matrix inequality (LMI) constraints.

System state observers based on TS fuzzy models are principally realized in the same structures as the linear observers [17–20], and design principles usually used techniques based on LMIs. Research in TS fuzzy observers application in fault detection and isolation has attracted many investigators and was the subject of widely scattered publications (see, e.g., [21–24]), mainly focused on the LMI-based observer design, to ensure the stability of the residuals and to optimize the quadratic performance of residual transfer matrix with respect to exogenous disturbance.

Because fault reconstruction provides a direct estimate of the size and severity of a fault, the location of the fault is so known, and the fault isolation step can be deleted. Establishing a general approach for fault reconstruction in systems described by TS models, or finding conditions under which fault reconstruction is well possible, is still an open task [25]. Sophisticated fault estimation schemes were proposed especially for systems with disturbances and uncertainties, where, for example, Gao et al. [26] propose the fuzzy descriptor observer, potentially applicable to sensor fault estimation. In contrast, Xu et al. [27] present an estimation algorithm, based on the integrated fuzzy observer and the inverse system model, for nonlinear actuator fault estimation. The principle of the inverse system model is combined with sliding mode also in [27], since sliding mode observers can be employed in fault estimation if systems are uncertain owing to their insensitivity to matched uncertainties or disturbances. Certainly, the basic approach to actuator faults estimation is based on TS adaptive observers [17], in spite of high-order observer dynamics. On the other hand, few results have been reported to reduced-order observer-based fault reconstruction [8, 28], despite the importance of relative-order dynamics of reduced-order observers for systems without disturbances.

Considering the author's previous work [29], the main contribution of the paper is to examine one principle for designing of reduced-order-observer-based actuator fault estimation for a class of continuous-time nonlinear MIMO systems, approximated by TS models. Comparing with the approach given in [14], a new design method is proposed to construct a set of linear reduced-order observers, combined by fuzzy rules, to estimate unmeasurable part of the system state vector. Based on the stable observer set, the actuator fault estimation scheme is developed to guarantee asymptotic estimation of actuator faults. The structure of the design conditions is motivated by the need for feasibility, while under defined matching conditions the stability of the reduced-order observer is assured.

The remainder of this paper is organized as follows. Sections 2 and 3 describe TS fuzzy model properties for given class of nonlinear systems and the design principle of the reduced-order observer based on TS model, respectively. The actuator fault reconstruction, using reduced-order fuzzy observer, is outlined in Section 4, especially with respect to observer design principle, matching condition, and stability. In Section 5, one illustrative example is given, and simulation results are presented to confirm the validity of the proposed fault reconstruction scheme. Finally, Section 6 draws some concluding remarks.

Throughout the paper, the following notations are used: $\mathbf{x}^T$, $\mathbf{X}^T$ denotes the transpose of the vector $\mathbf{x}$ and matrix $\mathbf{X}$, respectively, diag$[\cdot]$ denotes a block diagonal matrix, for a square matrix $\mathbf{X} = \mathbf{X}^T > 0$ (resp., $\mathbf{X} = \mathbf{X}^T < 0$) means that $\mathbf{X}$ is a symmetric positive definite matrix (resp., symmetric negative definite matrix), the symbol $\mathbf{I}_n$ represents the nth order unit matrix, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}^{n\times r}$ denotes the set of all $n \times r$ real matrices.

2. Takagi-Sugeno Fuzzy Models

The systems under consideration fall in a class of multi-input and multioutput (MIMO) nonlinear dynamic systems, which in the state-space form are represented as

$$\begin{gathered}\dot{\mathbf{q}}(t) = \mathbf{a}(\mathbf{q}(t)) + \mathbf{B}(\mathbf{q}(t))\mathbf{u}(t) + \mathbf{B}_f\mathbf{f}(t),\\ \mathbf{y}(t) = \mathbf{C}\mathbf{q}(t),\end{gathered} \tag{2.1}$$

where $\mathbf{q}(t) \in \mathbb{R}^n$, $\mathbf{u}(t)$, $\mathbf{u}_f(t) \in \mathbb{R}^r$, and $\mathbf{y}(t) \in \mathbb{R}^m$ are vectors of the state, input, actuator fault, and output variables, respectively, $\mathbf{C} \in \mathbb{R}^{m\times n}$, $\mathbf{B}_f \in \mathbb{R}^{n\times r}$ are real matrices, $\mathbf{a}(\mathbf{q}(t)) \in \mathbb{R}^n$, $\mathbf{B}(\mathbf{q}(t)) \in \mathbb{R}^{n\times r}$ are bounded nonlinear functions of $\mathbf{q}(t)$, and $\mathbf{f}(t) \in \mathbb{R}^r$ is an actuator fault. It is assumed that $\mathbf{a}(\mathbf{q}(t))$ is bounded in associated sectors, that is, in the regions within the system will operate, $\mathbf{a}(0) = 0$, only actuator faults can occur, and if no actuator fault occurs then $\mathbf{u}_f(t) = \mathbf{0}$, for all $t \ge 0$.

It is considered that the number of the nonlinear terms in the vector function $\mathbf{a}(\mathbf{q}(t))$ is p and there exists the set of the nonlinear sector functions $\{w_{lj}(\theta_j(t)),\ l = 1,2,\ldots,p,\ j = 1,2,\ldots,k\}$ such that

$$w_{l1}(\boldsymbol{\theta}(t)) = 1 - \sum_{j=2}^{k} w_{lj}(\theta_j(t)), \tag{2.2}$$

where k is the number of sector functions, and

$$\boldsymbol{\theta}(t) = \begin{bmatrix}\theta_1(t) & \theta_2(t) & \cdots & \theta_q(t)\end{bmatrix} \tag{2.3}$$

is the vector of premise variables. A premise variable represents any measurable variable (in a simple case it can be directly a state variable) and none of the premise variables depend on the inputs $\mathbf{u}(t)$.

Thus, constructing the set of membership functions $w_i(\boldsymbol{\theta}(t)) = \prod_{l=1|j}^{p} w_{lj}(\theta_j(t))$, $i = 1, 2, \ldots s$, $s = 2^k$ from all combinations of the sector functions, the states of the system with an actuator fault are inferred as follows:

$$\dot{\mathbf{q}}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\mathbf{A}_i\mathbf{q}(t) + \mathbf{B}_i\mathbf{u}(t)) + \mathbf{B}_f\mathbf{f}(t), \tag{2.4}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t), \tag{2.5}$$

where the system output is given by the relation (2.5) and

$$h_i(\boldsymbol{\theta}(t)) = \frac{w_i(\boldsymbol{\theta}(t))}{\sum_{i=1}^{s} w_i(\boldsymbol{\theta}(t))} \tag{2.6}$$

is the averaging weight for the ith rule, representing the normalized grade of membership (membership function). By definition, the membership functions satisfy the following convex sum properties:

$$0 \le h_i(\boldsymbol{\theta}(t)) \le 1, \quad \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) = 1, \quad \forall i \in \langle 1, \ldots, s \rangle, \tag{2.7}$$

$\mathbf{A}_i \in \mathbb{R}^{n \times n}$ is the Jacobian matrix of $\mathbf{a}(\mathbf{q}(t))$ with respect to $\mathbf{q}(t) = \mathbf{q}_i$, $\mathbf{B}_i \in \mathbb{R}^{n \times r}$ is the matrix equal to $\mathbf{B}(\mathbf{q}_i)$, and $\mathbf{q}_i$ is the center of the i-th fuzzy region, described by the associated sector function. It is evident that the fuzzy model is achieved by fuzzy amalgamation of the linear subsystem models.

Using a TS model, the conclusion part of a single rule consists no longer of a fuzzy set [19], but determines a function with state variables as arguments, and the corresponding function is a local function for the fuzzy region that is described by the premise part of the rule. Thus, using linear functions, a system state is described locally (in fuzzy regions) by linear models, and at the boundaries between regions the linear interpolation is used between the corresponding local models.

Note, the model (2.4), (2.5) is mostly considered for analysis, control, and state estimation of nonlinear systems.

It is supposed in the next that the aforementioned TS model does not include parameter uncertainties or external disturbances, and all premise and output variables are measurable.

3. Basic Preliminaries

Definition 3.1 (null space of the matrix). Let $\mathbf{E} \in \mathbb{R}^{h \times h}$, $\operatorname{rank}(\mathbf{E}) = k < h$ be a rank deficient matrix. Then the null space $N_{\mathbf{E}}$ of $\mathbf{E}$ is the orthogonal complement of the row space of $\mathbf{E}$.

Lemma 3.2 (orthogonal complement). *If* $\mathbf{E} \in \mathbb{R}^{h\times h}$, rank$(\mathbf{E}) = k < h$, *is a rank deficient matrix, then an orthogonal complement* $\mathbf{E}^{\perp}$ *of* $\mathbf{E}$ *is*

$$\mathbf{E}^{\perp} = \mathbf{E}^{\circ}\mathbf{U}_{E2}^{T}, \tag{3.1}$$

where $\mathbf{U}_{E2}^{T}$ *is the null space of* $\mathbf{E}$ *and* $\mathbf{E}^{\circ}$ *is an arbitrary matrix of appropriate dimension.*

*Proof (*see, e.g., [30]*).* The singular value decomposition (SVD) of $\mathbf{E}$ gives

$$\mathbf{U}_{E}^{T}\mathbf{E}\mathbf{V}_{E} = \begin{bmatrix}\mathbf{U}_{E1}^{T}\\ \mathbf{U}_{E2}^{T}\end{bmatrix}\mathbf{E}\begin{bmatrix}\mathbf{V}_{E1} & \mathbf{V}_{E2}\end{bmatrix} = \begin{bmatrix}\mathbf{S}_{E} & \mathbf{0}_{12}\\ \mathbf{0}_{21} & \mathbf{0}_{22}\end{bmatrix}, \tag{3.2}$$

where $\mathbf{U}_{E}^{T} \in \mathbb{R}^{h\times h}$ is the orthogonal matrix of the left singular vectors of $\mathbf{E}$, $\mathbf{V}_{E} \in \mathbb{R}^{h\times h}$ is the orthogonal matrix of the right singular vectors of $\mathbf{E}$, and $\mathbf{S}_{E} \in \mathbb{R}^{k\times k}$ is the diagonal positive definite matrix of the form

$$\mathbf{S}_{E} = \operatorname{diag}\begin{bmatrix}\sigma_{E1} & \cdots & \sigma_{Ek}\end{bmatrix}, \quad \sigma_{E1} \geq \cdots \geq \sigma_{Ek} > 0, \tag{3.3}$$

which diagonal elements are the singular values of $\mathbf{E}$. Using orthogonal properties of $\mathbf{U}_{E}$ and $\mathbf{V}_{E}$, that is, $\mathbf{U}_{E}^{T}\mathbf{U}_{E} = \mathbf{I}_{h}$, $\mathbf{V}_{E}^{T}\mathbf{V}_{E} = \mathbf{I}_{h}$, and

$$\begin{bmatrix}\mathbf{U}_{E1}^{T}\\ \mathbf{U}_{E2}^{T}\end{bmatrix}\begin{bmatrix}\mathbf{U}_{E1} & \mathbf{U}_{E2}\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{2}\end{bmatrix}, \quad \mathbf{U}_{E2}^{T}\mathbf{U}_{E1} = \mathbf{0}, \tag{3.4}$$

respectively, where $\mathbf{I}_{h} \in \mathbb{R}^{h\times h}$ is the identity matrix, then $\mathbf{E}$ can be written as

$$\mathbf{E} = \mathbf{U}_{E}\mathbf{S}_{E}\mathbf{V}_{E}^{T} = \begin{bmatrix}\mathbf{U}_{E1} & \mathbf{U}_{E2}\end{bmatrix}\begin{bmatrix}\mathbf{S}_{E} & \mathbf{0}_{12}\\ \mathbf{0}_{21} & \mathbf{0}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{V}_{E1}^{T}\\ \mathbf{V}_{E2}^{T}\end{bmatrix} = \begin{bmatrix}\mathbf{U}_{E1} & \mathbf{U}_{E2}\end{bmatrix}\begin{bmatrix}\mathbf{S}_{E1}\\ \mathbf{0}_{2}\end{bmatrix} = \mathbf{U}_{E1}\mathbf{S}_{E1}, \tag{3.5}$$

where $\mathbf{S}_{E1} = \mathbf{S}_{E}\mathbf{V}_{E1}^{T}$. Thus, (3.4) and (3.5) imply

$$\mathbf{U}_{E2}^{T}\mathbf{E} = \mathbf{U}_{E2}^{T}\begin{bmatrix}\mathbf{U}_{E1} & \mathbf{U}_{E2}\end{bmatrix}\begin{bmatrix}\mathbf{S}_{E1}\\ \mathbf{0}_{2}\end{bmatrix} = \mathbf{0}. \tag{3.6}$$

It is evident that for an arbitrary matrix $\mathbf{E}^{\circ}$ it is

$$\mathbf{E}^{\circ}\mathbf{U}_{E2}^{T}\mathbf{E} = \mathbf{E}^{\perp}\mathbf{E} = \mathbf{0}, \tag{3.7}$$

which implies (3.1). This concludes the proof. □

Lemma 3.3 (congruence transform). *Let the output matrix* $\mathbf{C}$ *be of full column rank,* rank $\mathbf{C} = m$, *then there exists a new coordinate system such that* $\mathbf{C}$ *takes the structure* $\mathbf{C}_{a} = [\mathbf{I}_{m} \;\; 0]$.

Proof. Applying SVD to $\mathbf{C}$ gives

$$\mathbf{C} = \mathbf{U}\left[\mathbf{S} \ \ \mathbf{0}\right]\mathbf{V}^T = \mathbf{U}\mathbf{S}\left[\mathbf{I}_m \ \ \mathbf{0}\right]\mathbf{V}^T, \tag{3.8}$$

where rows of $\mathbf{U}^T \in \mathbb{R}^{m\times m}$ are left singular vectors of $\mathbf{C}$, and columns of $\mathbf{V} \in \mathbb{R}^{n\times n}$ are right singular vectors of $\mathbf{C}$, all ordered in such way to be associated with the singular values of $\mathbf{C}$, written as diagonal elements of $\mathbf{S} \in \mathbb{R}^{m\times m}$,

$$\mathbf{S} = \operatorname{diag}\left[\sigma_1 \ \ \cdots \ \ \sigma_m\right], \quad \sigma_1 \geq \cdots \geq \sigma_m > 0. \tag{3.9}$$

Using the notations

$$\mathbf{W}^{-1} = \mathbf{U}\mathbf{S}, \qquad \mathbf{T}_a = \mathbf{V}^T, \qquad \mathbf{C}_a = \left[\mathbf{I}_m \ \ \mathbf{0}\right], \tag{3.10}$$

where

$$\mathbf{T}_a^T = \left[\mathbf{T}_{a1}^T \ \ \mathbf{T}_{a2}^T\right], \quad \mathbf{T}_{a1}^T \in \mathbb{R}^{n\times m}, \tag{3.11}$$

then (3.8) implies

$$\mathbf{C} = \mathbf{W}^{-1}\mathbf{C}_a\mathbf{T}_a, \quad \mathbf{C}_a = \mathbf{W}\mathbf{C}\mathbf{T}_a^{-1}. \tag{3.12}$$

Note if $\mathbf{C}$ is of rank m, $\mathbf{W} \in \mathbb{R}^{m\times m}$ is a regular matrix, and $\mathbf{T}_a \in \mathbb{R}^{n\times n}$ is an orthogonal matrix such that $\mathbf{T}_a^{-1} = \mathbf{T}_a^T = \mathbf{V}$. This concludes the proof. □

Lemma 3.4. *Using the congruence transform* (3.12), *each linear submodel of fault-free TS fuzzy model* (2.4), (2.5) *can be partitioned such that*

$$\begin{bmatrix}\dot{\mathbf{q}}_{a1i}\\ \dot{\mathbf{q}}_{a2i}\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{a11i} & \mathbf{A}_{a12i}\\ \mathbf{A}_{a21i} & \mathbf{A}_{a22i}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{a1}(t)\\ \mathbf{q}_{a2}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{B}_{a1i}\\ \mathbf{B}_{a2i}\end{bmatrix}\mathbf{u}(t) + \begin{bmatrix}\mathbf{B}_{af1}\\ \mathbf{B}_{af2}\end{bmatrix}\mathbf{f}(t), \tag{3.13}$$

$$\mathbf{y}(t) = \mathbf{W}^{-1}\mathbf{v}(t), \tag{3.14}$$

$$\mathbf{v}(t) = \left[\mathbf{I}_m \ \ \mathbf{0}\right]\mathbf{q}_a(t) = \left[\mathbf{I}_m \ \ \mathbf{0}\right]\begin{bmatrix}\mathbf{q}_{a1}(t)\\ \mathbf{q}_{a2}(t)\end{bmatrix}, \tag{3.15}$$

$$\mathbf{q}_a^T(t) = \left[\mathbf{q}_{a1}^T(t) \ \ \mathbf{q}_{a2}^T(t)\right], \quad \mathbf{q}_a(t) = \mathbf{T}_a\mathbf{q}(t), \tag{3.16}$$

where $\mathbf{q}_{a1}(t) \in \mathbb{R}^m$, $\mathbf{q}_{a2}(t) \in \mathbb{R}^{n-m}$, $\mathbf{B}_{a1i} \in \mathbb{R}^{m\times r}$, $\mathbf{B}_{af1} \in \mathbb{R}^{m\times r}$, *respectively.*

Proof. Substituting (3.11) into (2.5) gives

$$\mathbf{y}(t) = \mathbf{W}^{-1}\mathbf{C}_a\mathbf{T}_a\mathbf{q}(t) = \mathbf{W}^{-1}\mathbf{C}_a\mathbf{q}_a(t). \tag{3.17}$$

Thus, using

$$\mathbf{v}(t) = \mathbf{C}_a \mathbf{q}_a(t), \tag{3.18}$$

(3.17) implies (3.14), and with (3.11), (3.12), then (3.18) implies (3.15).

Substituting (3.16) in (2.4), it can be obtained

$$\dot{\mathbf{q}}_a(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{A}_{ai}\mathbf{q}_a(t) + \mathbf{B}_{ai}\mathbf{u}(t) + \mathbf{B}_{af}\mathbf{f}(t), \tag{3.19}$$

where

$$\mathbf{A}_{ai} = \mathbf{T}_a \mathbf{A}_i \mathbf{T}_a^{-1}, \qquad \mathbf{B}_{ai} = \mathbf{T}_a \mathbf{B}_i, \qquad \mathbf{B}_{af} = \mathbf{T}_a \mathbf{B}_f, \tag{3.20}$$

and partitioning accordingly to (3.15), (3.20) implies (3.13). This concludes the proof. □

Proposition 3.5 (matching condition). *The fault input matrix and the output matrix* $\mathbf{B}_{af}$, $\mathbf{C}$ *satisfies the conditions* rank $\mathbf{B}_{af} = r$, rank $\mathbf{C} >$ rank $\mathbf{B}_f$, *respectively, that is,* $m > r$, *and the matrix* $\mathbf{B}_{af}$ *takes the structure*

$$\mathbf{B}_{af} = \mathbf{C}_a^T \mathbf{B}_{af1}. \tag{3.21}$$

The matching condition, given in Proposition 3.5 seems to be restrictive theoretically, but fortunately, for many practical control systems it is satisfied. In addition, comparing with the static decoupling control principle [31], the condition reflects inserting at least one redundant output sensor into the sensor structure.

4. Full-Order TS Fuzzy Observer

Standard applications of TS fuzzy principle in nonlinear system fault diagnosis exploit the fuzzy observers as residual generators. The procedure of fault detection covers the residual generation by the fuzzy observers and their evaluation. Thus, the reconstruction error, or any function of it, is used as fault residual signal that is as a rule zero in the fault free case and nonzero otherwise [1, 2].

The fuzzy observer to the fault-free system (2.4), (2.5) is constructed as follows:

$$\dot{\mathbf{q}}_e(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\mathbf{A}\mathbf{q}_e(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{A}_i\mathbf{q}_e(t) + \mathbf{J}_i(\mathbf{y} - \mathbf{y}_e(t))), \tag{4.1}$$

$$\mathbf{y}_e(t) = \mathbf{C}\mathbf{q}_e(t), \tag{4.2}$$

where $\mathbf{q}_e(t) \in \mathbb{R}^n$ is the estimation of the system state vector, $\mathbf{J}_i \in \mathbb{R}^{n\times m}$, $i = 1, 2, \ldots, s$, is the set of the observer gain matrices. The design conditions are given by the next lemma.

Lemma 4.1. *The fuzzy observer* (4.1), (4.2) *is stable if there exist a positive definite symmetric matrix* $\mathbf{P} > 0$, $\mathbf{P} \in \mathbb{R}^{n\times n}$, *and matrices* $\mathbf{Z}_i \in \mathbb{R}^{n\times m}$, $i = 1, 2, \ldots, s$, *such that*

$$\mathbf{P} = \mathbf{P}^T > 0, \tag{4.3}$$

$$\mathbf{A}_i\mathbf{P} + \mathbf{P}\mathbf{A}_i - \mathbf{Z}_i^T\mathbf{C}^T - \mathbf{Z}_i\mathbf{C} < 0, \quad \forall i. \tag{4.4}$$

If the above conditions hold, the set of the observer gain matrices is given as

$$\mathbf{J}_i = \mathbf{P}^{-1}\mathbf{Z}_i. \tag{4.5}$$

Proof. Introducing the estimation error between the fault-free (2.4) and (4.1) as follows:

$$\mathbf{e}(t) = \mathbf{q}(t) - \mathbf{q}_e(t), \tag{4.6}$$

and taking into account the time derivative of $\mathbf{e}(t)$, it can be obtained

$$\dot{\mathbf{e}}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\mathbf{A}_i - \mathbf{J}_i\mathbf{C})\mathbf{e}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{A}_{ei}\mathbf{e}(t), \tag{4.7}$$

where the observer system matrices are

$$\mathbf{A}_{ei} = \mathbf{A}_i - \mathbf{J}_i\mathbf{C}, \quad i = 1, 2, \ldots, s. \tag{4.8}$$

Defining the quadratic positive definite Lyapunov function of the form

$$v(\mathbf{e}(t)) = \mathbf{e}^T(t)\mathbf{P}\mathbf{e}(t), \tag{4.9}$$

where $\mathbf{P} > 0$, then after evaluation of its derivative with respect to t it is obtained

$$\dot{v}(\mathbf{e}(t)) = \dot{\mathbf{e}}^T(t)\mathbf{P}\mathbf{e}(t) + \mathbf{e}^T(t)\mathbf{P}\dot{\mathbf{e}}(t). \tag{4.10}$$

Substituting (4.7) in (4.10) gives

$$\dot{v}(\mathbf{e}(t)) = \mathbf{e}^T(t)\mathbf{P}\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{A}_{ei}\mathbf{e}(t) + \mathbf{e}^T(t)\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{A}_{ei}^T\mathbf{P}\mathbf{e}(t), \tag{4.11}$$

$$\dot{v}(\mathbf{e}(t)) = \mathbf{e}^T(t)\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\left(\mathbf{P}\mathbf{A}_{ei} + \mathbf{A}_{ei}^T\mathbf{P}\right)\mathbf{e}(t), \tag{4.12}$$

respectively. It is evident that (4.12) is negative if there exist a set of gain matrices $\mathbf{J}_i \in \mathbb{R}^{n\times m}$, $i = 1, 2, \ldots, s$, and a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ such that

$$(\mathbf{A}_i - \mathbf{J}_i\mathbf{C})^T\mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{J}_i\mathbf{C}) < 0, \quad \forall i. \tag{4.13}$$

Setting

$$\mathbf{P}\mathbf{J}_i = \mathbf{Z}_i, \tag{4.14}$$

(4.13) implies (4.4). This concludes the proof. □

Note, to apply for actuator fault reconstruction, an adaptive structure of the full-order state observer can be used [12].

5. Reduced-Order TS Fuzzy Observer

Problem of the interest is to design the asymptotically stable reduced-order observer based on the TS fuzzy model of the fault-free nonlinear system (2.4), (2.5).

Theorem 5.1. *Considering the affine TS fuzzy system* (2.4), (2.5), *then the reduced-order TS fuzzy observer takes the form*

$$\dot{\mathbf{p}}_{2e}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{q}^{\circ}_{2ei}(t), \tag{5.1}$$

$$\mathbf{q}^{\circ}_{2ei}(t) = \mathbf{A}_{aei}\mathbf{p}_{2e}(t) + \mathbf{A}_{avi}\mathbf{v}(t) + \begin{bmatrix}-\mathbf{J}_i & \mathbf{I}_{n-m}\end{bmatrix}\mathbf{B}_{ai}\mathbf{u}(t), \tag{5.2}$$

$$\mathbf{q}_{a2e}(t) = \mathbf{p}_{2e}(t) + \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{J}_i\mathbf{v}(t), \tag{5.3}$$

where

$$\mathbf{A}_{avi} = \mathbf{A}_{a21i} - \mathbf{J}_i\mathbf{A}_{a11i} + (\mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i})\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{J}_j, \tag{5.4}$$

$$\mathbf{A}_{aei} = \mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i}, \tag{5.5}$$

and $\mathbf{J}_i \in \mathbb{R}^{(n-m)\times m},\ i = 1, 2, \ldots, s$ *is the set of gains.*

Proof. Since (3.13) can be partitioned as

$$\mathbf{A}_{a12i}\mathbf{q}_{a2}(t) = \dot{\mathbf{q}}_{a1i}(t) - \mathbf{A}_{a11i}\mathbf{v}(t) - \mathbf{B}_{a1i}\mathbf{u}(t), \tag{5.6}$$

$$\dot{\mathbf{q}}_{a2i}(t) = \mathbf{A}_{a21i}\mathbf{v}(t) + \mathbf{A}_{a22i}\mathbf{q}_{a2}(t) + \mathbf{B}_{a2i}\mathbf{u}(t), \tag{5.7}$$

then the TS fuzzy observer can be defined as follows:

$$\dot{\mathbf{q}}_{a2e}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{q}^{\bullet}_{a2ei}(t), \tag{5.8}$$

$$\mathbf{q}^{\bullet}_{a2ei}(t) = \mathbf{A}_{a21i}\mathbf{v}(t) + \mathbf{A}_{a22i}\mathbf{q}_{a2e}(t) + \mathbf{B}_{a2i}\mathbf{u}(t) + \mathbf{J}_i(\dot{\mathbf{q}}_{a1i}(t) - \mathbf{A}_{a11i}\mathbf{v}(t) - \mathbf{B}_{a1i}\mathbf{u}(t) - \mathbf{A}_{a12i}\mathbf{q}_{a2e}(t)), \tag{5.9}$$

where $\mathbf{q}_{a2e}(t) \in \mathbb{R}^{n-m}$ is an estimation of the unmeasurable part of system state vector, and $\mathbf{J}_i,\ i = 1, 2, \ldots, s,\ \mathbf{J}_i \in \mathbb{R}^{(n-m)\times m}$, is the set of the observer gain matrices. Now, (5.8), (5.9) can be rewritten as

$$\dot{\mathbf{q}}_{a2e}(t) - \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{J}_i\dot{\mathbf{q}}_{a1i}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{q}^{\diamond}_{a2i}(t), \tag{5.10}$$

$$\begin{aligned}\mathbf{q}^{\diamond}_{a2i}(t) = {}& \mathbf{B}_{a2i}\mathbf{u}(t) + \mathbf{A}_{a21i}\mathbf{v}(t) + \mathbf{A}_{a22i}\left(\mathbf{q}_{a2e}(t) - \textstyle\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{J}_j\mathbf{v}(t) + \textstyle\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{J}_j\mathbf{v}(t)\right) \\ & + \mathbf{J}_i\begin{pmatrix} -\mathbf{A}_{a11i}\mathbf{v}(t) - \mathbf{B}_{a1i}\mathbf{u}(t) \\ -\mathbf{A}_{a12i}\left(\mathbf{q}_{a2e}(t) - \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{J}_j\mathbf{v}(t) + \sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{J}_j\mathbf{v}(t)\right)\end{pmatrix}.\end{aligned} \tag{5.11}$$

Defining the new state variable

$$\mathbf{p}_{2e}(t) = \mathbf{q}_{a2e}(t) - \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{J}_i\mathbf{v}(t), \tag{5.12}$$

then (5.12) implies (5.3), and defining the left side of (5.10) as

$$\dot{\mathbf{p}}_{2e}(t) = \dot{\mathbf{q}}_{a2e}(t) - \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{J}_i\dot{\mathbf{q}}_{a1i}(t), \tag{5.13}$$

it can be obtained

$$\dot{\mathbf{p}}_{2e}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{q}^{\circ}_{2ei} \tag{5.14}$$

$$\begin{aligned}\mathbf{q}^{\circ}_{2ei} = {}& (\mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i})\mathbf{p}_{2e}(t) + (\mathbf{B}_{a2i} - \mathbf{J}_i\mathbf{B}_{a1i})\mathbf{u}(t) \\ & + (\mathbf{A}_{a21i} - \mathbf{J}_i\mathbf{A}_{a11i})\mathbf{v}(t) + (\mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i})\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{J}_j\mathbf{v}(t).\end{aligned} \tag{5.15}$$

Since

$$\mathbf{B}_{a2i} - \mathbf{J}_i\mathbf{B}_{a1i} = \begin{bmatrix} -\mathbf{J}_i & \mathbf{I}_{n-m} \end{bmatrix}\mathbf{B}_{ai} \tag{5.16}$$

with (5.4), (5.5), and (5.16) then (5.14), (5.15) implies (5.1), (5.2).

It is evident that

$$\mathbf{v}(t) = \mathbf{q}_{a1}(t) = \mathbf{p}_1(t). \tag{5.17}$$

This concludes the proof. □

Theorem 5.2 (reducer-order TS fuzzy observer stability). *The reduced-order TS fuzzy observer* (5.1), (5.2) *is asymptotically stable if there exist a symmetric positive definite matrix* $\mathbf{P}^{\circ} \in \mathbb{R}^{(n-m)\times(n-m)}$ *and matrices* $\mathbf{Z}_i^{\circ} \in \mathbb{R}^{(n-m)\times m}$, $i = 1, 2, \dots, s$ *such that*

$$\mathbf{P}^{\circ} = \mathbf{P}^{\circ T} > 0, \tag{5.18}$$

$$\mathbf{A}_{a22i}^{T}\mathbf{P}^{\circ} + \mathbf{P}^{\circ}\mathbf{A}_{a22i} - \mathbf{A}_{a12i}^{T}\mathbf{Z}_i^{\circ T} - \mathbf{Z}_i^{\circ}\mathbf{A}_{a12i} < 0. \tag{5.19}$$

If the above conditions hold, the set of the observer gain matrices is given as

$$\mathbf{J}_i = \left(\mathbf{P}^{\circ}\right)^{-1}\mathbf{Z}_i^{\circ}. \tag{5.20}$$

Proof. Using (5.1), (5.2), it can be rewritten as

$$\dot{\mathbf{p}}_{2e}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\left(\mathbf{A}_{aei}\mathbf{p}_{2e}(t) + \mathbf{A}_{avi}\mathbf{v}(t) + \left[-\mathbf{J}_i \quad \mathbf{I}_{n-m}\right]\mathbf{B}_{ai}\mathbf{u}(t)\right), \tag{5.21}$$

and, with (5.4), the autonomous part of (5.21) takes the form

$$\dot{\mathbf{p}}_{2e}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i})\mathbf{p}_{2e}(t). \tag{5.22}$$

Defining the quadratic positive definite Lyapunov function:

$$v(\mathbf{p}_{2e}(t)) = \mathbf{p}_{2e}^{T}(t)\mathbf{P}^{\circ}\mathbf{p}_{2e}(t), \tag{5.23}$$

where $\mathbf{P}^{\circ} = \mathbf{P}^{\circ T} > 0$, $\mathbf{P}^{\circ} \in \mathbb{R}^{(n-m)\times(n-m)}$ then, after evaluation of derivative with respect to t, it is obtained

$$\dot{v}(\mathbf{p}_{2e}(t)) = \mathbf{p}_{2e}^{T}(t)\mathbf{P}^{\circ}\dot{\mathbf{p}}_{2e}(t) + \dot{\mathbf{p}}_{2e}^{T}(t)\mathbf{P}^{\circ}\mathbf{p}_{2e}(t) < 0. \tag{5.24}$$

Substituting (5.22) into (5.24) gives

$$\dot{v}(\mathbf{p}_{2e}(t)) = \mathbf{p}_{2e}^{T}(t)\mathbf{P}^{\circ}\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{A}_{aei}\mathbf{p}_{2e}(t) + \mathbf{p}_{2e}^{T}(t)\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{A}_{aei}^{T}\mathbf{P}^{\circ}\mathbf{p}_{2e}(t) < 0, \tag{5.25}$$

$$\dot{v}(\mathbf{p}_{2e}(t)) = \mathbf{p}_{2e}^{T}(t)\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\left(\mathbf{P}^{\circ}\mathbf{A}_{aei} + \mathbf{A}_{aei}^{T}\mathbf{P}^{\circ}\right)\mathbf{p}_{2e}(t) < 0, \tag{5.26}$$

respectively. Thus, (5.26) is negative, if there exist a set of matrices $\mathbf{J}_i$, $i = 1, 2, \dots, s$, and a matrix $\mathbf{P}^{\circ}$ such that

$$(\mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i})^{T}\mathbf{P}^{\circ} + \mathbf{P}^{\circ}(\mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i}) < 0, \quad \forall i. \tag{5.27}$$

Setting

$$\mathbf{P}^{\circ}\mathbf{J}_i = \mathbf{Z}_i^{\circ} \tag{5.28}$$

(5.27) implies (5.19). This concludes the proof. □

Remark 5.3. If (5.19) is infeasible, then it can be set

$$\mathbf{A}_{a22i}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{a22i} - \mathbf{A}_{a12i}^T\mathbf{Z}_i^T - \mathbf{Z}_i\mathbf{A}_{a12i} < -\mathbf{Q}, \tag{5.29}$$

where $\mathbf{Q} = \mathbf{Q}^T > 0$, $\mathbf{Q} \in \mathbb{R}^{(n-m)\times(n-m)}$ is a symmetric positive definite matrix. It is obvious, based on the inequality (5.29), such design condition implies inherently the more conservative solution.

Theorem 5.4 (stability condition equivalency). *The asymptotic stability condition of the autonomous part of* (5.22) *is the same as the asymptotic stability condition of the error reference model as follows:*

$$\dot{\mathbf{e}}_{aq2}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i})\mathbf{e}_{aq2}(t), \tag{5.30}$$

where the estimation error of the unmeasurable part of state variables is

$$\mathbf{e}_{aq2}(t) = \mathbf{q}_{a2}(t) - \mathbf{q}_{a2e}(t). \tag{5.31}$$

Proof. Substituting (5.3) in (5.31) gives

$$\mathbf{e}_{a2}(t) = \mathbf{q}_{a2}(t) - \mathbf{p}_{2e}(t) - \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{J}_i\mathbf{v}(t). \tag{5.32}$$

Defining, in analogy with (5.1)–(5.3), the reference variable $\mathbf{p}_2(t)$ is as follows:

$$\mathbf{p}_2(t) = \mathbf{q}_{a2}(t) - \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{J}_i\mathbf{v}(t), \tag{5.33}$$

$$\dot{\mathbf{p}}_2(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{q}_{2i}^{\circ}(t), \tag{5.34}$$

$$\mathbf{q}_{2i}^{\circ}(t) = \mathbf{A}_{aei}\mathbf{p}_2(t) + \mathbf{A}_{avi}\mathbf{v}(t) + \begin{bmatrix}-\mathbf{J}_i & \mathbf{I}_{n-m}\end{bmatrix}\mathbf{B}_{ai}\mathbf{u}(t), \tag{5.35}$$

and substituting (5.33) in (5.32) gives

$$\mathbf{e}_{aq2}(t) = \mathbf{p}_2(t) - \mathbf{p}_{2e}(t) = \mathbf{e}_{p2}(t), \tag{5.36}$$

$$\dot{\mathbf{e}}_{aq2}(t) = \dot{\mathbf{e}}_{p2}(t) = \dot{\mathbf{p}}_2(t) - \dot{\mathbf{p}}_{2e}(t), \tag{5.37}$$

respectively. Thus, inserting (5.1), (5.2) and (5.34), (5.35) into (5.37) results in

$$\dot{\mathbf{e}}_{aq2}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{A}_{aei}(\mathbf{p}_2(t) - \mathbf{p}_{2e}(t)), \tag{5.38}$$

and with (5.5), (5.36) then (5.38) implies (5.30).

Since (5.22) and (5.36) are associated with the same system matrix $\mathbf{A}_{aei}$, this concludes the proof. □

Note, the form of the time derivative (5.13) is given by the definition.

Corollary 5.5 (error reference model). *The equalities* (5.17), (5.33) *can be compactly written as*

$$\begin{bmatrix}\mathbf{p}_1(t)\\ \mathbf{p}_2(t)\end{bmatrix} = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{T}_i(\mathbf{A}_{ai}\mathbf{q}_{ai}(t) + \mathbf{B}_{ai}\mathbf{u}(t)), \tag{5.39}$$

where

$$\begin{gathered}\mathbf{T}_i = \begin{bmatrix}\mathbf{I}_m & \mathbf{0}\\ -\mathbf{J}_i & \mathbf{I}_{n-m}\end{bmatrix},\\ \mathbf{T}_i\mathbf{B}_{ai}\mathbf{u}(t) = \begin{bmatrix}\mathbf{I}_m & \mathbf{0}\\ -\mathbf{J}_i & \mathbf{I}_{n-m}\end{bmatrix}\begin{bmatrix}\mathbf{B}_{a1i}\\ \mathbf{B}_{a2i}\end{bmatrix}\mathbf{u}(t),\\ \mathbf{T}_i\mathbf{A}_{ai}\mathbf{q}_a(t) = \begin{bmatrix}\mathbf{A}_{a11i} & \mathbf{A}_{a12i}\\ \mathbf{A}_{a21i} - \mathbf{J}_i\mathbf{A}_{a11i} & \mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i}\end{bmatrix}\begin{bmatrix}\mathbf{v}(t)\\ \mathbf{q}_{a2}(t)\end{bmatrix}.\end{gathered} \tag{5.40}$$

Ones explaining the variable $\mathbf{p}_2(t)$ *as a function of* $\mathbf{q}_{a2}(t)$ *using* (5.39)*, then the time derivative of* $\mathbf{p}_2(t)$ *can be obtained by the substitution* (5.33)*, that is,*

$$\dot{\mathbf{p}}_2(t) = \mathbf{p}_2(t)(\mathbf{q}_{a2}(t))_{\|\mathbf{q}_{a2}(t)\equiv\mathbf{p}_2(t)+\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t))\mathbf{J}_j\mathbf{v}(t)}. \tag{5.41}$$

Evidently, (5.39)–(5.41) can be adequately exploited to obtain the time derivative $\dot{\mathbf{p}}_{2e}(t)$ in the dependency on $\mathbf{p}_{2e}(t)$, $\mathbf{v}(t)$, and $\mathbf{u}(t)$.

Using the equivalency of the stability conditions, an actuator fault estimation structure based on reduced-order TS fuzzy observer can be discussed.

6. Estimation of Actuator Faults

To obtain an actuator fault estimation structure based on reduced-order TS fuzzy observer, the matching condition (3.21) has to be satisfied. This implies that a special observer gain matrices have to be chosen.

Theorem 6.1. *The estimation error dynamics of the reduced-order TS fuzzy observer* (5.1), (5.2) *is not affected by actuator faults if, with the matching condition* (3.21)*, there exists a symmetric positive definite matrix* $\mathbf{P}^\circ \in \mathbb{R}^{(n-m)\times(n-m)}$ *such that*

$$\mathbf{P}^\circ \mathbf{J} = \mathbf{B}_{af1}^{\perp}, \tag{6.1}$$

where $\mathbf{B}_{af1}^{\perp}$ *is the orthogonal complement to* $\mathbf{B}_{af1}$*, and* $\mathbf{J}_i = \mathbf{J}$ *for all* $i = 1, 2, \ldots, s$*.*

Proof. The system with an actuator fault is described as

$$\dot{\mathbf{q}}_{afi}(t) = \mathbf{A}_{ai}\mathbf{q}_{afi}(t) + \mathbf{B}_{ai}\mathbf{u}(t) + \mathbf{B}_{af}\mathbf{u}_f(t). \tag{6.2}$$

Since (3.13), (5.1), (5.2), and (5.34), (5.35) now implies

$$\mathbf{B}_{af} = \mathbf{C}_a^T \mathbf{B}_{af1} = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{0} \end{bmatrix} \mathbf{B}_{af1} = \begin{bmatrix} \mathbf{B}_{af1} \\ \mathbf{0} \end{bmatrix}, \tag{6.3}$$

$$\begin{aligned} \dot{\mathbf{p}}_{2e}(t) &= \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{q}_{2efi}^\circ(t), \\ \dot{\mathbf{p}}_{2}(t) &= \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{q}_{2i}^\circ(t), \end{aligned} \tag{6.4}$$

$$\mathbf{q}_{2efi}^\circ(t) = \mathbf{A}_{aei}\mathbf{p}_{2e}(t) + \mathbf{A}_{avi}\mathbf{v}(t) + \left[-\mathbf{J}_i \ \ \mathbf{I}_{n-m}\right]\left(\mathbf{B}_{ai}\mathbf{u}(t) + \mathbf{B}_{af}\mathbf{u}_f(t)\right), \tag{6.5}$$

$$\mathbf{q}_{2i}^\circ(t) = \mathbf{A}_{aei}\mathbf{p}_{2}(t) + \mathbf{A}_{avi}\mathbf{v}(t) + \left[-\mathbf{J}_i \ \ \mathbf{I}_{n-m}\right]\mathbf{B}_{ai}\mathbf{u}(t), \tag{6.6}$$

the dynamics of the error (5.37) can be rewritten as

$$\dot{\mathbf{e}}_{aq2}(t) = \dot{\mathbf{p}}_2(t) - \dot{\mathbf{p}}_{2e}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\mathbf{A}_{aei}\mathbf{e}_{aq2}(t) + \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\left[-\mathbf{J}_i \ \ \mathbf{I}_{n-m}\right]\mathbf{B}_{af}\mathbf{u}_f(t) \tag{6.7}$$

Defining the quadratic positive definite Lyapunov function as follows:

$$v\left(\mathbf{e}_{aq2}(t)\right) = \mathbf{e}_{aq2}^T(t)\mathbf{P}^\circ\mathbf{e}_{aq2}(t), \tag{6.8}$$

where $\mathbf{P}^\circ = \mathbf{P}^{\circ T} > 0$, $\mathbf{P}^\circ \in \mathbb{R}^{(n-m)\times(n-m)}$, after evaluation of derivative of (6.7) with respect to t it is obtained

$$\dot{v}\left(\mathbf{e}_{aq2}(t)\right) = \dot{\mathbf{e}}_{aq2}^T(t)\mathbf{P}^\circ\mathbf{e}_{aq2}(t) + \mathbf{e}_{aq2}^T(t)\mathbf{P}^\circ\dot{\mathbf{e}}_{aq2}(t). \tag{6.9}$$

From the expression (6.7) it follows that

$$\begin{aligned}\dot{v}\left(\mathbf{e}_{aq2}(t)\right) = {}& \mathbf{e}_{aq2}^{T}(t)\mathbf{P}^{\circ}\sum_{i=1}^{s}h_i(\boldsymbol{\theta}(t))\mathbf{A}_{aei}\mathbf{e}_{aq2}(t) + \mathbf{e}_{aq2}^{T}(t)\sum_{i=1}^{s}h_i(\boldsymbol{\theta}(t))\mathbf{A}_{aei}^{T}\mathbf{P}^{\circ}\mathbf{e}_{a2}(t) \\ & + \mathbf{e}_{aq2}^{T}(t)\mathbf{P}^{\circ}\sum_{i=1}^{s}h_i(\boldsymbol{\theta}(t))\left[-\mathbf{J}_i \ \ \mathbf{I}_{n-m}\right]\mathbf{B}_{af}\mathbf{u}_f(t) \\ & + \mathbf{u}_f^{T}(t)\sum_{i=1}^{s}h_i(\boldsymbol{\theta}(t))\mathbf{B}_{af}^{T}\left[-\mathbf{J}_i \ \ \mathbf{I}_{n-m}\right]^{T}\mathbf{P}^{\circ}\mathbf{e}_{aq2}^{T}(t),\end{aligned} \tag{6.10}$$

and with respect to the matching condition (3.21), it can be set

$$\mathbf{P}^{\circ}\left[-\mathbf{J}_i \ \ \mathbf{I}_{n-m}\right]\mathbf{B}_{af} = \mathbf{P}^{\circ}\left[-\mathbf{J}_i\mathbf{B}_{af1} \ \ \mathbf{0}_{n-m}\right] = \mathbf{0}, \tag{6.11}$$

which results in the equality:

$$\mathbf{P}^{\circ}\mathbf{J}_i\mathbf{B}_{af1} = \mathbf{0}, \quad \forall i. \tag{6.12}$$

Evidently, (6.12) can be satisfied if and only if $\mathbf{J}_i = \mathbf{J}$ for all i. With such $\mathbf{J}$, the equality (6.12) will be satisfied if (6.1) is satisfied. This concludes the proof. □

Evidently, $\mathbf{B}_{af1}$ may not be a square matrix.

Theorem 6.2. *The estimation error dynamic* (6.7) *is asymptotically stable, if there exists a symmetric positive definite* matrix $\mathbf{P}^{\circ} \in \mathbb{R}^{(n-m)\times(n-m)}$ *such that for* $i = 1, 2, \ldots, s$

$$\mathbf{P}^{\circ} = \mathbf{P}^{\circ T} > 0, \tag{6.13}$$

$$\mathbf{A}_{a22i}^{T}\mathbf{P}^{\circ} + \mathbf{P}^{\circ}\mathbf{A}_{a22i} - \mathbf{A}_{a12i}^{T}\mathbf{B}_{af1}^{\perp T} - \mathbf{B}_{af1}^{\perp}\mathbf{A}_{a12i} < 0, \tag{6.14}$$

where $\mathbf{B}_{af1}^{\perp}$ *is the orthogonal complement to* $\mathbf{B}_{af1}$.

If the above conditions hold, the observer gain matrix is given as

$$\mathbf{J} = \left(\mathbf{P}^{\circ}\right)^{-1}\mathbf{B}_{af1}^{\perp}. \tag{6.15}$$

Proof. Satisfying (6.11) then (6.10) implies

$$\dot{v}\left(\mathbf{e}_{aq2}(t)\right) = \mathbf{e}_{aq2}^{T}(t)\sum_{i=1}^{s}h_i(\boldsymbol{\theta}(t))\left(\mathbf{P}^{\circ}\mathbf{A}_{aei} + \mathbf{A}_{aei}^{T}\mathbf{P}^{\circ}\right)\mathbf{e}_{aq2}(t) < 0, \tag{6.16}$$

where $\mathbf{A}_{aei}$ is defined in (5.5). It is evident that (6.16) is negative if for all i

$$(\mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i})^T\mathbf{P}^\circ + \mathbf{P}^\circ(\mathbf{A}_{a22i} - \mathbf{J}_i\mathbf{A}_{a12i}) < 0. \tag{6.17}$$

Using (6.1) then (6.17) implies (5.19). This concludes the proof. □

Corollary 6.3. *Using* (5.20)*, then one has*

$$\begin{bmatrix} -\mathbf{J} & \mathbf{I}_{n-m} \end{bmatrix}\mathbf{B}_{af} = (\mathbf{P}^\circ)^{-1}\begin{bmatrix} -\mathbf{B}_{af1}^{\perp} & \mathbf{P}^\circ \end{bmatrix}\mathbf{C}^T\mathbf{B}_{af1} = (\mathbf{P}^\circ)^{-1}\begin{bmatrix} -\mathbf{B}_{af1}^{\perp}\mathbf{B}_{af1} & \mathbf{0} \end{bmatrix} = \mathbf{0}. \tag{6.18}$$

Equality given above implies that neither estimation error (6.7)*, nor reduced-order TS fuzzy observer equation* (6.5) *is affected by actuator faults.*

Corollary 6.4. *Since* $\mathbf{J}_i = \mathbf{J}$ *for all* i *and* $\sum_{j=1}^{s} h_j(\boldsymbol{\theta}(t)) = 1$, (5.12) *and* (5.19) *take the form*

$$\mathbf{p}_{2e}(t) = \mathbf{q}_{a2e}(t) - \mathbf{J}\mathbf{v}(t), \tag{6.19}$$

$$\mathbf{A}_{avi} = \mathbf{A}_{a21i} - \mathbf{J}\mathbf{A}_{a11i} + (\mathbf{A}_{a22i} - \mathbf{J}\mathbf{A}_{a12})\mathbf{J}. \tag{6.20}$$

Remark 6.5. If $\mathbf{B}_{ai} = \mathbf{B}_a = \mathbf{B}_{af}$ for all i, (6.18) implies that the reduced-order TS fuzzy observer will be independent on the input $\mathbf{u}(t)$ and will exploit only the vector variable $\mathbf{v}(t)$.

Considering the fact that the reduced-order TS fuzzy observer does not contain any information about actuator faults, the next reconstruction principle can be used.

Theorem 6.6. *Designed with respect to* $\mathbf{P}^\circ$ *satisfying* (5.18)–(5.20)*, the reduced-order TS fuzzy observer* (5.1), (5.2) *asymptotically estimates actuator faults.*

Proof. Since (3.11), (3.14) implies

$$\mathbf{q}_e(t) = \mathbf{T}_a^T\mathbf{q}_{ae}(t) = \mathbf{T}_{a2}^T\mathbf{q}_{a2e}(t) + \mathbf{T}_{a1}^T\mathbf{v}(t), \tag{6.21}$$

substituting (6.19) in (6.21) leads to

$$\mathbf{q}_e(t) = \mathbf{T}_{a2}^T\mathbf{p}_{2e}(t) + \left(\mathbf{T}_{a1}^T + \mathbf{T}_{a2}^T\mathbf{J}\right)\mathbf{W}\mathbf{y}(t), \tag{6.22}$$

$$\dot{\mathbf{q}}_e(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\mathbf{A}_i\mathbf{q}_e(t) + \mathbf{B}_i\mathbf{u}(t)) + \mathbf{B}_f\mathbf{u}_f(t). \tag{6.23}$$

Thus, using Moore-Penrose pseudoinverse $\mathbf{B}_f^{\ominus 1}$ of $\mathbf{B}_f$,

$$\mathbf{u}_{fe}(t) = \mathbf{B}_f^{\ominus 1}\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\dot{\mathbf{q}}_e(t) - \mathbf{A}_i\mathbf{q}_e(t) - \mathbf{B}_i\mathbf{u}(t)). \tag{6.24}$$

Explaining $\mathbf{u}_f(t)$ as follows:

$$\mathbf{u}_f(t) = \mathbf{B}_f^{\ominus 1}\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\dot{\mathbf{q}}(t) - \mathbf{A}_i\mathbf{q}_e(t) - \mathbf{B}_i\mathbf{u}(t)), \tag{6.25}$$

then for $\mathbf{e}_{uf}(t) = \mathbf{u}_f(t) - \mathbf{u}_{fe}(t)$ it yields

$$\mathbf{e}_{uf}(t) = \mathbf{B}_f^{\ominus 1}\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\left(\dot{\mathbf{e}}_q(t) - \mathbf{A}_i\mathbf{e}_q(t)\right). \tag{6.26}$$

Owing to that reduced-order observer is stable, $\mathbf{e}_{uf}(t)$ converges asymptotically to zero. This concludes the proof. □

Remark 6.7. Taking the actuator fault reconstructor as given by (6.24), it is necessary to note that $\dot{\mathbf{q}}_e(t)$ has to be computed numerically from (6.22), since (5.7) is affected by actuator faults and so cannot be used to the first state vector component derivative evaluation.

Note, matrix pseudoinverse in (6.24) is the reason that $\mathbf{B}_f$ has to be the same in all linear submodels of (2.4).

7. Illustrative Example

Referring to [32], a nonlinear hydrostatic transmission system is considered in this section for simulating the real environment. The proposed design method is applied to design an actuator fault estimation scheme based on a reduced-order fuzzy observer using TS model of this model. The hydrostatic transmission system is represented by the nonlinear state-space model of the form

$$\begin{aligned} \dot{q}_1(t) &= -a_{11}q_1(t) + b_{11}u_1(t) \\ \dot{q}_2(t) &= -a_{22}q_2(t) + b_{22}u_2(t), \\ \dot{q}_3(t) &= a_{31}q_1(t)p(t) - a_{33}q_3(t) - a_{34}q_2(t)q_4(t), \\ \dot{q}_4(t) &= a_{43}q_2(t)q_3(t) - a_{44}q_4(t), \end{aligned} \tag{7.1}$$

where $q_1(t)$ is the normalized hydraulic pump angle, $q_2(t)$ is the normalized hydraulic motor angle, $q_3(t)$ is the pressure difference [bar], $q_4(t)$ is the hydraulic motor speed [rad/s], $u_1(t)$ is the normalized control signal of the hydraulic pump, $u_2(t)$ is the normalized control signal of the hydraulic motor, and the external signal $p(t)$ represents speed of hydraulic pump [rad/s]. It is supposed that the external variable $p(t)$ and all state variables except $q_3(t)$ are measurable and the model parameters are

$$\begin{aligned}
a_{11} &= 7.6923, & a_{22} &= 4.5455, & a_{33} &= 7.6054 * 10^{-4}, \\
a_{31} &= 0.7877, & a_{34} &= 0.9235, & b_{11} &= 1.8590 * 10^{3}, \\
a_{43} &= 12.1967, & a_{44} &= 0.4143, & b_{22} &= 1.2879 * 10^{3}.
\end{aligned} \tag{7.2}$$

Since the variables $p(t) \in \langle 105, 300\rangle$ and $q_2(t) \in \langle 0, 1\rangle$ are bounded on the prescribed sectors, the vector of premise variables was chosen as follows:

$$\boldsymbol{\theta}(t) = \begin{bmatrix}\theta_1(t) & \theta_2(t)\end{bmatrix} = \begin{bmatrix}q_2(t) & p(t)\end{bmatrix}, \tag{7.3}$$

where the set of nonlinear sector functions:

$$\begin{aligned}
w_{11}(q_2(t)) &= \frac{b_1 - q_2(t)}{b_1 - b_2}, \quad w_{12}(q_2(t)) = 1 - w_{11}(q_2(t)), \quad b_1 = 0, \; b_2 = 1, \\
w_{21}(p(t)) &= \frac{c_1 - p(t)}{c_1 - c_2}, \quad w_{22}(p(t)) = 1 - w_{21}(p(t)), \quad c_1 = 105, \quad c_2 = 300
\end{aligned} \tag{7.4}$$

implies the next set of normalized membership functions:

$$\begin{aligned}
h_1(\boldsymbol{\theta}(t)) &= w_{11}(q_2(t))w_{21}(p(t)), & h_2(\boldsymbol{\theta}(t)) &= w_{11}(q_2(t))w_{22}(p(t)), \\
h_3(\boldsymbol{\theta}(t)) &= w_{12}(q_2(t))w_{21}(p(t)), & h_4(\boldsymbol{\theta}(t)) &= w_{12}(q_2(t))w_{22}(p(t)).
\end{aligned} \tag{7.5}$$

The overall TS fuzzy model (2.4), (2.5) with an actuator fault is represented as follows:

$$\begin{aligned}
\dot{\mathbf{q}}(t) &= \sum_{i=1}^{4} h_i(\boldsymbol{\theta}(t))(\mathbf{A}_i\mathbf{q}(t)) + \mathbf{B}(\mathbf{u}(t) + \mathbf{f}(t)), \\
\mathbf{y}(t) &= \mathbf{C}\mathbf{q}(t),
\end{aligned} \tag{7.6}$$

where

$$
\begin{gathered}
\mathbf{A}_1 = \begin{bmatrix} -7.6923 & 0 & 0 & 0 \\ 0 & -4.5455 & 0 & 0 \\ 82.7086 & 0 & -0.0008 & 0 \\ 0 & 0 & 0 & -0.4143 \end{bmatrix}, \\
\mathbf{A}_2 = \begin{bmatrix} -7.6923 & 0 & 0 & 0 \\ 0 & -4.5455 & 0 & 0 \\ 236.3103 & 0 & -0.0008 & 0 \\ 0 & 0 & 0 & -0.4143 \end{bmatrix}, \\
\mathbf{A}_3 = \begin{bmatrix} -7.6923 & 0 & 0 & 0 \\ 0 & -4.5455 & 0 & 0 \\ 82.7086 & 0 & -0.0008 & -0.9235 \\ 0 & 0 & 12.1967 & -0.4143 \end{bmatrix}, \\
\mathbf{A}_4 = \begin{bmatrix} -7.6923 & 0 & 0 & 0 \\ 0 & -4.5455 & 0 & 0 \\ 236.3103 & 0 & -0.0008 & -0.9235 \\ 0 & 0 & 12.1967 & -0.4143 \end{bmatrix}, \\
\mathbf{B}_i = \mathbf{B} = \mathbf{B}_f = \begin{bmatrix} 1.8590 & 0 \\ 0 & 1.2879 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \times 10^3, \quad i = 1,2,3,4, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{gathered}
\tag{7.7}
$$

Note that rank($\mathbf{C}$) > rank($\mathbf{B}$) and Proposition 3.5 are satisfied.

Using SVD of the output matrix $\mathbf{C}$, (3.10) implies

$$
\begin{gathered}
\mathbf{U} = \mathbf{S} = \mathbf{W}^{-1} = \mathbf{I}_3, \qquad \mathbf{V} = \mathbf{T}_a = \operatorname{diag}\left[\mathbf{I}_2 \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
\mathbf{T}_{a1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{T}_{a2} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},
\end{gathered}
\tag{7.8}
$$

and with such defined $\mathbf{T}_a$, for $i = 1, 2, 3, 4$, it yields

$$\mathbf{C}_a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B}_{a1} = \begin{bmatrix} 1.8590 & 0 \\ 0 & 1.2879 \\ 0 & 0 \end{bmatrix} \times 10^3, \qquad \mathbf{B}_{a2} = \begin{bmatrix} 0 & 0 \end{bmatrix},$$
$$\mathbf{A}_{a211} = \begin{bmatrix} 82.7086 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_{a212} = \begin{bmatrix} 236.3103 & 0 & 0 \end{bmatrix},$$
$$\mathbf{A}_{a213} = \begin{bmatrix} 82.7086 & 0 & -0.9235 \end{bmatrix}, \qquad \mathbf{A}_{a214} = \begin{bmatrix} 236.3103 & 0 & -0.9235 \end{bmatrix},$$
$$\mathbf{A}_{a121} = \mathbf{A}_{a122} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{A}_{a123} = \mathbf{A}_{a124} = \begin{bmatrix} 0 \\ 0 \\ 12.1967 \end{bmatrix}, \tag{7.9}$$
$$\mathbf{A}_{a11i} = \begin{bmatrix} -7.6923 & 0 & 0 \\ 0 & -4.5455 & 0 \\ 0 & 0 & -0.4143 \end{bmatrix}, \quad \mathbf{A}_{a22i} = \begin{bmatrix} -0.0008 \end{bmatrix}, \quad i = 1, 2, 3, 4.$$

Considering the conditions given in Theorem 6.1, the reduced-order observer (5.1)–(5.5) takes now the form

$$\dot{\mathbf{p}}_{2e}(t) = \sum_{i=1}^{4} h_i(\boldsymbol{\theta}(t))\left(\mathbf{A}_{aei}\mathbf{p}_{2e}(t) + \mathbf{A}_{avi}\mathbf{v}(t) + \begin{bmatrix} -\mathbf{J} & \mathbf{I}_{n-m} \end{bmatrix}\mathbf{B}_{ai}\mathbf{u}(t)\right), \tag{7.10}$$

where

$$\mathbf{A}_{avi} = \mathbf{A}_{a21i} - \mathbf{J}\mathbf{A}_{a11i} + (\mathbf{A}_{a22i} - \mathbf{J}\mathbf{A}_{a12i})\mathbf{J}, \qquad \mathbf{A}_{aei} = \mathbf{A}_{a22i} - \mathbf{J}\mathbf{A}_{a12i}, \tag{7.11}$$

and $\mathbf{J} \in \mathbb{R}^{1 \times 3}$ is given by (6.15) as follows:

$$\mathbf{J} = (\mathbf{P}^{\circ})^{-1}\mathbf{B}_{a1}^{\perp}, \tag{7.12}$$

where $\mathbf{B}_{a1}^{\perp}$ is an orthogonal complement to $\mathbf{B}_{a1}$.

The scalar LMI variable P° can be found by using the convex optimization techniques if $\mathbf{B}_{a1}^{\perp}$ is defined as a structured LM variable of the form

$$\mathbf{B}_{a1}^{\perp} = \begin{bmatrix} 0 & 0 & Z \end{bmatrix}, \quad Z \in \mathbb{R}, \tag{7.13}$$

where Z is an LMI variable. Note that a structured matrix variable can be specified only by including LMI matrix variables multiplied by a natural number or zero.

Thus, solving (6.13), (6.14) with respect to the LMI variables P°, Z using Self-Dual-Minimization (SeDuMi) package for Matlab [33], the reduced observer gain design problem was feasible with the results

$$\begin{gathered}
P^{\circ} = 1.0832, \qquad \mathbf{B}_{a1}^{\perp} = \begin{bmatrix} 0 & 0 & 0.0410 \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} 0 & 0 & 0.0378 \end{bmatrix}, \\
\mathbf{A}_{ae2} = \mathbf{A}_{ae1} = \mathbf{A}_{a22} - \mathbf{J}\mathbf{A}_{a121} = -7.6053 \times 10^{-4}, \\
\mathbf{A}_{ae4} = \mathbf{A}_{ae3} = \mathbf{A}_{b22} - \mathbf{J}\mathbf{A}_{a123} = -0.4610.
\end{gathered} \tag{7.14}$$

It is evident that the design of the stable reduced-order observer with suppressed input fault signals is now completely specified, and the system state can be reconstructed, using (6.22), from the estimated vector $\mathbf{p}_{2e}(t)$ and the output vector $\mathbf{y}(t)$ as

$$\mathbf{q}_e(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{p}_{2e}(t) + \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0.0378 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{y}(t). \tag{7.15}$$

Evidently, the final form of the state reconstruction equation is

$$\mathbf{q}_e(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{p}_{2e}(t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.0378 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{y}(t), \tag{7.16}$$

and actuator faults, if were occurred, can be computed by (6.24) as

$$\mathbf{u}_{fe}(t) = \mathbf{B}^{\ominus 1} \sum_{i=1}^{4} h_i(\boldsymbol{\theta}(t)) \left(\dot{\mathbf{q}}_e(t) - \mathbf{A}_i \mathbf{q}_e(t) - \mathbf{B}\mathbf{u}(t) \right), \tag{7.17}$$

where

$$\mathbf{B}^{\ominus 1} = \begin{bmatrix} 0.5379 & 0 & 0 & 0 \\ 0 & 0.7765 & 0 & 0 \end{bmatrix} \times 10^{-3}, \tag{7.18}$$

and $\boldsymbol{\theta}(t)$, $h_i(\boldsymbol{\theta}(t))(\dot{\mathbf{q}}_e(t))$, $\mathbf{A}_i$, and $\mathbf{B}$ are above specified. The derivative of the system state estimation $\dot{\mathbf{q}}_e(t)$ was computed by standard numerical method from the obtained $\mathbf{q}_e(t)$. Since the reduced-order observer is used only, for system without uncertainties no extra computation consumption is needed, comparing, for example, with the sliding mode approach.

For simulation purposes only, the equilibrium of the system was stabilized by the fuzzy feedback controller

$$\mathbf{u}(t) = -\sum_{j=1}^{4} h_j(\boldsymbol{\theta}(t)) \mathbf{K}_j \mathbf{q}(t), \tag{7.19}$$

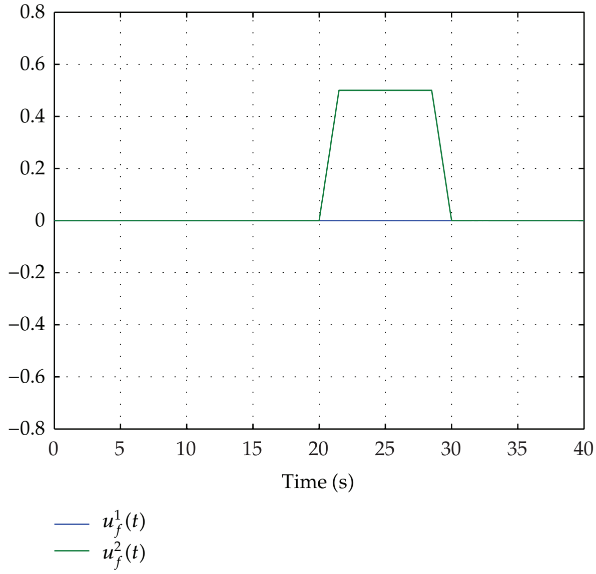

Figure 1: The second actuator fault signal.

where, using the method proposed in [34], offering the possibility to design the linear state controller for TS fuzzy system, the gain matrices were computed as

$$\mathbf{K} = \mathbf{K}_j = \begin{bmatrix} 0.2386 & 0.0000 & 0.0350 & 0.0075 \\ 0.0000 & 0.0207 & 0.0000 & 0.0000 \end{bmatrix}, \quad j = 1,2,3,4. \tag{7.20}$$

In simulations was considered the fault which does not cause closed-loop system instability, modeled by a fault starting at any time instant in the system equilibrium state. Applying the above-designed reduced-order observer-based actuator fault estimation, the fault responses for the nonlinear system are given in Figures 1 and 2. Thus, Figure 1 presents the fault signal reflecting single actuator fault in the the second actuator, starting at the time instant $t = 20\,\text{s}$ and continuing during the time 10 s, and Figure 2 illustrates the signals u_f^1, u_f^2 obtained from (6.26) as a reconstruction of the single fault. Note that equivalent results are obtained for the system working in a forced regime.

From the simulation results of Figures 1 and 2, it can be found that the errors between the signals reflecting a single actuator fault and the observer approximate ones tend to zero. Moreover, the states of the system converge to the equilibrium when the actuator fault disappeared, via the used fuzzy controller.

8. Concluding Remarks

Generalized design method of a reduced-order observer-based actuator fault estimation scheme is developed, as augmentation of unknown observers synthesis for one class of nonlinear systems described by TS fuzzy model. This is achieved by manipulation of observer asymptotic stability with respect to the proposed matching conditions. Design conditions for asymptotic estimation of actuator faults are derived in terms of LMI, using standard

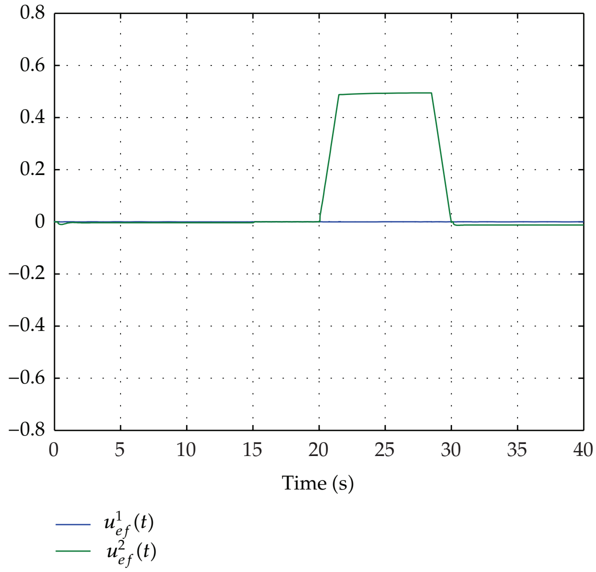

Figure 2: The reconstruction of the actuator fault signal.

LMI procedures to manipulate the reducer-order observer stability. Because of the specific observer gain matrix structure, the estimated unmeasurable part of the system state is free of actuators faults. By examining the estimated state vector, it is presented that using a numerical realization of time derivative of the state vector estimate, the actuator fault signals can be faithfully reconstructed.

Proposed scheme is able to simultaneously estimate the time-varying actuator faults, as well as the system state variables with a good accuracy, is easy to implement, and can be applied to a reasonably wide class of systems satisfying the matching condition. Presented simulations have shown that the proposed design task is feasible and effective.

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