MPE — Volume 2, pp. 333-365
Reprints available directly from the publisher
Photocopying permitted by license only

HIGH FREQUENCY ASYMPTOTIC SOLUTIONS OF THE REDUCED WAVE EQUATION ON INFINITE REGIONS WITH NON-CONVEX BOUNDARIES

CLIFFORD O. BLOOM

Department of Mathematics, S.U.N.Y. at Buffalo, Buffalo, New York 14214

(Received 2 September 1995)

The asymptotic behavior as $\lambda \to \infty$ of the function $U(x, \lambda)$ that satisfies the reduced wave equation $L_\lambda[U] = \nabla\cdot(E(x)\nabla U) + \lambda^2 N^2(x)U = 0$ on an infinite 3-dimensional region, a Dirichlet condition on ∂V, and an outgoing radiation condition is investigated. A function $U_N(x, \lambda)$ is constructed that is a global approximate solution as $\lambda \to \infty$ of the problem satisfied by $U(x, \lambda)$. An estimate for $W_N(x, \lambda) = U(x, \lambda) - U_N(x, \lambda)$ on V is obtained, which implies that $U_N(x, \lambda)$ is a uniform asymptotic approximation of $U(x, \lambda)$ as $\lambda \to \infty$, with an error that tends to zero as rapidly as λ^{-N} ($N = 1, 2, 3, \ldots$). This is done by applying a priori estimates of the function $W_N(x, \lambda)$ in terms of its boundary values, and the L_2 norm of $rL_\lambda[W_N(x, \lambda)]$ on V. It is assumed that $E(x)$, $N(x)$, ∂V and the boundary data are smooth, that $E(x) - I$ and $N(x) - 1$ tend to zero algebraically fast as $r \to \infty$, and finally that $E(x)$ and $N(x)$ are slowly varying; ∂V may be finite or infinite.

The solution $U(x, \lambda)$ can be interpreted as a scalar potential of a high frequency acoustic or electromagnetic field radiating from the boundary of an impenetrable object of general shape. The energy of the field propagates through an inhomogeneous, anisotropic medium; the rays along which it propagates may form caustics. The approximate solution (potential) derived in this paper is defined on and in a neighborhood of any such caustic, and can be used to connect local "geometrical optics" type approximate solutions that hold on caustic free subsets of V.

The result of this paper generalizes previous work of Bloom and Kazarinoff [C. O. BLOOM and N. D. KAZARINOFF, *Short Wave Radiation Problems in Inhomogeneous Media*: *Asymptotic Solutions*, SPRINGER VERLAG, NEW YORK, NY, 1976].

AMS Nos.: primary 35J25; secondary 35B40, 35B45, 78A05, 78A40, 78A45

Keywords: High frequency radiation, scattering, global approximate solution, uniform asymptotic approximation, caustics, geometrical optics, inhomogeneous medium, anisotropic medium, reduced wave equation.

1. INTRODUCTION

In this paper we investigate the asymptotic behavior as $\lambda \to \infty$ of the solution $U(x, \lambda)$ to the following radiating body problem:

$$\textbf{(P)}\quad \begin{aligned} &L_\lambda[U] = \nabla\cdot(E(x)U) + \lambda^2 N^2(x)U = 0 && (x = (x_1, x_2, x_3), x \in V),\\ &U(x', \lambda) = h(x') && (x' \in \partial V),\\ &\lim_{R\to\infty} \int_{r=R} r|D[U]|^2\, dS = 0 && (r = |x| = (x\cdot x)^{1/2}),\end{aligned}$$

where $D[U] = U_r - i\lambda U + r^{-1}U$.

We take V to be either (1) a region exterior to a smooth, simply connected, closed surface ∂V, or (2) a region bounded by a smooth, simply connected surface ∂V that extends to infinity. The function $h(x')$ is required to be smooth on ∂V, and to have compact support on ∂V if ∂V is infinite. We assume that:

H-(i) The 3×3 matrix $E(x)$ is symmetric, strictly positive definite, and smooth on $V \cup \partial V$.
H-(ii) The function $N(x)$ is smooth, and strictly positive on $V \cup \partial V$.
H-(iii) $N(x) - 1 = O(r^{-p})$ and $N(x) - 1 = O(r^{-p})$ for some $p > 2$, uniformly, as $r \rightarrow \infty$.
H-(iv) For $m \geq 1$ all m^{th} order derivatives of $E(x)$ and $N(x)$ are $O(r^{-p-m})$ for some $p > 2$, uniformly, as $r \rightarrow \infty$.

Note that hypotheses H-(iii) and H-(iv) on the far field behavior of $E(x)$ and $N(x)$ are consistent with the outgoing radiation condition imposed on $U(x, \lambda)$.

Additionally, we require that $E(x)$ and $N(x)$ be 'slowly varying' on $V \cup \partial V$. (See (viii) after equation (5.2), and (vi') after equation (5.5).)

The problem **P** is a mathematical model for the propagation of time-harmonic waves through an anisotropic (isotropic if $E(x) \equiv I$), inhomogeneous medium, filling the infinite region V. At each point x where $E(x) \neq I$ the direction of propagation of a wave front through x differs from the direction of flow of the energy carried by this front. If $E(x) \equiv I$ and $N(x) \not\equiv 1$, the solution $U(x, \lambda)$ of **P** can be interpreted as the amplitude of a time harmonic electromagnetic scalar potential, or as the amplitude of a component of the electric or magnetic field intensity (cf. [11] and [12]). Under this interpretation $N(x)$ is the index of refraction, and λ is the wave number ($\lambda = \omega/c$, ω = frequency, c = velocity of propagation in vacuo). If $E(x) \equiv N^2(x)I$ the solution of problem **P** can be interpreted as the amplitude of a time harmonic velocity potential of an acoustic field in a medium of density $\rho(x) = 1/N^2(x)$ (cf. [9]). We note that problems of the form **P** also arise in linear elasticity (cf. [1]), and in quantum mechanics (cf. [11]).

In applications where λ is large, the physically salient features of the radiation field are revealed by analyzing the asymptotic behavior of the solution as $\lambda \rightarrow \infty$. To this end we obtain a uniform asymptotic expansion as $\lambda \rightarrow \infty$ of the solution to problem **P**. We first construct an *approximate solution* to problem **P** for $\lambda >> 1$. The form of this approximate solution is a finite sum of surface integrals over certain subsets of the support of the boundary data. We then obtain order estimates for the difference between the exact and approximate solution of the problem **P**. These order estimates imply that our approximate solution of problem **P** is a *rigorous* asymptotic expansion of the exact solution, and are derived by applying previously obtained high frequency a priori estimates (cf. [5], [7], [13], [14]). The latter hold for the solution of the problem **P'** that is satisfied by the difference of the exact and approximate solution of problem **P**.

Results of this kind have been obtained by Bloom and Kazarinoff in [6] under the assumption that $E(x) \equiv I$. However, in [6] rather severe restrictions are imposed on the boundary; it is required either that ∂V be 'convex relative to $N(x)$', or at least that the support of the radiating boundary source distribution $h(x')$ ($U(x', \lambda) = h(x')$ if $x' \in \partial V$) be contained in disjoint patches of ∂V that are convex relative to $N(x)$. In [6] a surface is defined as convex relative to $N(x)$ if any two geodesics (rays) of the Riemannian metric $d\sigma = N(x)|dx|$ that emanate orthogonally from distinct points on the surface, extend to infinity without intersecting the surface or each other. This definition reduces to ordinary convexity if $N(x) \equiv 1$.

In the isotropic case ($E(x) \equiv I$) the energy of the geometrical optics field propagates along rays that emanate orthogonally from each point in the support of $h(x')$. *If the above mentioned convexity condition is eliminated, it is possible for the energy of the geometrical optics field to propagate along a family of rays that form caustics.*

In the anisotropic case ($E(x) \not\equiv I$), the energy of the geometrical optics field propagates along rays (geodesics) of the Riemannian metric $(d\sigma)^2 = N^2(x)dx\cdot(E^{-1}(x))dx$ that emanate orthogonally in a generalized sense from each point in the support of $h(x')$. A ray emanates orthogonally from a point x' of ∂V if $t\cdot(E^{-1}(x')t') = 0$ for any vector t tangent to ∂V at x', and any vector t' tangent to the ray at x'. Again in this more general case, the orthogonal rays from points in the support of $h(x')$ may form caustics. At points of V that do not lie on caustics formed by these rays, each surface integral in our asymptotic approximation of $U(x, \lambda)$ can be expanded asymptotically by the method of stationary phase. If $E(x) \equiv I$ and the support of $h(x')$ is contained in a finite union of disjoint subsets of ∂V that are convex relative to $N(x)$, then the leading terms of these asymptotic expansions can be shown to coincide with the leading terms of the asymptotic series expansions of $U(x, \lambda)$ constructed in [6]. They can also be shown to coincide with the leading terms of analogous ansetze for the case $E(x) \not\equiv I$.

The geometrical theory of high frequency wave propagation provides an algorithm for calculating the radiation field of a scattering problem at a typical point x in the transmitting medium without direct reference to the governing differential equations. A well-known conjecture of Luneburg [12] and Kline [11] asserts that the field calculated by applying the geometrical theory to problem **P** should be asymptotic to the exact solution as $\lambda \to \infty$. It can be shown that the leading terms of the above mentioned series ansetze are identical to the predicted "geometrical optics" field.

The approximate solution of problem **P** that we obtain in this paper does *not* depend on a priori knowledge of the structure and location of caustics that may be formed by the rays of $(d\sigma)^2 = N^2(x)dx\cdot(E^{-1}(x))dx$ that emanate orthogonally from points in the support of $h(x')$. On the other hand our method succeeds only if the following condition is satisfied: any two rays emanating in distinct directions from a point x^o in the support of $h(x')$ extend to infinity without intersecting. This condition is obviously met if $E(x) \equiv I$ and $N(x) \equiv 1$. It can be shown that it is also fulfilled if $E(x)$ and $N(x)$ are slowly varying, and if $E(x) - I$, $N(x) - 1$ with their derivatives approach zero as $r \to \infty$ algebraically fast as assumed above.

We remark that if Maslov's method (cf. [18, chap. V]) were applied to problem **P**, it would *in principle* be possible to construct a uniform asymptotic expansion of $U(x, \lambda)$ as $\lambda \to \infty$, without the above mentioned condition. However, this method can only be applied *after* the location and structure of each caustic formed by orthogonal rays from the support of $h(x')$ has been determined. For given coefficients $E(x)$ and $N(x)$ the structure and location of such caustics depend specifically on the geometry of ∂V, and obviously can be very complicated.

In a number of special cases scattering problems for the reduced wave equation have been solved explicitly in the form of integrals or infinite series, and high frequency asymptotic expansions of the geometrical optics field have been obtained from these solutions (cf. [17]). Explicit solutions are of course not available for general $E(x)$, $N(x)$ and ∂V.

It is sometimes possible using iteration to obtain a convergent series representation of the exact solution that is at the same time a (generalized) asymptotic expansion. For example, Ursell [16] derived the leading term of such a series expansion for the velocity potential ϕ of an acoustic field satisfying the Helmholtz expansion $(\Delta + \lambda^2)\phi = 0$ in the region exterior to an infinite cylinder of *convex* cross section, a Neumann condition on the boundary, and the Sommerfeld radiation condition. Using Ursell's technique Grimshaw [10] was able to construct a rigorous high frequency asymptotic series expansion of Green's function for the problem considered in [16]. The results of Ursell and Grimshaw are consistent with the geometrical theory of optics.

The asymptotic expansions derived in this paper are obtained directly starting with an appropriate integral ansatz. To achieve our goal we first construct an approximate solution of problem **P**, i.e., a smooth function $U_N(x, \lambda)$ such that as $\lambda \to \infty$

$$
\begin{aligned}
&L_\lambda[U_N(x, \lambda)] = O(r^{-3}\lambda^{-N}) && \text{(uniformly on } \partial V \cup V),\\
&U_N(x', \lambda) = h(x') + O(\lambda^{-N}) && \text{(uniformly in } x', x' \in \partial V),\\
&\lim_{R\to\infty} \int_{r=R} r|D[U_N]|^2\, dS = 0 &&
\end{aligned}
$$

for $N = 1, 2, 3, \ldots$.

Applying the high frequency a priori estimates obtained by Bloom and Kazarinoff in [7] we are able to conclude that if $E(x) \equiv I$, if $N(x)$ satisfies the hypothesis of problem **P**, and if ∂V can be illuminated from the exterior (see Section 5, or [6, chap. 1]), then $U(x, \lambda) - U_N(x, \lambda) = O(r^{-1}\lambda^{-N})$, uniformly, as $\lambda \to \infty$. In the special case $E(x) \equiv I$ and $N(x) \equiv 1$, the a priori estimates of Morawetz [13] can be used to prove that this result holds if ∂V is finite and "non-trapping" as defined in [13]. Applying the high frequency a priori estimates obtained by Bloom in [5] we conclude that $U(x, \lambda) - U_N(x, \lambda) = O(\lambda^{-N})$ uniformly on every bounded subset of $V \cup \partial V$ if (i) $E(x) \not\equiv I$, (ii) $N(x)$ and $E(x)$ satisfy the hypothesis of problem **P**, and (iii) ∂V is star-shaped.

We remark that it is possible to represent $U(x, \lambda)$ as follows:

$$
U(x, \lambda) = \sum_{k=1}^{K} U^k(x, \lambda),
$$

where $U^k(x, \lambda)$ is the solution of problem **P** with $h(x')$ replaced by $h^k(x') \equiv \chi^k(x')h(x')$. The functions $\chi^k(x')$ $(k = 1, 2, \ldots, K)$ define a smooth partition of unity subordinate to a covering of ∂V by open sets S^k $(k = 1, 2, \ldots, K)$. The diameter of each set S^k can be made arbitrarily small by taking K sufficiently large. Therefore, we consider without loss of generality, the problem of finding an approximate solution $U_N(x, \lambda)$ of the problem **P** if $U(x', \lambda) \equiv \chi(x')h(x')$, $x' \in \partial V$, where (i) $\chi(x')$ and $h(x')$ are smooth, (ii) supp $\chi(x') \subset S'$ where S' is an open set of arbitrarily small diameter. Furthermore, we assume without loss of generality, that the normals to S' are approximately vertical. For simplicity we restrict our analysis to scattering problems where the energy of the geometrical optics field propagates along orthogonal (in the generalized sense defined above) rays from the support of $h(x')$ that extend to infinity without being reflected by the boundary.

In Section 1 we present a detailed definition of our approximate solution $U_N(x, \lambda)$ to problem **P** with the boundary data, and the geometry of the boundary restricted in the manner described above. We prove in Section 2 that in addition to being continuous on $V \cup \partial V$, the function $L_\lambda[U_N(x, \lambda)]$ is $O(r^{-3}\lambda^{-N})$ as $\lambda \to \infty$, uniformly on $V \cup \partial V$ for $N = 1, 2, 3, \ldots$. In Section 3 we establish that $U(x', \lambda) - U_N(x', \lambda)$ is continuously differentiable on ∂V, and that this function and its first order tangential derivatives are $O(\lambda^{-N})$ as $\lambda \to \infty$, uniformly on ∂V for $N = 1, 2, 3, \ldots$. In Section 4 it is proved that $U_N(x, \lambda)$ satisfies the same radiation condition as $U(x, \lambda)$. In Section 5 we consider the problem **P'** satisfied by $U(x, \lambda) - U_N(x, \lambda)$. We first apply the pointwise a priori estimates obtained in [7] and [13] to the problem **P'** to prove under the conditions mentioned above, that $U(x, \lambda) - U_N(x, \lambda) = O(r^{-1}\lambda^{-N})$, uniformly on $V \cup \partial V$ as $\lambda \to \infty$. We next apply the pointwise a priori estimate obtained in [5] to the problem **P'** to prove under the conditions mentioned above that $U(x, \lambda) - U_N(x, \lambda) = O(\lambda^{-N})$, uniformly on bounded subsets of $V \cup \partial V$ as $\lambda \to \infty$. In Section 6 we describe how our global approximate solution of problem **P** can be used to connect local approximate solutions of $L_\lambda[U]$ that are valid only in caustic free subsets of $V \cup \partial V$, and that can be obtained by truncating certain formal asymptotic series solutions of $L_\lambda[U] = 0$. We conclude with a brief outline of some contemplated extensions and generalizations of the methods used in this paper.

1. Definition of $U_N(x, \lambda)$. Let $G_N(x, x'; \lambda)$ be an approximate free space Green's function (a high frequency parametrix for the differential operator L_λ):

$$L_\lambda[G_N(x, x'; \lambda)] = \delta(x, x') + \varepsilon_N(x, x'; \lambda)|x - x'|^{-1} \quad ((x, x') \in \mathbf{R}^3 \times \mathbf{R}^3),$$

$$\lim_{R\to\infty} \int_{r=R} r|D[G_N]|^2 \, dS = 0,$$

where $\varepsilon_N(x, x'; \lambda)$ is smooth in x and x', and $\varepsilon_N(x, x'; \lambda) = O((1 + r^2)^{-1}\lambda^{-(N+1)})$ as $\lambda \to \infty$, uniformly in x and x'. It can be shown that $G_N(x, x'; \lambda) = G_N(x', x; \lambda)$.

We next define a cut-off function $\psi(x, x')$. If ∂V extends to infinity we set $\psi(x, x') \equiv 1$. If ∂V is finite we set $\psi(x, x') \equiv \gamma(\cos\theta(x, x'))$ where $\theta(x, x')$ is the angle between $\nabla\sigma(x', x)$ and the outward unit normal to ∂V at x', and $\sigma(x', x)$ is the geodesic distance along the ray from x to x'. (See Theorem 1 below.) The function $\gamma(\xi)$ is defined as follows:

(a') $\gamma(\xi) \in C^\infty[-1, 1]$.
(b') $0 \le \gamma(\xi) \le 1$ for all $\xi \in [-1, 1]$.
(c') $\gamma(\xi) \equiv 1$ if $\cos(\pi/2 + \delta_1) \le \xi \le \cos 0$, $\delta_1 > 0$.
(d') $\gamma(\xi) \equiv 0$ if $\cos\pi \le \xi \le \cos(\pi - \delta_2)$, $\delta_2 > 0$, $\pi/2 + \delta_1 < \pi - \delta_2$.

The constants δ_1 and δ_2 are further specified in Section 2.

Let S be an open set containing S' such that the distance between boundary of S, and the boundary of S' is positive and small. It follows from above that for all $(x, x') \in V \times S$:

(a) $\psi(x, x')$ is as smooth as $\theta(x, x')$.
(b) $0 \leq \psi(x, x') \leq 1$.
(c) $\psi(x, x') \equiv 1$ if the angle $\theta(x, x')$ is greater than or equal to 0, and less than or equal to $\pi/2 + \delta_1$.
(d) $\psi(x, x') \equiv 0$ if the angle $\theta(x, x')$ is greater than or equal to $\pi/2 + \delta_2$ and less than or equal to π.
(e) $\nabla\psi(x, x') \equiv O(r^{-1})$, $\nabla\cdot(E(x)\nabla\psi(x, x')) \equiv O(r^{-2})$ as $r \to \infty$, uniformly with respect to x/r and x', $(x, x') \in V \times S$ (See Appendix).
(f) Derivatives of any order of $\nabla\psi(x, x')$ and $\nabla\cdot(E(x)\nabla\psi(x, x'))$ with respect to x_i $(i = 1, 2, 3)$ are $O(r^{-1})$ and $O(r^{-2})$ respectively, as $r \to \infty$, uniformly with respect to x/r and x', $(x, x') \in V \times S$ (See Appendix).

Note that by taking the diameter of S sufficiently small the subset of $\partial V \cup V$ where $\psi(x, x') \equiv 1$ for all $x' \in S$ contains the region T^+ filled by the geodesics (rays) of the Riemannian metric $(d\sigma)^2 = N^2(x)dx\cdot(E^{-1}(x)dx)$ that emanate orthogonally, in the sense of the inner product $v\cdot(E^{-1}(x')w)$ from points in S.

THEOREM 1. *For every positive integer N there exists a smooth function $u_N(x', \lambda)$ defined on ∂V, such that $u_N(x', \lambda) \equiv 0$ for $x' \in \partial V - S$, and*

$$U_N(x, \lambda) = \lambda \int_S \psi(x, x')\, G_N(x, x'; \lambda)u_N(x', \lambda)dS \tag{1.1}$$

is an approximate solution of the problem **P**. *As $\lambda \to \infty$*

$$L_\lambda[U_N(x, \lambda)] = R_N(x, \lambda) \qquad (x \in V), \tag{1.2}$$

$$U_N(x', \lambda) = \chi(x')h(x') + B_N(x', \lambda) \qquad (x' \in \partial V), \tag{1.3}$$

$$\lim_{R\to\infty} \int_{r=R} r|D[U_N]|^2\, dS = 0. \tag{1.4}$$

where $R_N(x, \lambda) \in C[\partial V \cup V]$ and $R_N(x, \lambda) = O(r^{-3}\lambda^{-N})$, uniformly on $\partial V \cup V$. Furthermore, $B_N(x', \lambda) \in C[\partial V]$ and $B_N(x', \lambda) = O(\lambda^{-N})$, uniformly on ∂V; the same is true for the first order tangential derivatives of $B_N(x', \lambda)$.

We prove that (1.2)–(1.4) hold under the assumption that any two geodesics of $(d\sigma)^2 = N^2(x)dx\cdot(E^{-1}(x)dx)$ that emanate from a point x' on ∂V in distinct directions, extend to infinity without intersecting.

Under this assumption it can be shown that

$$G_N(x, x'; \lambda) = \exp[i\lambda\sigma(x, x')]A_N(x, x'; \lambda) \qquad (x \in \partial V \cup V;\ x' \in \partial V).$$

Here $\sigma(x, x')$ is the smooth eikonal function satisfying

$$\nabla\sigma\cdot(E(x)\nabla\sigma) = N^2(x) \quad (x \in \mathrm{R}^3;\ x' \in \partial V),$$

$$\lim_{x \to x'} \frac{[(x - x')\cdot(E^{-1}(x')(x - x'))]^{\frac{1}{2}}}{\sigma(x, x')} = [N(x)]^{-1} \qquad (x' \in \partial V),$$

$$\lim_{r \to \infty} \frac{|x - x'|}{\sigma(x, x')} = 1 \qquad (x' \in \partial V).$$

The function $\sigma(x, x')$ is the distance traveled by a signal propagating along the path of least time from x' to x.

Furthermore,

$$A_N(x, x'; \lambda) = \sum_{n=0}^{N+1} a_n(x, x')\lambda^{-n}$$

where the smooth functions $|x - x'|a_0(x, x')$, $a_1(x, x')$, $a_2(x, x')$,..., $a_{N+1}(x, x')$ are defined recursively by the transport equations

$$2\nabla a_0\cdot(E(x)\nabla\sigma(x, x')) + \nabla\cdot(E(x)\nabla\sigma(x, x'))a_0 = 0, \tag{1.5}$$

$$2\nabla a_1\cdot(E(x)\nabla\sigma(x, x')) + \nabla\cdot(E(x)\nabla\sigma(x, x'))a_1 = -i(\delta(x, x') - \nabla\cdot(E(x)\nabla a_0)), \tag{1.6}$$

$$2\nabla a_n\cdot(E(x)\nabla\sigma(x, x')) + \nabla\cdot(E(x)\nabla\sigma(x, x'))a_n = i\nabla\cdot(E(x)\nabla a_{n-1}))\ (n = 2, 3, \cdots, N + 1), \tag{1.7}$$

and the following conditions:

$$\delta(x, x') - \nabla\cdot(E(x)\nabla a_0(x, x')),\ \nabla\cdot(E(x)\nabla a_{n-1}(x, x')) = O((1 + r^2)^{-1}|x - x'|^{-1}),$$

uniformly in x and x', $(x, x') \in \mathbf{R}^3 \times \mathbf{R}^3$ $(n = 2, 3,\ldots, N + 1)$.

With the $a_n(x, x')$'s as defined above it follows that as $\lambda \to \infty$

$$L_\lambda[G_N(x, x'; \lambda)] = \delta(x, x') + \nabla\cdot(E(x)\nabla a_{N+1}(x, x'))/\lambda^{N+1} \tag{1.8}$$

where $|x - x'|\nabla\cdot(E(x)\nabla a_{N+1}(x, x'))$ is smooth on $\mathbf{R}^3 \times \mathbf{R}^3$, and $\nabla\cdot(E(x)\nabla a_{N+1}(x, x')) = O((1 + r^2)^{-1}|x - x'|^{-1})$, uniformly on $\mathbf{R}^3 \times \mathbf{R}^3$. (See Avila and Keller [3], Babich [4], Duff [8], Vainberg [19], Whitehead [20].)

The function $u_N(x', \lambda)$ in equation (1.1) is expressible as a sum,

$$\sum_{m=0}^{N+1} v_{N,m}(x', \lambda)\lambda^{-m} \tag{1.9}$$

where

$$v_{N,0}(x', \lambda) = [H(x')]^{-1} \chi(x')h(x'), \tag{1.10}$$

and

$$v_{N,m}(x', \lambda) = [H(x')]^{-1} \sum_{p=1}^{m} i\partial_\sigma^p g(0; x', \lambda) \ (m = 1, 2, 3, \cdot \cdot \cdot, N + 1). \tag{1.11}$$

The functions $g_{N,n}(\sigma; x^o, \lambda)$ are determined recursively for $(n = 1, 2, 3, \ldots, N + 1)$ from the equations

$$g_{N,n}(\sigma; x^o, \lambda) = [H(x^o)]^{-1} \sum_{p=1}^{n} i^{p+1} \int_0^{2\pi} H_N(\sigma, \tau_1; x^o, \lambda)\, \partial_\sigma^p\, g_{N,n-p}(0; z(\sigma, \tau_1; x^o), \lambda) d\tau_1, \tag{1.12}$$

and

$$g_{N,n}(\sigma; x^o, \lambda) = -i[H(x^o)]^{-1} \int_0^{2\pi} H_N(\sigma, \tau_1; x^o, \lambda)\, \chi(z(\sigma, \tau_1; x^o))h(z(\sigma, \tau_1; x^o))d\tau_1, \tag{1.13}$$

for $\sigma \in [0, \sigma_1]$ and $x^O \in \partial V$. Here

$$H_N(\sigma, \tau_1; x^o, \lambda) = \psi(x^o, z(\sigma, \tau_1; x^o))J(\sigma, \tau_1; x^o)A_N(x^o, z(\sigma, \tau_1; x^o); \lambda),$$

$$J(\sigma, \tau_1; x^o) = |z_\sigma(\sigma, \tau_1; x^o) \times z_{\tau_1}(\sigma, \tau_1; x^o)|,$$

$$H(x^o) = 2\pi H_N(0, \tau_1; x^o, \lambda)$$

and $A_N(x, x'; \lambda)$ is the amplitude of $\exp[i\lambda\sigma(x, x')]$ in the definition of $G_N(x, x'; \lambda)$.

For every $x^o \in S$, the vector valued function $z(\sigma, \tau_1; x^O)$ in equations (1.12) and (1.13) is a parametric representation of a certain neighborhood $\mathcal{N}(x^o)$ of S on ∂V. This function and the set $\mathcal{N}(x^o)$ are defined more precisely in the proof of (1.3) given in Section 3. Also it is shown in Section 3 that $H(x^o) = 2\pi H_N(0, \tau_1; x^o, \lambda)$ is independent of τ_1 and λ and is strictly positive.

2. *Proof of (1.2).* For all $x \in V \cup \partial V$

$$L_\lambda[U_N] = \lambda \int_S \psi\, L_\lambda[G_N]u_N\, dS + \lambda \int_S \nabla\cdot[E\nabla\psi]G_N\, u_N\, dS + 2\lambda \int_S [E\nabla\psi]\cdot\nabla G_N\, u_N\, dS. \tag{2.1}$$

If ∂V is infinite, equation (2.1) reduces to

$$L_\lambda[U_N] = \lambda \int_S L_\lambda[G_N]u_N\, dS.$$

Since, as proved in Section 3, $u_N(x', \lambda)$ is continuous on S, it follows from (1.8) that the function $L_\lambda[U_N(x, \lambda)]$ is continuous on $V \cup \partial V$, and that as $\lambda \to \infty$

$$L_\lambda[U_N(x, \lambda)] = \lambda \int_S L_\lambda[G_N(x, x'; \lambda)]u_N(x', \lambda)dS = \lambda \int_S O(\lambda^{-(N+1)}(1 + r^2)^{-1}$$
$$|x - x'|^{-1})\, u_N(x', \lambda)dS = O(\lambda^{-N} r^{-3}),$$

uniformly in x, $x \in V \cup \partial V$.

Suppose now that ∂V is finite, i.e., the interior of ∂V is a closed bounded set, and let T be the subset of R^3 filled by the rays emanating orthogonally from both sides of S. The intersection of T with $V \cup \partial V$ consists of two disjoint closed subsets T^+ and T^-; the set T^+ is the family of orthogonal rays that emanate from the outside of S, and the set T^- consists of the parts of the orthogonal rays from the inside of S that lie in $V \cup \partial V$. As stated above, a ray emanates orthogonally from a point x' of ∂V if $t\cdot(E^{-1}(x')\, t') = 0$ for any vector t tangent to ∂V at x', and any vector t' tangent to the ray at x'. Also as mentioned in the introduction, we make a simplifying assumption that rules out reflection by ∂V of any orthogonal ray emanating from the outside of S, viz., each ray in T^+ extends to infinity without intersecting $\partial V - S$. Again in this more general case, the orthogonal rays from the support of $h(x')$ may form caustics.

We next define subsets V^+ and V^- of V as follows:

$$V^+ \equiv \{x{:}x \in V, 0 \le \theta(x, x^1) < \varepsilon_1 < \pi/2 \textit{ or } 0 \le \sigma(x, x^2) < \varepsilon_1 \text{ for any points } x^1, x^2 \in S\},$$

$$V^- \equiv \{x{:}x \in V, \pi - \varepsilon_2 \le \theta(x, x^3) \le \pi \text{ for some point } x^3 \in S\}.$$

In the above definitions it is assumed that $\varepsilon_1, \varepsilon_2 \approx 0$. It is clear that $V^+ \cup \partial V^+$ contains T^+ and that $V^- \cup \partial V^-$ contains T^-. We have (i) $\sup \theta(x, x') = \delta_1 + \pi/2$ on $(V^+ \cup \partial V^+) \times S$, and (ii) $\inf \theta(x, x') = \pi - \delta_2$ on $(V^- \cup \partial V^-) \times S$. We assume without loss of generality that the positive constants δ_1 and δ_2 are such that $\delta_1 + \pi/2 < \pi - \delta_2$, where δ_1 and δ_2 are the constants introduced in Section 1.

Note that the inequalities $\varepsilon_1 \le \sigma(x, x')$ and $\varepsilon_1 \le \theta(x, x') \le \pi - \varepsilon_2$ hold on $(V' \cup \partial V') \times S$ where $V' \equiv (V \cup \partial V) - (V^+ \cup \partial V^+) \cup (V^- \cup \partial V^-)$. Consequently, $t(x')\cdot\nabla\sigma(x', x)$ is bounded away from zero on $(V' \cup \partial V') \times S$ for every unit vector $t(x')$ tangent to S at x'.

If x lies in $V^- \cup \partial V^-$ we have by definition $\psi(x, x') \equiv 0$ for all $x' \in S$. Then obviously $\nabla\psi(x, x')$ and $\nabla\cdot[E(x)\, \psi(x, x')] \equiv 0$ for all $x' \in S$, and equation (2.1) reduces to $L_\lambda[U_N(x, \lambda)] \equiv 0$.

If x lies in $V^+ \cup \partial V^+$ we have by definition $\psi(x, x') \equiv 1$ for all $x' \in S$. Then as in the preceding case $\nabla\psi(x, x')$ and $\nabla\cdot[E(x)\, \psi(x, x')] \equiv 0$ for all $x' \in S$, and equation (2.1) reduces to:

$$L_\lambda[U_N(x, \lambda)] = \lambda \int_S L_\lambda[G_N(x, x'; \lambda)]u_N(x', \lambda)dS. \tag{2.2}$$

Using (1.8) we have as $\lambda \to \infty$, uniformly in x,

$$\lambda \int_S L_\lambda [G_N(x, x'; \lambda)]u_N(x', \lambda)dS =$$

$$\lambda \int_S O(\lambda^{-(N+1)}(1 + r^2)^{-1}|x - x'|^{-1})\, u_N(x', \lambda)dS = O(\lambda^{-N} r^{-3}). \tag{2.3}$$

Finally, suppose $x \in V' \cup \partial V'$. Again using (1.8), and recalling that $0 \le \psi(x, x') \le 1$ on $V \times \partial V$ it is clear that the first term on the r.h.s. of (2.1) is $O(\lambda^{-N} r^{-3})$ uniformly in x.

We next consider the second and third terms on the r.h.s. of (2.1) for $x \in V' \cup \partial V'$. Each of these terms is of the form

$$\lambda \int_{\tilde{S}} \exp[i\lambda\sigma(x, X(w))]F(w, x; \lambda)dw_1\, dw_2 \tag{2.4}$$

where $x = X(w) \equiv X(w_1, w_2)$ is a parametric representation of S, and $\tilde{S}$ is the pre-image of S under this mapping. Furthermore, $F(w, x; \lambda) = u_N(X(w), \lambda)\, f(x, X(w); w, \lambda)$ where

$$f(x, X; w, \lambda) = \nabla\cdot[E(x)\nabla\psi(x, X)]A_N(x, X; \lambda)|X_{w_1} \times X_{w_2}|$$

in the second term of (2.1), and

$$f(x, X; w, \lambda) = 2[i\lambda\, E(x)\nabla\sigma(x, X)A_N(x, X; \lambda) + E(x)\nabla A_N(x, X; \lambda)]\cdot\nabla\psi(x, X)|X_{w_1} \times X_{w_2}|$$

in the third term of (2.1).

It is shown in the Appendix that $\nabla\psi(x, x')$, $\nabla\cdot[E(x)\nabla\psi(x, x')]$, $\nabla\sigma(x, x')$, $A_N(x, x'; \lambda)$, $\nabla A_N(x, x'; \lambda)$, and all derivatives of these functions with respect to the primed variables are continuous in x and x' under our hypothesis on $E(x)$ and $N(x)$.

It is also shown there that (i) $\nabla\sigma(x, x')$, $\psi(x, x')$, $\nabla\psi(x, x')$, $\nabla\cdot[E(x)\nabla\psi(x, x')]$, and all derivatives of these functions with respect to the primed variables are respectively $O(1)$, $O(1)$, $O(r^{-1})$ and $O(r^{-2})$, uniformly in x and x' as $r \to \infty$, and that (ii) $A_N(x, x'; \lambda)$, $\nabla A_N(x, x'; \lambda)$ and all derivatives of these functions with respect to the primed variables are $O(r^{-1})$ and $O(r^{-2})$ as λ and $r \to \infty$, uniformly in x and x'.

It follows that the function $F(w, x; \lambda)$ in the integrands of the second and third terms on the r.h.s. of (2.1), and all derivatives of this function with respect to w_1 and w_2 are continuous on $S \times (V' \cup \partial V')$. Note that $X(w)$ is smooth on $\tilde{S}$ by hypothesis, and it is shown in Section 3 that $u_N(X(w), \lambda)$ is a smooth function of w on $\tilde{S}$. In the second term $F(w, x; \lambda)$ and its derivatives with respect to w_1 and w_2 are $O(r^{-3})$, while in the third term $F(w, x; \lambda)$ and its derivatives with respect to w_1 and w_2 are $O(\lambda\, r^{-3})$ as λ and $r \to \infty$.

Since $u_N(X(w), \lambda)$ and its partial derivatives with respect to w_1 and w_2 vanish on the boundary of $\tilde{S}$, the of same is true of $F(w, x; \lambda)$ and its partial derivatives with respect to w_1 and w_2.

Integrating (2.4) by parts we have as λ and $r \to \infty$:

$$\lambda \int_{\tilde{S}} \exp[i\lambda\sigma(x, X)]\, F(w, x; \lambda) dw_1\, dw_2 =$$

$$i \int_{\tilde{S}} \exp[i\lambda\sigma(X, x)]\, F_1\, (w, x; \lambda) dw_1\, dw_2 = O(\lambda^{m-2}\, r^{-3}), \tag{2.5}$$

where $m = 2$ for the second term on the r.h.s. of (2.4) and $m = 3$ for the third term. Here

$$F_1(w, x; \lambda) = \Gamma(w, x)[\nabla_w\, \sigma(X, x)\cdot\nabla_w\, F(w, x; \lambda)] + \nabla_w\cdot[\Gamma(w, x)\, \nabla_w\, \sigma(X, x)]\, F(w, x; \lambda),$$

$$\Gamma(w, x) \equiv |\nabla_w\, \sigma(X, x)|^{-2},$$

$$\nabla_w\, \sigma(X, x) = [\nabla\sigma(X, x)\cdot X_{w_1},\, \nabla\sigma(X, x)\cdot X_{w_2}],$$

$$\nabla_w\cdot[\Gamma(w, x)\, \nabla_w \sigma(X, x)] = \Gamma(w, x)[\Delta_w\, \sigma(X, x)$$

$$-\, 2\Gamma(w, x)\, \nabla_w\, \sigma(X, x)\cdot(M_1\, (X, x)\, \nabla_w\, \sigma(X, x))],$$

$$\sigma_{w_i w_j}\, (X, x) = X_{w_i}\cdot(M_2(X, x)X_{w_j}) + \nabla\sigma(X, x)\cdot X_{w_i w_j},$$

$$\Delta_w\, \sigma(X, x) = X_{w_1}\cdot(M_2(X, x)X_{w_1}) + X_{w_2}\cdot(M_2(X, x)X_{w_2}) + \nabla\sigma(X, x)\cdot\Delta_w X,$$

where

$$M_1(x', x) = \begin{bmatrix} \sigma_{w_1 w_1}\, (x, x') & \sigma_{w_1 w_2}\, (x, x') \\ \sigma_{w_2 w_1}\, (x, x') & \sigma_{w_2 w_2}\, (x, x') \end{bmatrix},$$

and

$$M_2(x', x) = \begin{bmatrix} \dfrac{\partial^2\sigma(x', x)}{\partial x_1'\partial x_1'} & \dfrac{\partial^2\sigma(x', x)}{\partial x_1'\partial x_2'} & \dfrac{\partial^2\sigma(x', x)}{\partial x_1'\partial x_3'} \\ \dfrac{\partial^2\sigma(x', x)}{\partial x_2'\partial x_1'} & \dfrac{\partial^2\sigma(x', x)}{\partial x_2'\partial x_2'} & \dfrac{\partial^2\sigma(x', x)}{\partial x_2'\partial x_3'} \\ \dfrac{\partial^2\sigma(x', x)}{\partial x_3'\partial x_1'} & \dfrac{\partial^2\sigma(x', x)}{\partial x_3'\partial x_2'} & \dfrac{\partial^2\sigma(x', x)}{\partial x_3'\partial x_3'} \end{bmatrix}.$$

The function $F_1(w, x, \lambda)$ and all its derivatives with respect to w_1 and w_2 in the second and third terms on the r.h.s. of (2.4) are continuous on $\tilde{S} \times (V' \cup \partial V')$, and are $O(\lambda^{m-2}\, r^{-3})$ as λ and r $\to \infty$. This is implied by: (i) the continuity of $\Gamma(w, x)$ on $\tilde{S} \times (V' \cup \partial V')$, (ii) the fact that this function is uniformly bounded away from zero on $\tilde{S} \times (V' \cup \partial V')$, and (iii) the above mentioned properties of $F(w, x, \lambda)$, $\sigma(X(w), x)$ and the derivatives of these functions with respect to w_1 and w_2.

In fact we may integrate (2.4) by parts $M + 1$ times to obtain the result that

$$\lambda \int_{\tilde{S}} \exp[i\lambda\sigma(X(w), x)]\, F(w, x; \lambda) dw_1\, dw_2 =$$

$$i^{(M+1)} \lambda^{-M} \int_{\tilde{S}} \exp[i\lambda\sigma(X(w), x)]\, F_{(M+1)}(w, x; \lambda) dw_1\, dw_2, \tag{2.6}$$

where

$$F_{(M+1)}(w, x; \lambda) = \Gamma(w, x)[\nabla_w \sigma(X, x) \cdot \nabla_w F_M (w, x; \lambda)] + \nabla_w \cdot [\Gamma(w, x) \nabla_w \sigma(X, x)]\, F_M (w, x; \lambda).$$

The function $F_{(M+1)}(w, x; \lambda)$ vanishes on $\tilde{S}$, and inherits the asymptotic behavior, smoothness, and continuity of $F_M(w, x; \lambda)$. It follows immediately from the above analysis that $L_\lambda [U_N (x, \lambda)] = O(r^{-3} \lambda^{-N})$ uniformly on $V \cup \partial V$ as $\lambda \to \infty$. Also since (i) $\Psi(x, X)$, $\nabla\Psi(x, X)$, $\nabla \cdot [E(x) \nabla\psi(x, X)]$, $|x - X|\, G_N (x, X; \lambda)$, $|x - X|\, L_\lambda [G_N (x, X; \lambda)] \in C[(V \cup \partial V) \times \tilde{S}\,]$, (ii) $u_N (X, \lambda) \in C[\tilde{S}\,]$ (see Section 3) and (iii) S is assumed to be smooth, it follows that $L_\lambda [U_N (x, \lambda)]$ is continuous on $V \cup \partial V$.

3. Proof of Equation (1.3). We prove in this section that with $u_N (x', \lambda)$ as defined above

$$\lim_{x \to x^0} U_N (x, \lambda) = \chi(x^0) h(x^0) + B_N (x^0, \lambda) \quad (x^0 \in \partial V) \tag{3.1}$$

where (i) $B_N (x^o, \lambda) \in C^1 [\partial V]$, (ii) $B_N (x^o, \lambda) = O(\lambda^{-N})$ as $\lambda \to \infty$, uniformly in x^o, $x^o \in \partial V$, (iii) all first order tangential derivatives of $B_N (x^o, \lambda)$ are $O(\lambda^{-N})$ as $\lambda \to \infty$, uniformly in x^o, $x^o \in \partial V$.

First, letting $x \to x^o$ ($\in S$) in (1.3), and recalling that $\sigma(x^o, x') = \sigma(x', x^o)$, we have

$$U_N (x^o, \lambda) = \lambda \int_S \psi(x^0, x')[\sigma(x', x^o)]^{-1} \exp[i\lambda\sigma(x', x^o)] A_N (x', x^o; \lambda)\, u_N (x', \lambda) dS \tag{3.2}$$

where $A_N (x', x^o; \lambda) = [\sigma(x', x^o)]^{-1} \mathscr{A}_N (x', x^o; \lambda)$.

For each ray (geodesic of $(d\sigma)^2 = N(x) dx \cdot (E^{-1}(x) dx)$ emanating from a point x^o of S we have the parametric representation

$$x = X(\sigma, \nu; x^o) = col(x_1(\sigma, \nu; x^o), x_2(\sigma, \nu; x^o), x_3(\sigma, \nu; x^o)) \quad (\sigma \geq 0, |\nu| \equiv 1, x^o \in S).$$

The vector valued functions $X(\sigma, \nu; x^o)$ and $P(\sigma, \nu; x^O)$ are defined by the differential equations

$$X_\sigma = [N(X)]^{-2} E(X) P,$$

$$P_\sigma = [N(X)]^{-1} \nabla N(X) - 2^{-1} [N(X)]^{-2} P \cdot (\nabla E(X) P), \tag{3.3}$$

and the initial conditions

$$X(0, \nu; x^o) = x^o$$

$$P(0, \nu; x^o) = col(p_1\,(0, \nu; x^0), p_2\,(0, \nu; x^o), p_3\,(0, \nu; x^0)) = N(x^o)\,E^{-1/2}(x^o)\,R(x^o)\nu, \tag{3.4}$$

where $\nabla E(x) = col(E_{x_1}(x), E_{x_2}(x), E_{x_3}(x))$ and $\nabla N(x) = col(N_{x_1}(x), N_{x_2}(x), N_{x_3}(x))$. The rotation matrix $R(x^o)$ is defined more precisely below.

Recall our hypothesis that for each point x^o on ∂V, the geodesics of the Riemannian metric $(d\sigma)^2 = N^2(x)dx\cdot(E^{-1}(x)dx)$ emanating from x^o form a field $\mathcal{F}(x^o)$(cf. [15]). It can be shown that such global geodesic fields exist under the following "physically reasonable" conditions on $N(x)$ and $E(x)$:

(*i*) $N(x) \geq N_0 < 0, \min_{|\xi|=1} \xi\cdot(\mathrm{E}(x)\bar{\xi}) \geq p_0 > 0\ (x \in \mathrm{R}^3)$.

(*ii*) $E(x), N(x)\ \varepsilon\ C^\infty[\mathrm{R}^3]$.

(*iii*) $|(E(x) - I)E^{-1}(x)|, |(N^2(x) - 1)\,N^{-2}(x)|, |\nabla E(x)E^{-1}(x)|$ and $|\nabla N(x)\,N^{-2}\,(x)|$ are sufficiently small on R^3.

(Note that the *local* existence of a geodesic field near x^o is guaranteed by a theorem of Whitehead [20].)

Under the above conditions on $E(x)$ and $N(x)$ the function $X(\sigma, \nu; x^o) \in C^\infty\,\tilde{\mathfrak{R}}$ where $\tilde{\mathfrak{R}} = \{(\sigma, \tau){:}\ \sigma \geq 0,\ |\nu| = 1\}$. *Furthermore, it can be shown under these conditions that as* $\sigma \rightarrow 0$

$$X(\sigma, \nu; x^o) \approx [N^o]^{-1}\,[E^o]^{1/2}\,R^o\,\nu\,\sigma + x^o,$$

and

$$X^\sigma(\sigma, \nu; x^o) \approx [N^o]^{-1}[E^o]^{1/2}\,R^o\nu,$$

uniformly with respect to ν, where $E^o = E(x^o)$, $N^o = N(x^o)$ and $R^o = R(x^o)$.

To every pair $(x, x^o) \in R^3 \times S$, there corresponds a unique 4-tuple $(\sigma(x, x^o), \nu(x, x^o))$ with $|\nu(x, x^o)| = 1$ such that $x = X(\sigma(x, x^o), \nu(x, x^o); x^o)$. Consequently, the equation $x = X(\sigma, \nu(\tau); x^o)$, where $\nu(\tau) = col[\cos\tau_1 \sin\tau_2, \sin\tau_1 \sin\tau_2, \cos\tau_2]$, defines a one to one mapping of the region $\mathfrak{R} = \{(\sigma, \tau){:}\ \sigma \geq 0, 0 \leq \tau_1 < 2\pi, 0 < \tau_2 < \pi\}$ onto $\mathrm{R}^3 - (\mathscr{C}_1 \cup \mathscr{C}_2)$, where

$$\mathscr{C}_1 = \{x{:}x = X(\sigma, col[0, 0, 1]; x^0), \sigma \geq 0\},$$

and

$$\mathscr{C}_2 = \{x{:}x = X(\sigma, col[0, 0, -1]; x^0), \sigma \geq 0\}.$$

Under our assumption that the normals to S are approximately vertical we can be sure that the curve $\mathscr{C}_2$ lies entirely in $R^3 - V^+$, and that the curve $\mathscr{C}_1$ lies entirely in $R^3 - V^-$. If the diameter of S is sufficiently small and the elements of $R(x^0)$ are chosen so that

$$\frac{R(x^0)col[F_{3,1}(x^0), F_{3,2}(x^0), F_{3,3}(x^0)]}{[F_{3,1}^2(x^0) + F_{3,2}^2(x^0) + F_{3,3}^2(x^0)]^{1/2}} = col[0, 0, 1]$$

where $F(x^0) = E^{1/2}(x^0)$, then there exists a function $\tau_2(\tau_1, \sigma; x^0)$ and a positive constant σ_1 such that $\mathscr{N}(x^0) = \{x{:}x = X(\sigma, \nu(\tau_1, \tau_2(\tau_1, \sigma; x^0)); x^0), \tau_1 \in [0, 2\pi], \sigma \in [0, \sigma_1]\}$ is a neighborhood of S on ∂V for every $x^0 \in S$.

We set $z(\sigma, \tau_1; x^0) = X(\sigma, \nu(\tau_1, \tau_2(\tau_1, \sigma; x^0); x^0)$. Note that for each fixed x^0, and $\sigma_0 \in [0, \sigma_1]$, the equation $x = z(\sigma_0, \tau_1; x^0)$, $\tau_1 \in [0, 2\pi]$, is a parametric representation of the curve $\{x{:}x \in \mathscr{N}(x^0), \sigma(x, x^0) = \sigma_0\}$.

By taking the diameter of the open set S to be sufficiently small we can guarantee that for any x' lying in $\mathscr{N}(x^0)$ the angle between $\nabla \sigma(x', x^0)$ and the normal to ∂V at x' that points into V is close to $\pi/2$, and certainly lies in the interval $[0, \pi/2 + \delta_1]$ so that $\psi(x^0, x') \equiv 1$ for all $x' \in S$.

We can therefore rewrite the integral in (1.1) in terms of σ and τ_1 as follows:

$$U_N(x^0, \lambda) = \lambda \int_0^{2\pi} \int_0^{\sigma_1} \exp[i\lambda\sigma] H_N(\sigma, \tau_1; x^0, \lambda) u_N(z(\sigma, \tau_1; x^0), \lambda) d\sigma\, d\tau_1 \tag{3.5}$$

where

$$H_N(\sigma, \tau_1; x^0, \lambda) = \sigma^{-1} \psi(x^0, z(\sigma, \tau_1; x^0)) J(\sigma, \tau_1; x^0) A_N(z(\sigma, \tau_1; x^0), x^0; \lambda),$$

and

$$J(\sigma, \tau_1; x^0) = |z_\sigma(\sigma, \tau_1; x^0) \times z_{\tau_1}(\sigma, \tau_1; x^0)|.$$

The function $H_N(\sigma, \tau_1; x^0, \lambda)$ defined after equation (3.5), is a smooth function of σ and τ_1 for all $0 \leq \sigma \leq \sigma_1$ and $0 \leq \tau_1 \leq 2\pi$. Furthermore, the integrand of (3.5) and its σ and τ_1 derivatives vanish if $z(\sigma, \tau_1; x^0) \in \mathscr{N}(x^0) - S$ due to the factor $u_N(z(\sigma, \tau_1; x^0), \lambda)$. By construction the factor $\psi(x^0, z(\sigma, \tau_1; x^0)) \equiv 1$ as $\sigma \to 0+$ for every $x^0 \in S$.

Integrating (3.5) by parts $N + 1$ times with respect to σ we obtain

$$U_N(x^0, \lambda) = \sum_{n=0}^{N} \lambda(-1)^{n+1} \partial_\sigma^n g_N(0; x^0, \lambda)(i\lambda)^{-n-1} +$$

$$(-1)^{N+1}(i\lambda)^{-N} \int_0^{2\pi} \int_0^{\sigma_1} \exp[i\lambda\sigma] \partial_\sigma^{N+1}[H_N(\sigma, \tau_1; x^0, \lambda) u_N(z(\sigma, \tau_1; x^0), \lambda)] d\sigma d\tau_1 \tag{3.6}$$

where

$$g_N(\sigma; x^0, \lambda) = \int_0^{2\pi} H_N(\sigma, \tau_1; x^0, \lambda) u_N(z(\sigma, \tau_1; x^0), \lambda) d\tau_1. \tag{3.7}$$

If in (3.5) we set

$$u_N(x', \lambda) = \sum_{m=0}^{N+1} v_{N,m}(x', \lambda)\lambda^{-m}, \tag{3.8}$$

then

$$H_N(\sigma, \tau_1; x^0, \lambda) u_N(z(\sigma, \tau_1; x^0), \lambda) = \sum_{m=0}^{N+1} H_{N,m}(\sigma, \tau_1; x^0, \lambda)\lambda^{-m}, \tag{3.9}$$

where $H_{N,m}(\sigma, \tau_1; x^0, \lambda) = H_N(\sigma, \tau_1; x^0, \lambda)\, v_{N,m}(z(\sigma, \tau_1; x^0), \lambda)$. Moreover, by virtue of equation (3.7), the defining equation for $g_N(\sigma; x^0, \lambda)$, and equation (3.9) we have

$$\partial_\sigma^n g_N(\sigma; x^0, \lambda) = \sum_{m=0}^{N+1} \partial_\sigma^n g_{N,m}(\sigma; x^0, \lambda)\lambda^{-m}, \tag{3.10}$$

where

$$\partial_\sigma^n g_{N,m}(\sigma; x^0, \lambda) = \int_0^{2\pi} \partial_\sigma^n H_{N,m}(\sigma, \tau_1; x^0, \lambda)\, d\tau_1 \; (m = 0, 1, 2, \cdots, N+1). \tag{3.11}$$

Setting $\sigma = 0$ in (3.10) and substituting into the sum

$$\sum_{n=0}^{N} \lambda(-1)^{n+1} \partial_\sigma^n g_N(0; x^0, \lambda)(i\lambda)^{-n-1}$$

on the r.h.s. of (3.6), this sum becomes

$$\sum_{n=0}^{N} \sum_{m=0}^{N+1} i^{n+1} \partial_\sigma^n g_{N,m}(0; x^0, \lambda)(i\lambda)^{-m-n},$$

which can be rewritten as

$$\sum_{n=0}^{N} \sum_{m=n}^{N+1} i^{n+1} \partial_\sigma^n g_{N,m-n}(0; x^0, \lambda)(i\lambda)^{-m}.$$

Interchanging the order of summation in the above sum we get

$$\sum_{m=0}^{N} \left[\sum_{n=0}^{m} i^{n+1} \partial_\sigma^n g_{N,m-n}(0; x^0, \lambda) \right] \lambda^{-m}$$

$$+ \sum_{m=N+1}^{2N+1} \left[\sum_{n=m-(N+1)}^{N} i^{n+1} \partial_\sigma^n g_{N,m-n}(0; x^o, \lambda) \right] \lambda^{-m} \tag{3.12}$$

If we now set

$$g_{N,0}(0; x^o, \lambda) = -i\chi(x^o)h(x^o), \tag{3.13}$$

$$ig_{N,m}(0; x^o, \lambda) + \sum_{n=1}^{m} i^{n+1} \partial_\sigma^n g_{N,m-n}(0; x^o, \lambda) = 0 \ (m = 1, 2, \cdot \cdot \cdot, N), \tag{3.14}$$

and

$$ig_{N,N+1}(0; x^o, \lambda) + \sum_{n=1}^{N} i^{n+1} \partial_\sigma^n g_{N,(N+1)-n}(0; x^o, \lambda) = 0, \tag{3.15}$$

then (3.6) reduces to

$$U_N(x^o, \lambda) = \chi(x^o)h(x^o) + \sum_{m=N+2}^{2N+1} \left[\sum_{n=m-(N+1)}^{N} i^{n+1} \partial_\sigma^n g_{N,m-n}(0; x^o, \lambda) \right] \lambda^{-m} +$$

$$(-1)^{N+1}(i\lambda)^{-N} \int_0^{2\pi} \int_0^{\sigma_1} \exp[i\lambda\sigma] \partial_\sigma^{N+1} [H_N(\sigma, \tau_1; x^o, \lambda) u_N(z(\sigma, \tau_1; x^o), \lambda)] d\sigma d\tau_1. \tag{3.16}$$

As explained above we have assumed without loss of generality that supp $\chi(x^o) \in S' \subset S$. Additionally, we require that the function $h(x^o) \in C^\infty[\partial V]$.

Recall from (3.11) that

$$g_{N,p}(\sigma; x^o, \lambda) = \int_0^{2\pi} H_N(\sigma, \tau_1; x^o), \lambda) v_{N,p}(z(\sigma, \tau_1; x^o), \lambda) d\tau_1$$

$$(0 \le \sigma \le \sigma_1, x^o \in S; p = 0, 1, 2, \cdot \cdot \cdot, N + 1). \tag{3.17}$$

It can be shown that $A_N(z(\sigma, \tau_1; x^o), x^o; \lambda) \approx a_0(z(\sigma, \tau_1; x^o), x^o)$ as $\sigma \to 0+$ where

$$a_0(z(\sigma, \tau_1; x^o), x^o) = (4\pi)^{-1} N(x^o)[\det E(x^o)]^{-1/2} [2^{-1} L_{\sigma\sigma}(0, \tau_1; x^o)]^{1/2} [L(\sigma, \tau_1; x^o)]^{-1/2},$$

$$L(\sigma, \tau_1; x^o) = -[N(z(\sigma, \tau_1; x^o))]^2 K(\sigma, \tau_1, \tau_2(\tau_1, \sigma; x^o); x^o),$$

and

$$K(\sigma, \tau; x^o) = X_\sigma(\sigma, \nu(\tau); x^o)\cdot[X_{\tau_1}(\sigma, \nu(\tau); x^o) \times X_{\tau_2}(\sigma, \nu(\tau); x^o)].$$

Furthermore, we have

$$z_\sigma(\sigma, \tau_1; x^o) = X_\sigma(\sigma, \nu(\tau); x^o) + X_{\tau_2}(\sigma, \nu(\tau); x^o)[\tau_2(\tau_1, \sigma; x^o)]_\sigma,$$

and

$$z_{\tau_1}(\sigma, \tau_1; x^o) = X_{\tau_1}(\sigma, \nu(\tau); x^o) + X_{\tau_2}(\sigma, \nu(\tau); x^o)[\tau_2(\tau_1, \sigma; x^o)]_{\tau_1},$$

where $\tau = (\tau_1, \tau_2(\tau_1, \sigma; x^o))$. The function $\tau_2(\tau_1, \sigma; x^o)$ is defined by the equation

$$x_3(\sigma, \nu(\tau_1, \tau_2); x^o) = f(x_1(\sigma, \nu(\tau_1, \tau_2); x^o), x_2(\sigma, \nu(\tau_1, \tau_2); x^o)),$$

where $x_3 = f(x_1, x_2)$ is a representation of some neighborhood of $\mathcal{N}(x^o)$ on ∂V.

As a consequence of the preceding equations, our definition of $R(x^o)$, and the asymptotic formulas

$$X(\sigma, \nu(\tau); x^o) \approx [N^o]^{-1} [E^o]^{1/2} R^o \nu(\tau)\sigma + x^o,$$

$$X_\sigma(\sigma, \nu(\tau); x^o) \approx [N^o]^{-1} [E^o]^{1/2} R^o \nu(\tau),$$

$$X_{\tau_i}(\sigma, \nu(\tau); x^o) \approx [N^o]^{-1} [E^o]^{1/2} R^o \nu_{\tau_i}(\tau)\sigma + x^o \quad (i = 1, 2),$$

that holds as $\sigma \to 0+$, it follows that $\tau_2(\tau_1, 0+; x^o) = \pi/2$,

$$\lim_{\sigma\to 0+} \sigma A_N(z(\sigma, \tau_1; x^o), x^o; \lambda) = (4\pi)^{-1} N^o[\det E^o]^{-1/2},$$

$$\lim_{\sigma\to 0+} \sigma^{-2} J(\sigma, \tau_1; x^o) = [N^o]^{-1}|([E^o]^{1/2} R^o \nu(\tau_1, \pi/2)) \times ([E^o]^{1/2} R^o \nu_{\tau_2}(\tau_1, \pi/2))| =$$

$$[N^o]^{-1}[\det E^o]^{1/2}[((R^o)^t \nu_{\tau_2}(\tau_1, \pi/2))\cdot([E^o]^{-1}((R^o)^t \nu_{\tau_2}(\tau_1, \pi/2)))]^{1/2} =$$

$$[N^o]^{-1}[\det E^o]^{1/2}[R^o_{3,*}\cdot([E^o]^{-1} R^o_{3,*})]^{1/2},$$

where $(R^o)^t$ = transpose of R^o, and $R^o_{3,*} = col[R^o_{3,1}, R^o_{3,2}, R^o_{3,3}]$.

We conclude that

$$H_N(0, \tau_1; x^o) = (4\pi)^{-1}[R^o_{3,*}\cdot([E^o]^{-1}R^o_{3,*})]^{1/2} > 0.$$

It now follows from (3.17) that

$$\mathrm{v}_{N,p}(x^o, \lambda) = [H(x^o)]^{-1} g_{N,p}(0; x^o, \lambda), \tag{3.18}$$

where $H(x^o) = \int_0^{2\pi} H_N(0, \tau_1; x^o, \lambda)d\tau_1 = 2^{-1}[R^o_{3,*}\cdot([E^o]^{-1}R^o_{3,*})]^{1/2}$, and consequently that

$$g_{N,p}(\sigma; x^o, \lambda) = [H(x^o)]^{-1}\int_0^{2\pi} H_N(\sigma, \tau_1; x^o, \lambda)g_{N,p}(0; z(\sigma, \tau_1; x^o), \lambda)d\tau_1.$$

Using (3.14) this becomes

$$g_{N,p}(\sigma; x^o, \lambda) = -[H(x^o)]^{-1}\sum_{n=1}^{p} i^p \int_0^{2\pi} H_N(\sigma, \tau_1; x^o, \lambda)\partial_\sigma^n g_{N,p-n}(0; z(\sigma, \tau_1; x^o), \lambda)d\tau_1 \tag{3.19}$$

for $p = 1, 2, 3, \ldots, N$. Furthermore, using (3.15) we get

$$g_{N,N+1}(\sigma; x^o, \lambda) = -[H(x^o)]^{-1}\sum_{n=1}^{N} i^p \int_0^{2\pi} H_N(\sigma, \tau_1; x^o, \lambda)\partial_\sigma^n g_{N,(N+1)-n}$$
$$(0; z(\sigma, \tau_1; x^o), \lambda)d\tau_1. \tag{3.20}$$

Equations (3.19) and (3.20) recursively determine the $g_{N,p}(\sigma; x^o, \lambda)$'s for all $p = 0, 1, 2, \ldots, N + 1$, $\sigma \in [0, \sigma_1]$ and $x^o \in S$.

It can be shown by mathematical induction that these functions are smooth with respect to σ and x^o on $[0, \sigma_1] \times \partial V$. By virtue of (3.18) the $\mathrm{v}_{N,p}(x^o, \lambda)$'s are smooth functions of x^o and vanish identically for all $x^o \in \partial V - S$. Consequently, the same is true of $u_N(x^o, \lambda)$ as originally assumed.

It can also be shown inductively that as $\lambda \to \infty$

$$\partial_\sigma^n g_{N,j}(\sigma; x^o, \lambda) = O(1)\ (j = 1, 2, \cdots, N; n = 0, 1, 2, \cdots, N + 1),$$

uniformly in σ and x^o. Therefore, the sum in (3.16) is a smooth function of x^o and is $O(\lambda^{-(N+2)})$ as $\lambda \to \infty$, uniformly in x^o.

The integral in (3.16) is also a smooth function of x^o and is $O(\lambda^{-N})$, uniformly in x^o. This is because:

(i) $H_N(\sigma, \tau_1; x^o, \lambda)$ and the σ derivatives up to order $N + 1$ of this function are continuous in σ, τ_1 and x^o and are $O(1)$ uniformly in σ, τ_1 and x^o.

(ii) The σ derivatives up to order $N + 1$ of

$$u_{N,p}(z(\sigma, \tau_1; x^o), \lambda) = [H(x^o)]^{-1} \sum_{m=0}^{N+1} g_{N,m}(0; z(\sigma, \tau_1; x^o), \lambda)\lambda^{-m},$$

are continuous in σ, τ_1 and x^o and are $O(1)$ uniformly in σ, τ_1 and x^o.

We next consider the case where $x \to x^o \in \partial V - S$. We have

$$\lim_{x\to x^o} U_N(x, \lambda) = \lambda \int_{\tilde{S}} \exp[i\lambda\sigma(x^o, X(w))]F(w, x^o; \lambda)dw_1\, dw_2 \; (x \in \partial V - S) \tag{3.21}$$

where

$$F(w, x^o; \lambda) = u_N(X(w), \lambda)f(X(w), x^o; w, \lambda),$$

$$f(X, x^o; w, \lambda) = \Psi(x^o, X)A_N(x^o, X; \lambda)|X_{w_1} \times X_{w_2}|,$$

and as above the equation $x = X(w)$, $w \in \tilde{S}$, is a parametric representation of S.

The function $F(w, x^o; \lambda)$, and its derivatives with respect to w_1 and w_2, are continuous in x^o and smooth with respect to w on $\tilde{S} \times (\partial V - S)$. Furthermore, by virtue of our assumption that supp $X(x') \subseteq S'$ where $S \subset S'$ and the distance from $\partial S'$ to ∂S is positive, it follows that the function $u_N(X(w), \lambda)$ and consequently $F(w, x^o; \lambda)$ vanishes outside $\tilde{S}'$ where $\tilde{S}'$ is the preimage of S' under the mapping $x = X(w)$. Therefore, (3.21) reduces to

$$\lim_{x\to x^o} U_N(x, \lambda) = \lambda \int_{\tilde{S}'} \exp[i\lambda\sigma(x^o, X(w))]F(w, x^o; \lambda)dw_1\, dw_2, \tag{3.22}$$

for all $x^o \in \partial V - S$. Note that $\sigma(X(w), x^o)$ and $|\nabla_w \sigma(X(w), x^o)|$ are uniformly bounded away from zero on $\tilde{S} \times (\partial V - S)$. These functions are also continuous in x^o, and smooth with respect to w_1 and w_2.

If we integrate the r.h.s. of equation (3.22) by parts $N + 1$ times we obtain the result that if $x^o \in \partial V - S$, then

$$\lim_{x\to x^o} U_N(x, \lambda) \doteq i^{N+1} \lambda^{-N} \int_{\tilde{S}'} \exp[i\lambda\sigma(x^o, X(w))]F_{(N+1)}(w, x^o; \lambda)dw_1\, dw_2, \tag{3.23}$$

where

$$F_{(N+1)}(w, x^o; \lambda) = \Gamma(w, x^o)[\nabla_w \sigma(X, x^o) \cdot \nabla_w F_N(w, x^o; \lambda)] +$$
$$\nabla_w \cdot [\Gamma(w, x^o) \nabla_w \sigma(X, x^o)] F_N(w, x^o; \lambda),$$

and $F_0(w, x^o; \lambda) = F(w, x^o; \lambda)$.

We may therefore use mathematical induction to conclude that $F_N(w, x^o; \lambda)$ is a smooth function of w_1 and w_2, and is continuous in x^o on $\tilde{S} \times (\partial V - S)$. It follows from (3.23) that $U_N(x^o, \lambda)(= B_N(x^o, \lambda))$ is continuous on $\partial V - S$ and is $O(\lambda^{-N})$ as $\lambda \to \infty$, uniformly in x^o, $x^o \in \partial V - S$.

We next consider the tangential derivatives of $U_N(x, \lambda)$ on ∂V. On S these derivatives are linear combinations of the functions $[U_N(X(w), \lambda)]_{w_i}$ $(i = 1, 2)$ with coefficients that are smooth functions of w, $w \in \tilde{S}$. Therefore, to establish the smoothness and the asymptotic behavior of the tangential derivatives of $U_N(x, \lambda)$ on S as $\lambda \to \infty$, it suffices to consider the functions $[U_N(X(w), \lambda)]_{w_i}$ $(i = 1, 2)$. It follows from the preceding discussion that for all $w \in \tilde{S}$

$$\begin{aligned}[U_N(X(w), \lambda)]_{w_i} &= [\chi(X(w))h(X(w))]_{w_i} + \\ &(-1)^{N+1}(i\lambda)^{-N} \int_0^{2\pi} \int_0^{\sigma_1} \exp[i\lambda\sigma]\, \partial_\sigma^{N+1}[H_N(\sigma, \tau_1; X(w), \lambda) u_N \\ &(z(\sigma, \tau_1; X(w)), \lambda)]_{w_i}\, d\sigma\, d\tau_1 \end{aligned} \tag{3.24}$$

The partial derivatives of $H_N(\sigma, \tau_1; X(w), \lambda)$ and $u_N(z(\sigma, \tau_1; X(w)), \lambda)$ with respect to w_1 and w_2 are continuous in σ, τ_1 and w. Also these derivatives are $O(1)$ as $\lambda \to \infty$, uniformly with respect to σ, τ_1 and w. Consequently,

$$[U_N(X(w), \lambda)]_{w_i} = [\chi(X(w))h(X(w))]_{w_i} + [B_N(X(w), \lambda)]_{w_i}$$

where $[B_N(X(w), \lambda)]_{w_i}$ is continuous in w, and is $O(\lambda^{-N})$ as $\lambda \to \infty$, uniformly on $\tilde{S}$. This implies that the tangential derivatives of $U(x, \lambda) - U_N(x, \lambda)$ on S are continuous and are $O(\lambda^{-N})$ as $\lambda \to \infty$, uniformly on S. In particular the tangential derivative $[\nabla - n(n \cdot \nabla)][U(x, \lambda) - U_N(x, \lambda)]|_S$ where $n = n(x)$ is the outward unit normal to S at x, is continuous and is $O(\lambda^{-N})$ as $\lambda \to \infty$, uniformly on S. We finally consider the tangential derivatives of $U_N(x, \lambda)$ on $\partial V - S$. On a typical coordinate patch of $\partial V - S$ represented parametrically by $y = Y(\mathrm{v})$, the tangential derivatives of $U_N(x, \lambda)$ are linear combinations of the partial derivatives of $U_N(Y(\mathrm{v}), \lambda)$ with respect to v_1 and v_2. It follows from (3.22) that

$$[U_N(Y(\mathrm{v}), \lambda)]_{\mathrm{v}_i} = \lambda \int_{\tilde{S}} \exp[i\lambda\sigma(Y(\mathrm{v}), X(w))]\, F(\mathrm{v}, w; \lambda) dw_1\, dw_2 \; (Y(\mathrm{v}) \in \partial V - S), \tag{3.25}$$

where $F(\mathrm{v}, w; \lambda) = u_N(X(w), \lambda) f(X(w), Y(\mathrm{v}); \mathrm{v}, w, \lambda)$ and

$$f(X, Y; \mathrm{v}, w, \lambda) = [(\nabla\psi(Y, X)\cdot Y\mathrm{v}_i)\, \mathscr{A}_N(Y, X; \lambda) + \psi(Y, X)(\nabla\mathscr{A}_N(Y, X; \lambda)\cdot Y\mathrm{v}_i) +$$
$$i\lambda(\nabla\sigma(Y, X)\cdot Y\mathrm{v}_i)\psi(Y, X)\mathscr{A}_N(Y, X; \lambda)]|X_{w_1} \times X_{w_2}|.$$

Integrating the r.h.s. of (3.25) by parts $N + 2$ times we obtain the result that $[U_N(Y(\mathrm{v}), \lambda)]_{\mathrm{v}_i} = O(\lambda^{-N})$ as $\lambda \to \infty$ uniformly on each coordinate patch of $\partial V - S$. It follows that in addition to being continuous on $\partial V - S$, the tangential derivatives of $U(x, \lambda) - U_N(x, \lambda)$ are $O(\lambda^{-N})$ as $\lambda \to \infty$ uniformly on $\partial V - S$. In particular the tangential derivative

$$[\nabla - n(n\cdot\nabla)][U(x, \lambda) - U_N(x, \lambda)]|_{\partial V - S}$$

is continuous and is $O(\lambda^{-N})$ as $\lambda \to \infty$, uniformly on $\partial V - S$.

4. Proof of (1.4). To prove (1.4) we start with the following expression obtained from (1.1).

$$D[U_N(x, \lambda)] = \lambda \int_S [\psi(x, x')]_r\, G_N(x, x'; \lambda) u_N(x', \lambda) dS +$$
$$\lambda \int_S \psi(x, x')\, D[G_N(x, x'; \lambda)] u_N(x', \lambda) dS. \tag{4.1}$$

Recalling that $G_N(x, x'; \lambda) = \exp[i\lambda\sigma(x, x')][\sigma(x, x')]^{-1} A_N(x, x'; \lambda)$ we have

$$D[G_N(x, x'; \lambda) = \exp[]i\lambda\sigma(x, x')][\sigma(x, x')]^{-1}\, [[A_N(x, x'; \lambda)]_r +$$
$$[-i\lambda(1 - [\sigma(x, x')]_r) + (r^{-1} - [\sigma(x, x')]^{-1}[\sigma(x, x')]_r)]A_N(x, x'; \lambda)].$$

Using the asymptotic formulas $\sigma(x, x') = r + O(1)$, $[\sigma(x, x')]_r = 1 + O(r^{-2})$, $A_N(x, x'; \lambda) = O(1)$, $[A_N(x, x'; \lambda)]_r = O(r^{-2})$ that hold as $r \to \infty$, uniformly in x', $x' \in S$, to estimate the r.h.s. of the preceding equation we find that $G_N(x, x'; \lambda) = O(r^{-1})$ and $D[G_N(x, x'; \lambda)] = O(r^{-3})$ as $r \to \infty$, uniformly in x/r and x'. Also recall that $\psi(x, x') = O(1)$ and $[\psi(x, x')]_r = O(r^{-1})$ as $r \to \infty$, uniformly in x/r and x'.

Using the results obtained above to estimate the integral in (4.1) we conclude that $D[U_N(x, \lambda)] = O(r^{-2})$, uniformly in x/r, as $r \to \infty$. This estimate immediately implies (1.4).

5. Pointwise estimates for $U(x, \lambda) - U_N(x, \lambda)$. In [7] the a priori estimate

$$|U(x, \lambda)| \leq C\lambda^2 r^{-1}\Bigg[\left\{\int_V r^2|f|^2\, dV\right\}^{1/2} +$$
$$\left\{\int_{\partial V}\left|\left[\nabla g - n(n\cdot\nabla g)\right]\right|^2 dS\right\}^{1/2} + \lambda \max_{\partial V}|g|\Bigg] \tag{5.1}$$

for the solution $U(x, \lambda)$ of the following exterior boundary value problem is obtained, where C is a constant independent of λ and x, and n is the outward unit normal to ∂V:

$$[\Delta + \lambda^2 N^2(x)]U = f(x, \lambda) \; (x \in V),$$

$$U(x', \lambda) = g(x', \lambda) \; (x' \in \partial V),$$

$$\lim_{R\to\infty} \int_{r=R} r|D[U(x, \lambda)]|^2 \, dS = 0. \qquad (5.2)$$

Inequality (5.1) is derived under the following assumptions:

(i) $f(x, \lambda)$ is piecewise continuous on $\partial V \cup V$ and $\int_V r^2|f|^2 \, dV \leq \infty$.
(ii) $g(x, \lambda)$ and its tangential derivatives are piecewise continuous on ∂V.
(iii) ∂V is finite and can be illuminated from the exterior: there exists a (smooth) closed, convex surface Σ containing ∂V such that each point of ∂V lies on exactly one perpendicular from Σ. For every $x \in V \cup \partial V$ let $\sigma(x)$ be the arc length along the normal from Σ that passes through x; $\sigma(x)$ is smooth, $|\nabla\sigma(x)| = 1$, and $\sigma(x)$ is asymptotic to r as $r \to \infty$. Note that $\sigma_o - \sigma(x)$ is bounded away from zero on the inside of Σ for some positive constant σ_o.
(iv) $N(x) \in C^2[V] \cap C[V \cup \partial V]$.
(v) $N(x) \geq \varepsilon_0 > 0$ on $V \cup \partial V$ for some positive constant ε_o.
(vi) $|\nu\cdot\nabla N(x)|$, $|N(x) - 1| = O((\sigma(x) + \sigma_o)^{-p})$ uniformly on $\partial V \cup V$ for some positive constant p exceeding two, and some positive constant σ_o such that $\sigma(x) + \sigma_o$ is positive for all $x \in V \cup \partial V$. Here ν is the outward unit normal to Σ.
(vii) For $m = 1, 2$ all m^{th} order derivatives of $N(x)$ are $O(r^{-p-m})$, uniformly on $V \cup \partial V$.
(viii) The function $N(x)$ is slowly varying: for some positive constant ε_1 we have $N(x) + (\sigma(x) + \sigma_o)(\nabla\sigma(x)\cdot\nabla N(x)) \geq \varepsilon_1$ on the exterior of Σ, and $N(x) + (\sigma(x) - \sigma_o)(\nabla\sigma(x)\cdot\nabla N(x)) \geq \varepsilon_1$, on the interior of Σ.

If $E(x) \equiv I$ the function $U(x, \lambda) - U_N(x, \lambda)$ satisfies equations (5.2) with $f(x, \lambda) = \lambda^{-N} (1 + r^3)^{-1} \hat{R}_N(x, \lambda)$ and $g(x', \lambda) = \lambda^{-N} \hat{S}_N(x', \lambda)$, where as a consequence of what we have shown above $\hat{R}_N(x, \lambda)$ is continuous on $V \cup \partial V$, and $\hat{S}_N(x', \lambda)$ together with its tangential derivatives are continuous on ∂V. Furthermore, as $\lambda \to \infty$ we have $\hat{R}_N(x, \lambda) = O(1)$ uniformly on $V \cup \partial V$, $\hat{S}_N(x', \lambda) = O(1)$ and $[\nabla - n(n\cdot\nabla)]\hat{S}_N(x', \lambda) = O(1)$ uniformly on ∂V. It follows that if ∂V satisfies (iii), and if $N(x)$ satisfies conditions (iv)–(viii) then,

$$|U(x, \lambda) - U_N(x, \lambda)| \leq C\lambda^2 r^{-1} [\{\int_V r'^2 (1 + r'^3)^{-2} |\hat{R}_N(x', \lambda)|^2 \, dV\}^{1/2} +$$

$$\lambda^{-N} \{\int_{\partial V} |[\nabla - n(n\cdot\nabla)]\hat{S}_N(x', \lambda)|^2 \, dS\}^{1/2} + \lambda^{1-N} \max_{\partial V} |\hat{S}_N(x', \lambda)|]. \qquad (5.3)$$

We note that (5.1) and consequently (5.3) can be derived if ∂V is a smooth surface extending to infinity provided ∂V can be illuminated from the exterior by an infinite, smooth, convex surface Σ. This means that ∂V lies entirely on the concave side of Σ, and that every pair of normals from the concave side of Σ passes through ∂V before intersecting.

In [5] it is shown that for some constant C independent of λ and x,

$$|U(x, \lambda)| \leq C\lambda^2 \left[\left\{\int_V r^2|f|^2\, dV\right\}^{1/2} + \left\{\int_{\partial V} |[\nabla g - n(n\cdot\nabla g)]|^2\, dS\right\}^{1/2} + \lambda \max_{\partial V} |g|\right] \tag{5.4}$$

on every bounded subset of $V \cup \partial V$ if $U(x, \lambda)$ is the solution of the following exterior boundary value problem:

$$\begin{gathered} L_\lambda[U] = f(x, \lambda) \quad (x \in V), \\ U(x', \lambda) = g(x', \lambda) \ (x' \in \partial V), \\ \lim_{R\to\infty} \int_{r=R} r|D[U(x, \lambda)]|^2\, dS = 0. \end{gathered} \tag{5.5}$$

The pointwise, a priori estimate given by (5.4) holds if the following conditions are satisfied:

(i') $f(x, \lambda)$ is piecewise continuous on $V \cup \partial V$, and $\int_V r^2|f|^2\, dV \leq \infty$.
(ii') $g(x', \lambda)$ and its tangential derivatives are piecewise continuous on ∂V.
(iii') ∂V is finite, smooth, and star-shaped.
(iv') $N(x)$ is smooth and strictly positive on $V \cup \partial V$. $E(x)$ is smooth and strictly positive definite on $V \cup \partial V$.
(v') $[N(x)]^{-2} E(x) - I = O(r^{-p})$, $[E(x)]_r = O(r^{-p+1})$ and $\nabla N(x) = O(r^{-p+1})$ as $r \to \infty$, uniformly, for some constant p, $p > 2$.
(vi') The function $[N(x)]^2$ and the matrix $A(x) \equiv [N(x)]^{-2} E(x)$ are slowly varying:

$$\min_{V\cup\partial V}\left[\min_{|\xi|=1}\left(\xi\cdot(A(x)\bar{\xi})\right)\right] \geq \max_{V\cup\partial V}\left|2[N(x)]^{-1}\nabla N(x)\cdot A(x)\right| + \frac{1}{2}\max_{V\cup\partial V}\left|r[A(x)]_r\right|.$$

The function $U(x, \lambda) - U_N(x, \lambda)$ satisfies equations (5.5) if we set $f(x, \lambda) = \lambda^{-N}(1 + r^3)^{-1}\hat{R}_N(x, \lambda)$ and $g(x', \lambda) = \lambda^{-N}\hat{S}_N(x', \lambda)$, where as a consequence of what we have shown above $\hat{R}_N(x, \lambda)$ is continuous on $V \cup \partial V$, and $\hat{S}_N(x', \lambda)$ together with its tangential derivatives are continuous on ∂V. Furthermore, as $\lambda \to \infty$ we have $\hat{R}_N(x, \lambda) = O(1)$ uniformly on $V \cup \partial V$, $\hat{S}_N(x', \lambda) = O(1)$ and $[\nabla - n(n\cdot\nabla)]\,\hat{S}_N(x', \lambda) = O(1)$, uniformly on ∂V. It follows from (5.4) that if ∂V satisfies (iii'), and if $E(x)$ and $N(x)$ satisfies conditions (iv')–(vi'), then

$$\begin{aligned} &|U(x, \lambda) - U_N(x, \lambda)| \leq C\lambda^2 \left[\left\{\int_V r'^2(1 + r'^3)^{-2}|\hat{R}_N(x', \lambda)|^2\, dV\right\}^{1/2} + \right. \\ &\left.\lambda^{-N}\left\{\int_{\partial V} |[\nabla - n(n\cdot\nabla)]\hat{S}_N(x', \lambda)|^2\, dS\right\}^{1/2} + \lambda^{1-N}\max_{\partial V}|\hat{S}_N(x', \lambda)|\right] = O(\lambda^{3-N}) \end{aligned} \tag{5.6}$$

on every bounded subset of $V \cup \partial V$.

We note that (5.4), and consequently (5.6) can be derived if ∂V is a smooth, star-shaped surface extending to infinity.

6. Matching Across Caustics. If the subset of $V \cup \partial V$ covered by the orthogonal rays from supp $\chi(x')h(x')$ is caustic free then we may assume that the rays in T^+ form a normal congruence (cf. [15]), and it can be shown that

$$U_N(x, \lambda) = \exp[i\lambda\hat{\sigma}(x)]\hat{a}_0(x) + O(\lambda^{-1}) \tag{6.1}$$

as $\lambda \to \infty$. Here $x'(x)$ is the point where the "optical path" from x intersects ∂V. The function $\hat{\sigma}(x) = \sigma(x, x'(x))$ gives the "optical distance" from x to ∂V, ie., the distance along the orthogonal ray from $x'(x)$ on ∂V to x, and

$$\hat{a}_0(x) = \chi(x'(x))h(x') \exp$$

$$[-\int_0^{\hat{\sigma}(x)} [N(X(\sigma; x'(x)))]^{-2}\nabla\cdot[E(X(\sigma; x'(x)))\cdot\nabla(\hat{\sigma}(X(\sigma; x'(x)))]d\sigma]. \tag{6.2}$$

In the preceding equation $X(\sigma; x'(x))$ is a parametric representation of the optical path from $x'(x)$ to x. The functions $X(\sigma; x')$ and $P(\sigma; x')$ are uniquely determined by equations (3.3), and the initial conditions $X(0) = x'$, $P(0) = N(x')E^{-1/2}(x')\hat{n}(x')$, where $\hat{n}(x')$ is the unit vector pointing into V such that $t\cdot(E^{-1/2}(x')\, \hat{n}(x')) = 0$ for every vector t tangent to ∂V at x'. Note that $\hat{n}(x')$ is the outward unit normal to ∂V at x' if $E(x') \equiv I$.

The function $\hat{\sigma}(x)$ satisfies the eikonal equation $\nabla\sigma\cdot[E(x)\cdot\nabla\sigma]=[N(x)]^2$. Furthermore, for every $x' \in \partial V$ we have

$$[N(X(\sigma; x'))]^{-2}\nabla\hat{\sigma}(X(\sigma; x')) = P(\sigma; x'),$$

and

$$X_\sigma(\sigma; x') = [N(X(\sigma; x'))]^{-2}\, E(X(\sigma; x'))P(\sigma; x').$$

The r.h.s. of (6.1) is obtained if the integral representing $U_N(x, \lambda)$ is expanded asymptotically by the method of stationary phase with an error that is $O(\lambda^{-1})$. However, it is rather difficult show that the amplitude function $\hat{a}_0(x)$ that is first obtained from the stationary phase analysis of (1.1) is identical to r.h.s. of (6.2); a proof will be presented in a sequel to this paper.

We remark that the asymptotic expansion given by (6.1) is the leading term of the ansatz

$$\hat{U}_N(x, \lambda) = \exp[i\lambda\hat{\sigma}(x)] \sum_{m=0}^{N} \hat{a}_m(x)\lambda^{-m}, \tag{6.3}$$

where $\hat{a}_0(x)$ is given by (6.2), and the $\hat{a}_m(x)$'s are defined recursively for $m = 1, 2, 3,\ldots, N$, by the following transport equations and initial conditions:

$$2\nabla \hat{a}_m \cdot (E(x)\nabla\hat{\sigma}(x)) + \nabla\cdot(E(x)\nabla\hat{\sigma}(x))\hat{a}_m = i\nabla\cdot(E(x)\nabla\hat{a}_{m-1}), \tag{6.4}$$

$$\lim_{x\to x'(x)} \hat{a}_m = 0. \tag{6.5}$$

It can be shown that $L_\lambda\,[\hat{U}_N(x, \lambda)]$ is continuous on $V \cup \partial V$, and that $L_\lambda\,[\hat{U}_N(x, \lambda)] = O(\lambda^{-N}r^{-3})$, uniformly on $\hat{V} \cup \partial V$, as $\lambda \to \infty$. Furthermore, by virtue of (6.2) and (6.5) we have $\hat{U}_N(x', \lambda) = \chi(x')h(x')$ for all $x' \in \partial V$.

Within caustic free subsets of T^+ it is possible to construct *local*, high frequency approximate solutions of the problem **P**. Recall that T^+ is the subset of $V \cup \partial V$ filled by the orthogonal rays from points in points in $S \supset \operatorname{supp} \chi(x')h(x')$. The global approximate solution given by equation (1.1) which satisfies the given boundary and radiation conditions, "connects" these approximate solutions.

For example, suppose $\hat{T}^+$ is a caustic free subset of T^+, and assume for simplicity that only one ray from the support of $\chi(x')h(x')$ passes through each point of $\hat{T}^+$. We may assume without loss of generality that $\hat{T}^+$ is a subset of T^+ covered by orthogonal rays from the surface $S(\sigma_o) = \{y{:}y = X(\sigma_0; x'), x' \in \mathcal{N}\}$ where $\mathcal{N}$ is an open, simply connected, subset of S, and σ_0 is a non-negative constant. We have $\hat{T}^+ = \{x{:}\ x = X(\sigma + \sigma_0; x'), x' \in \mathcal{N}, \sigma \in [\sigma_0, \sigma_1], \sigma_1 > \sigma_0\}$.

Consequently, the function $\hat{U}_N(x', \lambda)$ given by equation (6.6) below is a local approximate solution that is asymptotically equivalent to the global approximate solution given by (1.1) as $\lambda \to \infty$, with an error that is $O(\lambda^{-(N+1)}\, r^{-1})$ uniformly on $\hat{T}^+$:

$$\hat{U}_N(x, \lambda) = \exp[i\lambda\sigma^{>}(x)] \sum_{m=0}^{N} b_m(x)\lambda^{-m} \tag{6.6}$$

The eikonal function $\sigma^{>}(x)$ appearing in (6.6) is the unique solution of the equation $x = X^{>}(\sigma^{>}; x'(x))$ for all x such that $x'(x) \in \mathcal{N}$, where $X^{>}(\sigma; x') \equiv X(\sigma + \sigma_0; x')$. The function $\exp[i\lambda\sigma^{>}(x)]\, b_0(x)$ is the leading term of the stationary phase expansion of the integral in (1.1) for $x \in \hat{T}^+$. The $b_m(x)$'s are defined recursively, for $m = 1, 2, 3,\ldots, N$, by the transport equations

$$2\nabla b_m \cdot (E(x)\nabla\sigma^{>}(x)) + \nabla\cdot(E(x)\nabla\sigma^{>}(x))b_m = i\nabla\cdot(E(x)\nabla b_{m-1}) \tag{6.7}$$

and the requirement that each $b_m(x)$ should vanish on $S(\sigma_0)$. Since

$$[N(X^{>}(\sigma; x'))]^{-2}\, \nabla\sigma^{>}(X^{>}(\sigma; x')) = E^{-1}(X^{>}(\sigma; x'))\, X^{>}_{\sigma}(\sigma; x')$$

we can rewrite (6.7) as

$$2(b_m)_\sigma + [N(X^{>}(\sigma; x'))]^{-2}\, \nabla\cdot(E(X^{>}(\sigma; x'))\nabla\sigma^{>}(X^{>}(\sigma; x'))b_m = i\nabla\cdot(E(X^{>}(\sigma; x'))\nabla b_{m-1}), \tag{6.8}$$

to be solved for $b_m(\sigma)$ ($\sigma > 0$) subject to the initial conditions $b_m(0) = 0$ for $m \geq 1$, and with $b_0(\sigma) = b_0(X^{>}(\sigma; x'))$.

7. CONCLUSION

In a sequel we intend to apply the techniques of this paper to radiating body problems where each of the orthogonal rays from the support of $h(x')$ intersects the boundary nontangentially at one or more points on ∂V outside the support of $h(x')$. If this occurs the radiation from the boundary source distribution is reflected by the boundary one or more times before propagating to infinity. The family of orthogonal rays from the support of $h(x')$ may form caustics before being reflected. Additional caustics may be formed after the first reflection.

We also intend to consider radiating body problems where the boundary is smooth, but not C^∞, problems where the boundary is only required to be piecewise smooth, and problems where $E(x)$, $N(x)$ or derivatives of these functions have jump discontinuities across surfaces lying in $V \cup \partial V$.

Additionally, we plan to consider radiation problems where the radiation emanates from a distributed source of compact support in $V \cup \partial V$, or a point source located at x^o in $V \cup \partial V$. We note that the techniques used in this paper can be applied directly to problems where (i) ∂V is infinite, and (ii) none of the rays from the point source is tangentially incident on the boundary. Under these conditions the geometrical theory of high frequency wave propagation predicts that every point x in the medium filling V is illuminated by radiation from x^o that travels along the optical path of least time from x^o to x and radiation that travels along the optical path of least time from x^o to x, and that includes a point of reflection on ∂V. For the special case $E(x) \equiv I$, $N(x) \equiv 1$ a rigorous high frequency asymptotic expansion of Green's function for the Dirichlet problem, valid in the illuminated portion of the region exterior to a finite smooth convex body, has been obtained by Morawetz and Ludwig in [14]; they used a pointwise a priori bound to estimate to the difference of the exact Green's function and a globally valid approximate Green's function. Subsequently, Alber [2] established a similar result for exterior regions with finite, smooth boundaries of more general shape, and used it to obtain a rigorous asymptotic expansion as $\lambda \to \infty$ of the solution of problem **P** with $E(x) \equiv I$ and $N(x) \equiv 1$. The approach in [2] is to represent the Green's function for the Helmholtz operator as the Fourier transform of a corresponding Riemann function. The local, asymptotic behavior of the Green's function as $\lambda \to \infty$ is established by analyzing corresponding local approximations of the Riemann function.

We remark that approximate solutions of problem **P** can be constructed by the method of this paper if the Dirichlet boundary condition is replaced by a boundary condition of the second or third kind.

Finally, we are also interested in extending the methods of this paper to Maxwell's equations, and the equations of linear elasticity.

APPENDIX.

In this appendix we establish the continuity in x and x', and the asymptotic behavior for large λ as $r \to \infty$ of the following functions and their derivatives with respect to the primed variables: $\mathcal{A}_N(x, x'; \lambda)$, $\nabla\mathcal{A}_N(x, x'; \lambda)$, $\Psi(x, x')$, $\nabla\Psi(x, x')$ and $\nabla\cdot(E(x)\nabla\Psi(x, x'))$. It is assumed here that x lies in $(V \cup \partial V) - V^+$ where $\sigma(x, x')$, and a fortiori $|x - x'|$ is bounded away from zero for all $x' \in S \subset \partial V$.

We first note that $\mathcal{A}_N(x, x'; \lambda) = a_N(\sigma(x, x'), \tau_1(x, x'), \tau_2(x, x'); x', \lambda)$, where the functions $\sigma(x, x')$, $\tau_1(x, x')$ and $\tau_2(x, x')$ are uniquely defined by the vector equation $x = X(\sigma, \tau_1, \tau_2; x')$. Recall that $x = X(\sigma, \tau_1, \tau_2; x')$ is a parametric representation of a typical ray emanating from x', and is defined by equations (3.3) and (3.4). Consequently, the continuity in x and x', and the asymptotic behavior of $\mathcal{A}_N(x, x'; \lambda)$, $\nabla\mathcal{A}_N(x, x'; \lambda)$ and the derivatives of these functions with respect to the primed variables for λ, $r >> 1$ is determined by:

(i) the continuity in x and x', and the asymptotic behavior as $r \to \infty$, of $\sigma(x, x')$, $\tau_1(x, x')$, $\tau_2(x, x')$, $\nabla\sigma(x, x')$, $\nabla\tau_1(x, x')$, $\nabla\tau_2(x, x')$, and the derivatives of these functions with respect to the primed variables.

(ii) the smoothness of $a_N(\sigma, \tau_1, \tau_2; x', \lambda)$ with respect to σ, τ_1, τ_2, x', and the asymptotic behavior of this function and its derivatives with respect to σ, τ_1, τ_2, x' for large λ as $\sigma \to \infty$.

The continuity of $\sigma(x, x')$ and $\tau_i(x, x')$ $(i = 1, 2)$ with respect to x and x' is implied by the continuity of the vector valued function $x = X(\sigma, \tau_1, \tau_2; x')$, and its first derivatives with respect to σ, τ_1, τ_2 and x', via the implicit function theorem. The continuity in x and x' of the derivatives of $\sigma(x, x')$ and $\tau_i(x, x')$ $(i = 1, 2)$ with respect to the primed variables is implied (inductively) by the continuity of the derivatives of $X(\sigma, \tau_1, \tau_2; x')$ with respect to σ, τ_1, τ_2 and x'. For example the continuity of first derivatives is established by differentiating both sides of the defining equation $x = X(\sigma(x, x'), \tau_1(x, x'), \tau_2(x, x'); x')$ with respect to the primed variables. We obtain the following formulas after some straightforward vector algebra:

$$\sigma_{x_j}(x, x') = K^{-1}[X_{\tau_1} \times X_{\tau_2}] \cdot X_{x_j} = [E(x)]^{-1} X_\sigma \cdot X_{x_j} \quad (j = 1, 2, 3), \tag{A.1}$$

$$[\tau_i(x, x')]_{x_j} = (-1)^{i+1} K^{-1} [X_{\tau_1} \times X_{\tau_{3-i}}] \cdot X_{x_j} \quad (i = 1, 2; j = 1, 2, 3), \tag{A.2}$$

$$\nabla\sigma(x, x') = -K^{-1}[X_{\tau_1} \times X_{\tau_2}], \tag{A.3}$$

$$\nabla\tau_i(x, x') = (-1)^{i+1} K^{-1} [X_{\tau_{3-i}} \times X_\sigma] \quad (i = 1, 2), \tag{A.4}$$

where $K(\sigma, \tau; x') = [X_{\tau_1} \times X_{\tau_2}] \cdot X_\sigma$. In the preceding formulas $\sigma = \sigma(x, x')$ and $\tau = \tau(x, x') \equiv (\tau_1(x, x'), \tau_2(x, x'))$.

Note that $K(\sigma, \tau; x') \neq 0$ if $\sigma > 0$, which is the case if $\sigma = \sigma(x, x')$ and $(x, x') \in ((V \cup \partial V) - V^+) \times S$. This follows from our hypothesis about the rays emanating from x', which guarantees that the equation $x = X(\sigma, \tau_1, \tau_2; x')$ defines a global coordinate transformation for every $x' \in S$.

The continuity in x and x' of higher order derivatives of $\sigma(x, x')$, $\tau_i(x, x')$ $(i = 1, 2)$, and of the functions $\nabla\sigma(x, x')$, $\nabla\tau_i(x, x')$ $(i = 1, 2)$ with respect to the primed variables for $x \neq x'$ ($x \neq x'$ if $x \in (V \cup \partial V) - V^+$ and $x' \in S$) is established inductively by repeated differentiation of the above equations, and making use of the fact that $X_\sigma(\sigma, \tau; x')$, $X_{\tau_i}(\sigma, \tau; x')$ $(i = 1, 2)$, $X_{x'_j}(\sigma, \tau; x')$ $(j = 1, 2, 3)$ are smooth functions of σ, τ_1, τ_2 and x' for all $\sigma \geq 0$, $0 \leq \tau_1 < 2\pi$, $0 \leq \tau_2 < \pi$, $x' \in S$.

To establish the asymptotic behavior of the aforementioned functions and their derivatives for large r, we make use of the following results that can be derived from equations (3.3) and (3.4) under our hypothesis on $E(x)$ and $N(x)$:

(i) $X(\sigma, \nu(\tau); x') = \sigma\mu(\tau; x') + c(\tau; x') + O(\sigma^{-p})$, $p > 2$, uniformly in τ_1, τ_2 and x', as $\sigma \to \infty$, where the unit vector $\mu(\tau; x')$ and the vector $c(\tau; x')$ are smooth functions of τ_1, τ_2 and x'.

(ii) $X_{\tau_i}(\sigma, \nu(\tau); x') = \sigma\mu_{\tau_i}(\tau; x') + c_{\tau_i}(\tau; x') + O(\sigma^{-p})$ $(i = 1, 2; p > 2)$ uniformly in τ_1, τ_2 and x' as $\sigma \to \infty$.

(iii) $X_{x_j}(\sigma, \nu(\tau); x') = \sigma\mu_{x'_j}(\tau; x') + C_{x_j}(\tau; x') + O(\sigma^{-p})$ $(j = 1, 2, 3; p > 2)$ uniformly in τ_1, τ_2 and x' as $\sigma \to \infty$.

(iv) $X_\sigma(\sigma, \nu(\tau); x') = \mu(\tau; x') + O(\sigma^{-p-1})$ $(p > 2)$ uniformly in τ_1, τ_2 and x' as $\sigma \to \infty$.

(v) $\sigma(x, x') = r - 2\mu(\tau; x')\cdot c(\tau; x') + O(r^{-1})$ uniformly in x' as $r \to \infty$.

(vi) Derivatives of $X_{\tau_i}(\sigma, \nu(\tau); x')$ $(i = 1, 2)$, $X_{x_j}(\sigma, \nu(\tau); x')$ $(j = 1, 2, 3)$ and $X_\sigma(\sigma, \nu(\tau); x')$ with respect to the primed variables and/or σ, τ_1, τ_2 are asymptotic to the corresponding derivatives of $\sigma\mu_{\tau_i}(\tau; x') + c_{\tau_i}(\tau; x')$, $\sigma\mu_{x'_j}(\tau; x') + c_{x'_j}(\tau; x')$ and $\mu(\tau; x')$ uniformly in τ_1, τ_2 and x' as $\sigma \to \infty$, with errors that are $O(\sigma^{-p})$, $O(\sigma^{-p})$ and $O(\sigma^{-p-1})$ respectively.

(vii) Derivatives of $\sigma(x, x')$ with respect to x_j $(j = 1, 2, 3)$ are asymptotic to the corresponding derivatives of $-2\mu(\tau; x')\cdot c(\tau; x')$ uniformly in x' as $r \to \infty$, with an error that is $O(r^{-1})$.

Using the above asymptotic formulas in (A.1)–(A.4) it follows that as $r \to \infty$

$$\sigma_{x_j}(x, x') \approx \sigma\mu\cdot\mu_{x_j} + \mu\cdot c = \mu\cdot c \; (j = 1, 2, 3), \tag{A.5}$$

$$[\tau_i(x, x')]_{x_j} \approx (-1)^{i+1}\sigma^{-1}(\mu \times \mu_{\tau_{3-i}})\cdot\mu_{x_j} K_\infty^{-1} \; (i = 1, 2; j = 1, 2, 3), \tag{A.6}$$

$$\nabla\sigma(x, x') \approx (\mu_{\tau_1} \times \mu_{\tau_2})K_\infty^{-1}, \tag{A.7}$$

$$\nabla\tau_i(x, x') \approx (-1)^i\sigma^{-1}(\mu \times \mu_{\tau_{3-i}})K_\infty^{-1} \; (i = 1, 2), \tag{A.8}$$

uniformly in x'. In the preceding formulas $K_\infty = \mu\cdot(\mu_{\tau_1} \times \mu_{\tau_2})$, $\mu = \mu(\tau(x, x'); x')$ and $c = c(\tau(x, x'); x')$. Uniform asymptotic formulas as $r \to \infty$ for derivatives of $\sigma(x', x)$, $\nabla\sigma(x', x)$,

$\tau_i(x', x)$ $(i = 1, 2)$ and $\nabla\tau_i(x', x)$ $(i = 1, 2)$ with respect to the primed variables are obtained by differentiating both sides of the above asymptotic equivalencies with respect to the primed variables.

Next, consider $a_N(\sigma, \tau; x', \lambda)$. We wish to study the smoothness and asymptotic behavior of this function as $\sigma \to \infty$. Integrating the transport equations (1.5), (1.6) and (1.7) along a typical characteristic curve given parametrically by $x = X(\sigma, \nu(\tau); x')$, it can be shown that

$$a_N(\sigma, \tau; x', \lambda) = \sum_{n=0}^{N+1} \alpha_n(\sigma, \tau; x')\lambda^{-n}. \qquad (A.9)$$

Clearly, $a_n(x, x') = \alpha_n(\sigma(x, x'), \tau(x, x'); x')$, where the $a_n(x, x')$'s are solutions of the transport equations (1.5), (1.6) and (1.7). The function $\alpha_0(\sigma, \tau; x')$ is given by

$$\alpha_0(\sigma, \tau; x') = (4\pi)^{-1}[\det E(x')]^{-1/2} N(x')[c_0(\tau; x') N^{-2}(X(\sigma, \nu(\tau); x'))K^{-1/2}(\sigma, \tau; x')],$$

where

$$c_0(\tau; x') = \lim_{\sigma\to 0+} \sigma^{-2} N^2(X(\sigma, \nu(\tau); x'))K(\sigma, \tau; x') =$$

$$2^{-1}[N^2(X(0, \nu(\tau); x'))K(0, \tau; x')]_{\sigma\sigma} = -[\det E(x')]^{1/2} \sin \tau_2.$$

A recursive formula for the functions $\alpha_n(\sigma, \tau; x')$ $(n = 1, 2, 3, \ldots, N + 1)$ implied by the inhomogenous transport equations (1.6) and (1.7), is given by

$$\alpha_n(\sigma, \tau; x') = \alpha_0(\sigma, \tau; x') \int_0^\sigma [\alpha_0(\tilde{\sigma}, \tau; x')]^{-1} \nabla\cdot(E\nabla\alpha_{n-1})(\tilde{\sigma}, \tau; x')d\tilde{\sigma}. \qquad (A.10)$$

Here

$$\nabla\cdot(E\nabla\alpha_{n-1}) = N^2[(\alpha_{n-1})_{\sigma\sigma} + K^{-1} K_\sigma (\alpha_{n-1})_\sigma] +$$
$$N^2[\nabla\tau_1\cdot(E\nabla\tau_1)(\alpha_{n-1})_{\tau_1\tau_1} + 2\nabla\tau_1\cdot(E\nabla\tau_2)(\alpha_{n-1})_{\tau_1\tau_2} +$$
$$\nabla\tau_2\cdot(E\nabla\tau_2)(\alpha_{n-1})_{\tau_2\tau_2} + \nabla\cdot(E\nabla\tau_1)(\alpha_{n-1})_{\tau_1} + \nabla\cdot(E\nabla\tau_2)(\alpha_{n-1})_{\tau_2}],$$

and

$$\nabla\cdot(E\nabla\tau_i) = \nabla\sigma\cdot(E\nabla\tau_i)_\sigma + \nabla\tau_1\cdot(E\nabla\tau_i)_{\tau_1} + \nabla\tau_2\cdot(E\nabla\tau_i)_{\tau_2}.$$

To derive (A.10) from the transport equations (1.5), (1.6) and (1.7) the formulas $\nabla\cdot(E\nabla\sigma) = N^2[(N^2 K)^{-1} (N^2 K)_\sigma]$ and $E\nabla\sigma = N^2 X_\sigma$ are used.

Recursive formulas for the derivatives of $\alpha_n(\sigma, \tau; x')$ with respect to σ, τ_1, τ_2 and x_j' $(j = 1, 2, 3)$ can be derived by repeated differentiation of equation (A.10) with respect to σ, τ_1, τ_2 and x_j' $(j = 1, 2, 3)$. The continuity and the asymptotic behavior as $\sigma \to \infty$ of $\alpha_n(\sigma, \tau; x')$, and its derivatives for $n = 1, 2, 3, \ldots, N + 1$, can then be established by mathematical induction. It can be shown that as $\sigma \to \infty$:

$$(i)\ \alpha_0(\sigma, \tau; x') \approx f_0(\tau; x')/\sigma,\ (ii)\ \alpha_n(\sigma, \tau; x') \approx \alpha_0(\sigma, \tau; x') f_n(\tau; x'),$$

uniformly in τ_1, τ_2 and x', where the $f_n(\tau; x')$'s are smooth functions of their arguments. It can also be shown that the asymptotic equivalencies given by (i) and (ii) are preserved under differentiation with respect to σ, τ_1, τ_2 and x_j' $(j = 1, 2, 3)$. It follows that $a_n(\sigma, \tau; x', \lambda)$ is a smooth function of σ, τ_1, τ_2 and x'. Furthermore, as $\sigma \to \infty$

$$a_N(\sigma, \tau; x', \lambda) \approx f_0(\tau; x')\sigma^{-1}\left[1 + \sum_{n=0}^{N+1} f_n(\tau; x')\lambda^{-n}\right], \tag{A.11}$$

uniformly in τ_1, τ_2 and x'. Uniform asymptotic formulas for the derivatives of $a_n(\sigma, \tau; x', \lambda)$ are given by the corresponding derivatives of the right hand side of (A.11). In particular it follows from (A.11) that as $\sigma \to \infty$:

$$a_N(\sigma, \tau; x', \lambda) = O(\sigma^{-1}) \text{ and } \partial_\sigma^m a_N(\sigma, \tau; x', \lambda) = O(\sigma^{-1-m})\ (m = 1, 2, 3, \cdots). \tag{A.12}$$

All derivatives of $\alpha_n(\sigma, \tau; x', \lambda)$ with respect to τ_1 and/or τ_2 are $O(\sigma^{-1})$. (A.13)

All derivatives of $\alpha_n(\sigma, \tau; x', \lambda)$ with respect to the primed variables are $O(\sigma^{-1})$. (A.14)

From the above discussion we conclude that $\mathcal{A}_N(x, x'; \lambda)$ and its derivatives with respect to the primed variables are continuous in x and x', and that these functions are $O(\sigma^{-1}(x, x')) = O(r^{-1})$ as $r \to \infty$.

To establish the continuity in x and x', and the asymptotic behavior as $r \to \infty$ of $\nabla\mathcal{A}_N(x, x'; \lambda)$ and its derivatives with respect to the primed variables, we simply note that

$$\nabla\mathcal{A}_N(x, x'; \lambda) = \nabla a_N = (\partial_\sigma a_N)\nabla\sigma + (\partial_{\tau_1} a_N)\nabla\tau_1 + (\partial_{\tau_2} a_N)\nabla\tau_2, \tag{A.15}$$

where $a_N = a_N(\sigma, \tau; x', \lambda)$, $\sigma = \sigma(x, x')$ and $\tau = \tau(x, x')$. In view of the above discussion we may conclude that $\nabla\mathcal{A}_N(x, x'; \lambda)$ and derivatives of this function with respect to the primed variables are continuous in x and x', and that these functions are $O(\sigma^{-2}(x, x')) = O(r^{-2})$ as $r \to \infty$, uniformly in x'.

We next investigate the function $\psi(x, x') = \gamma(\cos\theta(x, x'))$ where $\cos\theta(x, x') = n(x')\cdot\nabla\sigma(x, x')|\nabla\sigma(x, x')|^{-1}$. We recall that $n(x')$ is the outward unit normal to ∂V at x', and that (a') $\gamma(\xi) \in C^\infty([-1, 1])$, (b') $0 \leq \gamma(\xi) \leq 1$ for all $\xi \in [-1, 1]$, (c') $\gamma(\xi) \equiv 1$ if $\cos(\pi/2 + \delta_1) \leq \xi \leq \cos 0$, $\delta_1 > 0$, (d') and $\gamma(\xi) \equiv 0$ if $\cos\pi \leq \xi \leq \cos(\pi - \delta_2)$, $\delta_1 + \delta_2 <$

$\pi/2$. We need to know that the functions $\psi(x, x')$, $\nabla\psi(x, x')$, $\nabla\cdot(E(x)\nabla\psi(x, x'))$ and their derivatives with respect to the primed variables are continuous in x and x', and we need to determine their asymptotic behavior as $r \to \infty$.

Since $\psi(x, x') = \gamma(\xi(x, x'))$ and $\nabla\psi(x, x') = \gamma_\xi(\xi(x, x'))\nabla\xi(x, x')$ where

$$\xi(x, x') = \cos\theta(x, x') = n(x')\cdot\zeta(\tau(x, x'); x'), \tag{A.16}$$

and

$$\zeta(\tau; x') = [E^{-1/2}(x')\,\nu(\tau)][\nu(\tau)\cdot[E^{-1}(x')\,\nu(\tau)]]^{-1/2} \tag{A.17}$$

for all $x' \in \partial V$, the continuity of $\psi(x, x')$, $\nabla\psi(x, x')$ and the derivatives of these functions with respect to the primed variables is implied by the defined smoothness of $\gamma(\xi)$ on $[-1, 1]$, and the continuity of $\xi(x, x')$, $\nabla\xi(x, x')$ and the derivatives of these functions with respect to the primed variables. By assumption $n(x')$, the outward unit normal to ∂V, is a smooth is a function of x'. The components of the unit vector $\nu(\tau(x, x'))$ are the functions $\cos\tau_1(x, x')\sin\tau_2(x, x')$, $\sin\tau_1(x, x')\sin\tau_2(x, x')$, and $\cos\tau_2(x, x')$. Consequently, the continuity of the functions $\xi(x, x')$, $\nabla\xi(x, x')$ and the derivatives of these functions with respect to the primed variables is an immediate consequence of the continuity in x and x' of $\tau_i(x, x')$ $(i = 1, 2)$, $\nabla\tau_i(x, x')$ $(i = 1, 2)$, $E^{-1/2}(x')$ and the derivatives of these functions with respect to the primed variables.

Furthermore, $\psi(x, x') = O(1)$, uniformly in x and x', since by definition $0 \le \gamma(\xi) \le 1$ for all $-1 \le \xi \le 1$. To establish the asymptotic behavior of $\nabla\psi(x, x')$ as $r \to \infty$ we first recall that $\nabla\psi(x, x') = \gamma_\xi(\xi(x, x'))\nabla\xi(x, x')$ where

$$\begin{aligned}\nabla\xi(x, x') &= \nabla\tau_1\, n(x')\cdot[\zeta(\tau; x')]_{\tau_1} + \nabla\tau_2\, n(x')\cdot[\zeta(\tau; x')]_{\tau_2}, \text{ and } \tau = \tau(x, x') \\ &= (\tau_1(x, x'), \tau_2(x, x')).\end{aligned} \tag{A.18}$$

It follows that $\nabla\psi(x, x') = O(r^{-1})$ as $r \to \infty$ since $\nu(\tau)\cdot[E^{-1}(x')\,\nu(\tau)]$ is bounded away from zero uniformly in τ and x', $|\nu_{\tau_1}(\tau)|, |\nu_{\tau_2}(\tau)| \equiv 1$, $\gamma_\xi(\xi) = O(1)$, $\nabla\tau_1(x, x')$, $\nabla\tau_2(x, x') = O(\sigma^{-1}(x, x')) = O(r^{-1})$ as $r \to \infty$, uniformly in x', $x' \in \partial V$.

Finally, we consider the function $\nabla\cdot(E(x)\nabla\psi(x, x'))$. Under our hypothesis that $E(x) - I = O(r^{-p})$ and $\nabla\cdot E(x) = O(r^{-p-1})$ as $r \to \infty$, we have

$$\begin{aligned}\nabla\cdot(E(x)\nabla\psi(x, x')) = N^2[&(\nabla\tau_1)\cdot(E\nabla\tau_1)\Psi_{\tau_1\tau_1} + (\nabla\tau_2)\cdot(E\nabla\tau_2)\Psi_{\tau_2\tau_2} + 2\nabla\tau_1\cdot(E\nabla\tau_2)\Psi_{\tau_1\tau_2} \\ &+ \nabla\cdot(E\nabla\tau_1)\Psi_{\tau_1} + \nabla\cdot(E\nabla\tau_1)\Psi_{\tau_2}]\end{aligned} \tag{A.19}$$

where

$$\Psi \equiv \Psi(\tau; x') = \gamma(n(x')\cdot\zeta(\tau; x')),$$

$$\Psi_{\tau_i} = \gamma_\xi\,[n(x')\cdot\zeta_{\tau_i}]\ (i = 1,2),$$

$$\Psi_{\tau_i\tau_i} = \gamma_{\xi\xi}[n(x')\cdot\zeta_{\tau_i}] + \gamma_{\xi}[n(x')\cdot\zeta_{\tau_i\tau_i}]\ (i = 1,2),$$

$$\Psi_{\tau_1\tau_2} = \gamma_{\xi\xi}[n(x')\cdot\zeta_{\tau_1}][n(x')\cdot\zeta_{\tau_2}] + \gamma_{\xi}[n(x')\cdot\zeta_{\tau_1\tau_2}].$$

Furthermore,

$$\nabla\cdot(E\nabla\tau_1) = (\nabla\sigma)\cdot(E_\sigma\nabla\tau_1) + (\nabla\sigma)\cdot(E(\nabla\tau_1)_\sigma) + (\nabla\tau_1)\cdot(E_{\tau_1}\nabla\tau_1) + (\nabla\tau_1)\cdot(E(\nabla\tau_1)_{\tau_1}) + (\nabla\tau_2)\cdot(E_{\tau_2}\nabla\tau_1) + (\nabla\tau_2)\cdot(E(\nabla\tau_1)_{\tau_2}), \tag{A.20}$$

and

$$\nabla\cdot(E\nabla\tau_2) = (\nabla\sigma)\cdot(E_\sigma\nabla\tau_2) + (\nabla\sigma)\cdot(E(\nabla\tau_2)_\sigma) + (\nabla\tau_1)\cdot(E_{\tau_1}\nabla\tau_2) + (\nabla\tau_1)\cdot(E(\nabla\tau_2)_{\tau_1}) + (\nabla\tau_2)\cdot(E_{\tau_2}\nabla\tau_2) + (\nabla\tau_2)\cdot(E(\nabla\tau_2)_{\tau_2}) \tag{A.21}$$

where $E_\sigma = (\nabla E)\cdot X_\sigma$, $E_{\tau_1} = (\nabla E)\cdot X_{\tau_1}$ and $E_{\tau_2} = (\nabla E)\cdot X_{\tau_2}$. It follows from the above discussion that $\nabla\cdot(E(x)\nabla\psi(x, x'))$ and the derivatives of this function with respect to the primed variables are continuous in x and x'. It also follows that $\nabla\cdot(E(x)\nabla\psi(x, x')) = O(r^{-2})$ as $r \to \infty$, since as $\sigma(x, x') \to \infty$:

$$\Psi_{\tau_1\tau_1}, \Psi_{\tau_2\tau_2}, \Psi_{\tau_1\tau_2} = O(1),$$

$$K^{-1}K_\sigma = O(\sigma^{-1}) = O(r^{-1}),$$

$$(\nabla\tau_1)\cdot(E\nabla\tau_1), (\nabla\tau_2)\cdot(E\nabla\tau_2), (\nabla\tau_1)\cdot(E\nabla\tau_2) = O(\sigma^{-2}) = O(r^{-2}),$$

$$\nabla\cdot(E\nabla\tau_1), \nabla\cdot(E\nabla\tau_2) = O(r^{-2}).$$

To obtain the last two order estimates in the preceding list, we make use of the following order estimates in (A.20) and (A.21). As $r \to \infty$

$$(\nabla\sigma)\cdot(E_\sigma\nabla\tau_1) = O(1)O(\sigma^{-1})O(\sigma^{-1}) = O(r^{-2}),$$

$$(\nabla\tau_1)\cdot(E_{\tau_1}\nabla\tau_1), (\nabla\tau_2)\cdot(E_{\tau_2}\nabla\tau_1) = O(\sigma^{-1})O(1)O(\sigma^{-1}) = O(r^{-2}),$$

$$(\nabla\tau_1)\cdot(E(\nabla\tau_1)_{\tau_1}), (\nabla\tau_2)\cdot(E(\nabla\tau_1)_{\tau_2}) = O(\sigma^{-1})O(1)O(\sigma^{-1}) = O(r^{-2}),$$

$$(\nabla\tau_1)\cdot(E(\nabla\tau_2)_{\tau_1}), (\nabla\tau_2)\cdot(E(\nabla\tau_2)_{\tau_2}) = O(\sigma^{-1})O(1)O(\sigma^{-1}) = O(r^{-2}).$$

Note that asymptotic formulas for the functions $(\nabla\tau_1)_{\tau_1}$, $(\nabla\tau_2)_{\tau_1}$, $(\nabla\tau_1)_{\tau_2}$ and $(\nabla\tau_2)_{\tau_2}$ that occur above can be formally obtained by differentiating equations (A.8) with respect to τ_i $(i = 1, 2)$, and then setting $\tau = \tau(x, x')$.

References

1. J. D. Achenbach and A. K. Gautesen, *Ray methods for waves in elastic solids*, Pitman, Boston, 1982.
2. H. D. Alber, *Justification of geometrical optics for non-convex obstacles*, J. Math. Anal. Appl. **80** (1981), pp. 372–386.
3. G. S. S. Avila and J. B. Keller, *The high-frequency asymptotic field of a point source in an inhomogeneous medium*, Comm. Pure Appl. Math. **16** (1963), pp. 363–381.
4. V. M. Babic, *On the short-wave asymptotic behavior of the solution of the problem of a point source in an inhomogeneous medium*, U.S.S.R. Comp. Math. and Math. Phys. **5**:5 (1965), pp. 247–251.
5. C. O. Bloom, *Estimates for solutions of reduced hyperbolic equations of the second order with a large parameter*, J. Math. Anal. Appl. **44** (1973), pp. 310–332.
6. C. O. Bloom and N. D. Kazarinoff, *Short wave radiation problems in inhomogeneous media: asymptotic solutions*, Lecture Notes in Math., vol. **22**, Springer, New York, 1976.
7. C. O. Bloom and N. D. Kazarinoff, *A priori bounds for solutions of the Dirichlet problem for* $[\Delta + \lambda^2 n(x)]u = f(x, \lambda)$ *on an exterior domain*, J. Differential Equations **24** (1977), pp. 437–465.
8. G. F. D. Duff, *Partial Differential Equations*, University of Toronto Press, Toronto, 1956.
9. F. G. Friedlander, *Sound Pulses*, Cambridge University Press, London, 1958.
10. R. Grimshaw, *High-frequency scattering by finite convex regions*, Comm. Pure Appl. Math. **19** (1966), pp. 167–198.
11. M. Kline and I. W. Kay, *Electromagnetic theory and geometrical optics*, Wiley-Interscience, New York, 1965.
12. R. K. Luneburg, *Mathematical theory of optics*, University of California Press, Berkley & Los Angeles, 1964.
13. C. S. Morawetz, *Decay for solutions of the Dirichlet exterior problem for the wave equation*, Comm. Pure Appl. Math. **28** (1975), pp. 229–264.
14. C. S. Morawetz and D. Ludwig, *An inequality for the reduced wave operator and the justification of geometrical optics*, Comm. Pure Appl. Math. **21** (1968), pp. 187–203.
15. O. N. Stavroudis, *The optics of rays, wave fronts and caustics*, Academic Press, New York, 1972.
16. F. Ursell, *On the short-wave asymptotic theory of the wave equation* $(\nabla^2 + k^2)\phi = 0$, Proc. Cambridge Philos. Soc. **53** (1957), pp. 115–133.
17. P. L. E. Uslenghi, *Electromagnetic scattering*, Academic Press, New York, 1978.
18. B. R. Vainberg, *Asymptotic methods in equations of mathematical physics*, Gordon and Breach Science Publishers, New York, 1988.
19. B. R. Vainberg, *On a point source in an inhomogeneous medium*, Math. U.S.S.R Sbornik **23** (1974), no. 1, pp. 128–147.
20. J. H. C. Whitehead, *Convex regions in the geometry of paths*, Quart. J. of Math. (Oxford) **3** (1932), pp. 33–42.