Mathematical Problems in Engineering
Volume 3. pp. 1–25
Reprints available directly from the publisher
Photocopying permitted by license only

Lyapunov Analysis of Sliding Motions: Application to Bounded Control

W. PERRUQUETTI, J.P. RICHARD and P. BORNE*

Ecole Centrale de LILLE, LAIL-URA CNRS D 1440, BP 48, 59651 Villeneuve d'Ascq CEDEX—FRANCE

(Received 15 January 1995)

The results concern the fundamental problem of Lyapunov analysis of sliding motions. It consist first to estimate the useful part of the sliding surface (the so-called "sliding domain") and second to estimate the useful part of the state domain that is the domain of all initial conditions for which the corresponding solutions converge to the sliding domain. The application of such results concern the design of a realistic bounded control. Several examples are exposed in order to illustrate the obtained results.

Keywords: sliding modes control; lyapunov functions; bounded control

Classification Categories: 34A34, 34A40, 93C15, 93D15

1. INTRODUCTION

Sliding mode control has been widely investigated since the first results given by Andronov [1]. In particular, V.I. Utkin [9, 10] gave a general framework for such a control based on variable structure system theory and the leading notion of sliding motions: the control commutes between two values in order to force the system's motions to evolve on a desired surface which is called the sliding surface. The existence conditions of sliding motions (this is, motions belonging to the sliding surface) given by Filipov [2, 3] are only local and unfortunately without much of practical interest from the engineer's point of view for the following reasons:

*Corresponding author.

i) in order to obtain global convergence onto the whole sliding surface, one needs to implement an unbounded control (see for example Section 3.2) which is not realistic due to the physical constraints of the system,
ii) a solution to i) is to implement a saturated control, but in that case what is the validity of the new control? In particular what are the admissible initial conditions which ensure the "desired" sliding motion, this is on a restricted part of the sliding surface?

Let us consider such problems for systems modelled by:

$$\frac{dx}{dt} = f(t, x) + g(t, x)\,u, \tag{1}$$

where:
$x \in \mathbb{R}^n$ is the **state vector**,
$\underline{x}(t;\, t_0,\, x_0;\, u)$ is the **system motion** (in short $\underline{x}(t)$),
$u \in \mathbb{R}$ is the **control**,
$t \in [t_0, +\infty[$.

Furthermore, the sliding motions are studied for a control defined as follows:

$$u = \begin{cases} u^+(t, x) \text{ if } s > 0 \\ u^-(t, x) \text{ if } s < 0 \end{cases}, \text{ with } s = s(x). \tag{2}$$

After some notations in Section 2, Section 3 provides a general framework for the above mentionned problems, that are the estimations of:

–the useful part of a given sliding surface called sliding domain,
–the initial state for which the motions converge to the above estimate of the sliding domain.

Section 4 gives several results for solving the key problems. These results are based on the use of two Lyapunov functions, at each step we relax some hypothesis on the used Lyapunov functions. Some examples illustrate the results and their different contexts.

Then, Section 5 gives conditions for the estimation of the domain of asymptotic stability of the origin. Lastly, Section 6 provides a methodology in order to design a bounded control ensuring the desired sliding properties and asymptotical stability of the origin.

In each section, the obtained results are illustrated with examples.

2. NOTATIONS

In the following, the two vector fields (f, g) are assumed to be smooth enough (for example satisfy Lipschitz's condition) and the two controls $(u^+(t, x), u^-(t, x))$ satisfy Filipov's conditions such that the solution of (1) exists, is continuous w.r.t. time and defined for $t \in [t_0, +\infty]$.

- $\overline{(.)}$, $\overset{\circ}{(.)}$, ∂ (.) respectively denote the closure, the interior and the boundary of the set (.).
- ρ, the Euclidean distance.
- $\mathcal{N}(\mathcal{A}; \varepsilon) = \{x \in \mathbb{R}^n: \rho(\mathcal{A}; x) < \varepsilon\}$, the ε-neighbourhood of $\mathcal{A}$.
- $D_t^+ V(x) = \operatorname*{Lim}_{\theta \downarrow 0^-} \dfrac{V[x(t+\theta)] - V[x(t)]}{\theta}$, the right-hand time derivative (Dini derivative).
- $\nabla(V; z)$, the usual gradient of V with respect to the variable z.
- $C^\alpha(S, \mathbb{R}^k)$, the set of α-times continuously differentiable functions from S into $\mathbb{R}^k$.
- Sgn(x) is the signum function defined as follow:

$$\text{Sgn}(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0 \end{cases}.$$

3. FRAMEWORK

3.1. Definitions

The sliding modes approach has been widely investigated (see [9, 10] and references herein) and, obviously, a tight connection between the study of sliding motions and stability theory has been pointed out, which leads to the definition of sliding domains [9, 10]. Roughly speaking, the sliding domain (D_s) is the useful part of the sliding surface: the sliding motions can only "appear" on D_s. For a single input system, D_s is defined as follow:

DEFINITION 1 (see [9, 10] and figure 1) A domain D_s of dimension $(n - 1)$ included in the manifold $\{s = 0\}$ is a sliding domain for system (1) with the control law u, if assumption P is true:

$$P\colon [\forall\ \varepsilon > 0,\ \exists\ \delta(\varepsilon) > 0, \text{ such that } \forall\ x_0 \in \mathcal{N}(D_s;\delta), \text{ the solution}$$
$$\underline{x}(t) = \underline{x}(t;\, t_0, x_0;\, u) \text{ can only leave } \mathcal{N}(D_s;\varepsilon) \text{ through } \mathcal{N}(\partial D_s;\varepsilon)] \quad (3)$$

One can notice that assumption P is a stability-like assumption (replace the end of P by "$\underline{x}(t)$ cannot leave $\mathcal{N}(D_s;\varepsilon)$"). Utkin [9, 10] gives results on the estimate of D_s using Lyapunov function of variable s (the other are "extra-variables"). As in stability domain theory [4, 6, 7], the estimation of initial state (x_0) such that assumption P is true, is of practical interest, leading for sliding modes theory to the following notion:

DEFINITION 2 (see figure 1) A domain $D_i(D_s)$ of dimension n included in

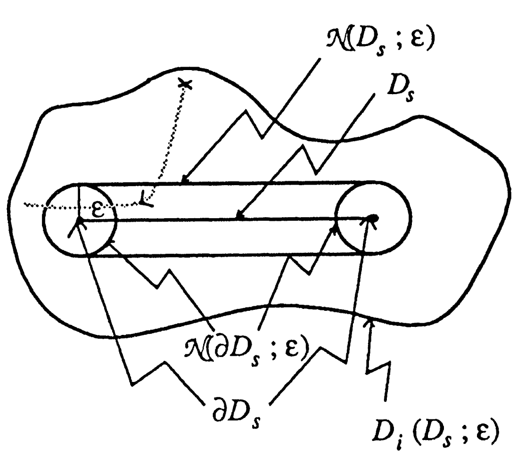

FIGURE 1 Illustration of definitions.

the state space is the initial domain of sliding motions for system (1) with the control law u, if:

i) $\forall \varepsilon > 0$, let $D_i(D_s; \varepsilon)$ be a neighbourhood of D_s such that the solution $x(t)$ can only leave $\mathcal{N}(D_s; \varepsilon)$ through $\mathcal{N}(\partial D_s; \varepsilon)$ if and only if $x_0 \in D_i(D_s; \varepsilon)$,

ii) $D_i(D_s) = \bigcup_{\varepsilon>0} D_i(D_s; \varepsilon)$.

This is to say, that $D_i(D_s)$ is the set of initial state x_0 such that assumption P is true.

Definition 3 A domain $E(D_s)$ of dimension $(n - 1)$ included in the manifold $\{s = 0\}$ (respectively $E(D_i(D_s))$ of dimension n included in the state space) is an estimate of D_s (respectively $D_i(D_s)$) if $E(D_s) \subset D_s$ (respectively $E(D_i(D_s)) \subset D_i(D_s)$).

3.2. Practical Interest

One can notice that the classical sliding condition ($s \frac{\mathrm{d}s}{\mathrm{d}t} < 0$) can be fulfilled using the following control (if $\nabla(s; x)^{\mathrm{T}} \cdot g(t, x) \neq 0$):

$$u_{\mathrm{eq}} = -\frac{\nabla (s; x)^{\mathrm{T}} \cdot f(t, x)}{\nabla (s; x)^{\mathrm{T}} \cdot g(t, x)}, \tag{4}$$

$$u = u_{\mathrm{eq}} - \frac{k}{\nabla (s; x)^{\mathrm{T}} \cdot g(t, x)} \operatorname{Sgn}(s). \tag{5}$$

In that case, it is obvious that the entire sliding surface $\{s = 0\}$ is a sliding domain and that the entire state-space is an initial domain of sliding motions. But, unfortunately the control (5) is unbounded, which in practice is not realistic: this motivates the above definitions and the following results.

4. DOMAINS ESTIMATION

4.1. Problem Formulation

According to Section 3, it comes out that the practical point of view leads to the following problems:

–find the efficient sliding surface D_s or an estimate $E(D_s)$,
–find an estimation of the initial states such that the motions “converge” to D_s and can only leave D_s through a neighbourhood of its boundaries: $E(D_i(D_s))$.

In the following the sliding surface is chosen as an hyperplan: $s = c^{\mathrm{T}} x$. The vector c is suppose to be a non zero vector. The choice of an hyperplan simplifies the development, but the method can be extended to any differentiable manifold.

As C is a non zero vector, one can find a nonsingular change of coordinate defined as:

$$w = \begin{bmatrix} C^{\mathrm{T}} \\ N \end{bmatrix} x = Fx = \begin{bmatrix} s \\ s* \end{bmatrix}. \tag{6}$$

By construction, matrix F is nonsingular and thus system (1) can be rewritten in the following form:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = F\,[f(t, F^{-1}\,w) + g(t, F^{-1}\,w)\,u(t, F^{-1}\,w)]. \tag{7}$$

4.2. The Main Results

The first step is to caracterize D_s and an under-estimate (S_{12}) of the domain of initial conditions such that the solution $\underline{x}(t;\, t_0;\, x_0,\, u)$ tends to D_s and can only leave it through a neighbourhood its boundary: it is to say the domain $D_i(D_s)$. This is achieved by using two Lyapunov functions:

–one is a function of the variable s (V_1) leading to a more general condition analogous to $(s\,\frac{s}{\mathrm{d}t},\, < 0)$ ensuring the attractivity of a part of the “sliding surface” (the sliding domain),
–the other one is a function of all the variables $w = [s,\, s*]^{\mathrm{T}}$ (V_2) leading to an invariance result.

The combination of the attractivity and invariance leads to the desired result.

THEOREM 1 *If there exist two continuous functions $V_1(s)$ and $V_2(w)$ verifying H1 and H2:*

H1) $V_1 : \mathbb{R} \to \mathbb{R}_+$

$s \to V_1(s)$, *with* $[V_1(s) = 0 \Leftrightarrow s = 0]$,

for $s \neq 0$: V_1 *is differentiable with respect to s.*

$V_2 : \mathbb{R}^n \to \mathbb{R}_+$

$w \to V_2(w)$,

for $s \neq 0$: V_2 *is differentiable with respect to w.*

H2) there exist α_1 *(finite or infinite) and* α_2 *such that:*

$$S_2(\alpha_2 = \{w \in \mathbb{R}^n: V_2(w) \leq \alpha_2\}, \alpha_2 > 0, \tag{8}$$

$$\mathring{S}_2(\alpha_2) \cap \{w \in \mathbb{R}^n: s = 0\} \neq \emptyset, \tag{9}$$

$$D_s = S_2(\alpha_2) \cap \{w \in \mathbb{R}^n: s = 0\}, \tag{10}$$

$$S_1(\alpha_1) = \{w \in \mathbb{R}^n: V1(s) \leq \alpha_1\}, \alpha_1 > 0, \tag{11}$$

$$S_1(\alpha_1) \textit{ contains a neighbourhood of } s = 0, \tag{12}$$

$$S_{12}(\alpha_1; \alpha_2) = S_1(\alpha_1) \cap S_2(\alpha_2), \tag{13}$$

$$w \in \partial S_{12}(\alpha_1; \alpha_2) - [\partial D_s \cup \partial S_1(\alpha_1)]:$$

$$\nabla(V_2; s^*)^{\mathrm{T}} \cdot \frac{\mathrm{d}s^*}{\mathrm{d}t} < 0, \tag{14}$$

$$\nabla(V_1; s)\ \nabla(V_2; s) \geq 0. \tag{15}$$

$$w \in S_{12}(\alpha_1; \alpha_2) - D_s: \nabla(V_1; s) \cdot \frac{\mathrm{d}s}{\mathrm{d}t} < 0. \tag{16}$$

Then:

C1) D_s *is a sliding domain for system (1),* *C2)* $S_{12}(\alpha_1; \alpha_2)$ *is an under-estimate of* $D_i(D_s)$, *that is for every initial* x_0 condition such that $w_0 = \begin{bmatrix} C^T \\ N \end{bmatrix} x_0$ is in $S_{12}(\alpha_1; \alpha_2)$, the solution $\underline{x}(t; t_0, x_0; u)$ tends to D_s and can only leave it through its boundary. ■

Proof see appendix.

Remarks

1) Notice that conditions (14) and (15) can be replaced by:

$$\nabla(V_2; w)^T \cdot \frac{\mathrm{dw}}{\mathrm{d}t} < 0. \tag{17}$$

2) This result and the following results can be easily extended to non autonomous Lyapunov functions ($V(t, w)$).

Example 1 Let us consider a non-linear system having the following state-space representation:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -(x_1 + x_2)^2 x_1, \tag{18}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = (u + {}^2x_1)(x_1 + x_2)^2$$

Using the following transformation $w = Fx = \begin{bmatrix} s \\ s* \end{bmatrix}$, with $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, leads to:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = (u + s*) s^2, \tag{19}$$

$$\frac{\mathrm{d}s*}{\mathrm{d}t} = -s^2 s*. \tag{20}$$

A bounded control law is selected as follow:

$$u = \begin{cases} -k \text{ if } s > 0 \\ k \text{ if } s < 0 \end{cases}, \text{ with } k > 0. \tag{21}$$

Let $V_1(s) = \frac{1}{2}s^2$, then $\frac{\mathrm{d}s}{\mathrm{d}t} \cdot \nabla(V_1; s) = s^3(u - s^*)$. Using $V_2(s^*) = \frac{1}{2}s^{*2}$ and Theorem 1 leads to the conclusion that $D_s = \{s = 0 \text{ and } |s^*| < k\}$ is a sliding domain for system (18) with control defined by (21), and for every initial condition x_0 such that $w_0 = Fx_0$ is in $S = \{|s^*| < k\}$ (unbounded set), the solution $\underline{x}(t)$ tends to D_s and can only leave it through its boundaries.

From the previous result we can derive a special case which is very useful in practice since the two Lyapunov functions depend respectively on s and s^*.

COROLLARY 1 *If there exist two continuous functions $V_1(s)$ and $V_2(s^*)$ verifying H1 and H2:*

H1) $V_1 : \mathbb{R} \to \mathbb{R}_+$

$s \to V_1(s)$, *with* $[V_1(s) = 0 \Leftrightarrow s = 0]$,

for $s \neq 0$: V_1 is differentiable with respect to s.

$V_2 : \mathbb{R}^{n-1} \to \mathbb{R}_+$

$s^* \to V_2(s^*)$,

V_2 is differentiable with respect to s^*.

H2) there exist α_1 (finite or infinite) and α_2 such that:

$$S_2(\alpha_2) = \{w \in \mathbb{R}^n: V_2(s^*) \leq \alpha_2\}, \alpha_2 < 0, \tag{22}$$

$$\mathring{S}_2(\alpha_2) \cap \{w \in \mathbb{R}^n: s = 0\} \neq \emptyset, \tag{23}$$

$$D_s = S_2(\alpha_2) \cap \{w \in \mathbb{R}^n: s = 0\}, \tag{24}$$

$$S^1(\alpha_1) = \{w \in \mathbb{R}^n: V_1(s) \leq \alpha_1\}, \alpha_1 > 0, \tag{25}$$

$$S_1(\alpha_1) \text{ contains a neigbourhood of } s = 0, \tag{26}$$

$$S_{12}(\alpha_1; \alpha_2) = S_1(\alpha_1) \cap S_2(\alpha_2), \tag{27}$$

$$w \in \partial S^{12}(\alpha_1; \alpha_2) - [\partial D_s \cup \partial S_1(\alpha_1)]: \nabla(V_2; s^*)^T \cdot \frac{\mathrm{d}s^*}{\mathrm{d}t} < 0, \tag{28}$$

$$w \in S^{12}(\alpha_1; \alpha_2) - D_s\colon \nabla(V_1; s) \cdot \frac{\mathrm{d}s}{\mathrm{d}t} < 0. \tag{29}$$

Then:

C1) D_s is a sliding domain for system (1),

C2) for every initial condition x_0 such that $w_0 = \begin{bmatrix} C^T \\ N \end{bmatrix} x_0$ is in $S_{12}(\alpha_1; \alpha_2)$, the solution $\underline{x}(t;\, t_0,\, x_0;\, u)$ tends to D_s and can only leave it through its boundary. ■

Proof it is a direct conclusion from Theorem 1.

Example 2 Let a linear system be defined by the transfert function $\frac{Y(p)}{U(p)} = \frac{p+1}{p^2+p+1}$, then a companion form and $w = Fx = \begin{bmatrix} s \\ s^* \end{bmatrix}$, with $F = F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, leads to:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -s^* + u,$$

$$\frac{\mathrm{d}s^*}{\mathrm{d}t} = (s - s^*), \text{ notice that the output variable is } y = s. \tag{30}$$

A bounded control law is selected as follow:

$$u = \begin{cases} -k \text{ if } s > 0 \\ k \text{ if } s < 0 \end{cases}, \text{ with } k > 0. \tag{31}$$

Thus using Corollary 1 with functions $V_1(s) = \frac{1}{2}s^2$, $V_2(s^*) = \frac{1}{2}s^{*2}$, leads to the conclusion that $D_s = \{s = 0 \text{ and } |s^*| < k\}$, is a sliding domain for system (30) with control defined by (31), and for every initial condition x_0 such that $w_0 = Fx_0$ is in $S = \{|s| < k \text{ and } |s^*| < k\}$, the solution $\underline{x}(t)$ tends to D_s and can only leave it through its boundaries.

It is of importance to obtain sufficient condition for global attractivity of the sliding surface. This can be achivied in the following result using only one Lyapunov function of the variable s.

THEOREM 2 *If there exist a continuous function $V_1(s)$ and a real α_1 (finite or infinite):*

$$V_1\colon \mathbb{R} \to \mathbb{R}_+$$
$$s \to V_1(s),$$
$$\text{with } [V_1(s) = 0 \Leftrightarrow s = 0],$$

such that:

$$\text{for } s \neq 0\text{: } V_1 \text{ is differentiable with respect to } s, \tag{32}$$

$$S_1(\alpha_1) = \{w \in \mathbb{R}^n\colon V_1(s) \leq \alpha_1\},\ \alpha_1 > 0, \tag{33}$$

$$S_1(\alpha_1) \text{ contains a neigborhood of } s = 0, \tag{34}$$

$$w \in S_1(\alpha_1) - \{w \in \mathbb{R}^n\colon s = 0\})\colon \nabla(V_1; s) \cdot \frac{\mathrm{d}s}{\mathrm{d}t} < 0. \tag{35}$$

Then:

C1) $D_s = \{w \in \mathbb{R}^n\colon s = 0\}$, is a sliding domain for system (1),

C2) for every initial x_0 condition such that $w_0 = \begin{bmatrix} C^T \\ N \end{bmatrix} x_0$ is in $S_1(\alpha_1)$, the solution $\underline{x}(t;\ t_0,\ x_0;\ u)$ tends to D_s asymptotically (or hits it and stays on D_s). ■

Proof see appendix.

Example 3 Notice that, for the model given in example 2 by (30), one can select an unbounded control such as:

$$u = s^* + \begin{cases} -k \text{ if } s > 0 \\ k \text{ if } s < 0 \end{cases},\ \text{with } k > 0. \tag{36}$$

Then using Theorem 2 with functions $V_1(s)$ defined in example 2, leads to the conclusions that $D_s = \{w \in \mathbb{R}^n\colon s = 0\}$ is a sliding domain and for every initial condition x_0 the solution $\underline{x}(t;\ t_0,\ x_0;\ u)$ tends to D_s. This is in accordance with Section 3.2.

The previous results require strong condition on differentiability of the Lyapunov functions. In order to relax this condition and to allow the use of Lypunov function of Höder norm type such as $|s|$, we can obtain the following result:

THEOREM 3 and 4 *Conclusions of Theorem 1 (or Theorem 2) holds if the hypothesis on the differentiability of V_1 and V_2 (or V_1) is transformed into differentiability of V_1 and V_2 (or V_1) almost everywhere and the existence of their right and left gradients.* ■

Proof see appendix.

THEOREM 5 *If:*

H1) the system (S^*) (7) (which is system (1) rewritten with variable w) leads to:

$$\frac{d|s|}{dt} = \mathrm{Sgn}(s)\, C^T\, [f(t, F^{-1} w) + g(t, F^{-1} w)\, u(t, F^{-1} w)] \leq h(t, |s|), \tag{37}$$

for $w \in S - \{w \in \mathbb{R}^n\colon s = 0\}$, with:

$$\overset{\circ}{S} \cap \{w \in \mathbb{R}^n\colon s = 0\} \neq \varnothing.$$

H2) there exists $D =]0, \alpha_1]$ included in S, positively invariant for the system:

$$\frac{dz}{dt} = h(t, z), \tag{38}$$

such that:

$$\lim_{t \to +\infty} \underline{z}\,(t; t_0, z_0) = 0, \tag{39}$$

where $\underline{z}(t; t_0, z_0)$ is the solution of system (38) for $z_0 \in]0, \alpha_1]$.

Let us define:

$$S_1(\alpha_1) = \{w \in \mathbb{R}^n\colon s \in [-\alpha_1, \alpha_1]\}. \tag{40}$$

H3) there exists a continuous function $V_2(w)$:

$$V_2 \colon \mathbb{R}^n \to \mathbb{R}_+$$

$$w \to V_2(w),$$

for $s \neq 0$: V_2 is differentiable with respect to w,

and there exists a positive real α_2 such that:

$$S_2(\alpha_2) = \{w \in \mathbb{R}^n \colon V_2(w) \leq \alpha_2\}, \alpha_2 > 0, \tag{41}$$

$$\overset{\circ}{S}_2(\alpha_2) \cap \{w \in \mathbb{R}^n \colon s = 0\} \neq \emptyset, \tag{42}$$

$$D_s = S_2(\alpha_2) \cap \{w \in \mathbb{R}^n \colon s = 0\}, \tag{43}$$

$$S_{12}(\alpha_1; \alpha_2) = S_1(\alpha_1) \cap S_2(\alpha_2) \subset S, \tag{44}$$

$$\text{for } w \in \partial S_{12}(\alpha 2) - [\partial D_s \cup \partial S_1] \colon \nabla(V_2; w)^T \cdot \frac{\mathrm{d}w}{\mathrm{d}t} < 0. \tag{45}$$

Then:

C1) D_s is a sliding domain for system (1),

C2) for every initial x_0 condition such that $w_0 = \begin{bmatrix} C^T \\ N \end{bmatrix} x_0$ is in $S_{12}(\alpha_1; \alpha_2)$, the solution $\underline{x}(t; t_0, x_0; u)$ tends to D_s asymptotically and can only leave it through its boundary. ■

Proof see appendix.

Remark If (38) has a locally asymptotically stable equilibrium point $\{z = 0\}$ with a positively invariant set D containing a neighborhood of $\{z = 0\}$ and included both in S and in the domain of asymptotic stability of $\{z = 0\}$, then H2 is satisfied.

Example 4 Let us consider system (30) of example 2 with control given by (31). Then, one can obtain:

$$\frac{\mathrm{d}|s|}{\mathrm{d}t} \leq -k + |s^*|, \tag{46}$$

and thus Theorem 5, with $V_2(s^*) = \frac{1}{2} s^{*2}$, leads to a sliding domain D_s for system (30): $D_s = \{s = 0 \text{ and } |s^*| < k\}$, and for every initial condition w_0 in $S = \{|s| < k \text{ and } |s^*| < k\}$, the solution $\underline{x}(t)$ tends to D_s and can only leave it through its boundaries.

5. DESIGN OF CONTROL

A classical approach for designing a sliding modes control is to separate the control u into two functions one, u_{eq}, defined when $s = 0$ (the "equivalent control") and the other, $u_{\neq}$, defined when $s \neq 0$ (most of the time it is chosen as $-k\ \mathrm{Sgn}(s)$). Let us recall that u_{eq} and $u_{\neq}$ were assumed to satisfied Fillipov's conditions on continuation of solution (see Introduction or Hypothesis H3 in the following theorem). Evaluating the convenience of a given designed control, the estimation of the domain of asymptotic stability for the origin appears as a crucial problem.

THEOREM 6 *Let us suppose that:*

H1) $D \subset \{w \in \mathbb{R}^n: s = 0\}$,

H2) for every initial $x_0 \in S$, the solution $\underline{x}(t;\ t_0,\ x_0;\ u)$ reaches $\mathring{D}$ in finite time.

H3) $u_{eq} \in \overline{co}\{\lim_{w_i \to w} u_{\neq}(w_i)\!: s_i \neq 0 \text{ and } w \in D\}$, (the closure of the convexe cone generated by "$u_{\neq}$" on D).

H4) the origin $\{O^*\} \in \{w \in \mathbb{R}^n\!: s = 0\} \subset \mathbb{R}^{n-1}$, is locally asymptotically stable for system:

$$\frac{\mathrm{d}s^*}{\mathrm{d}t} = \begin{bmatrix} 0 \\ N \end{bmatrix} [f(t, F^{-1} \begin{bmatrix} 0 \\ s^* \end{bmatrix}) + g(t, F^{-1} \begin{bmatrix} 0 \\ s^* \end{bmatrix})\, u_{eq}], \tag{47}$$

with D positively invariant w.r.t. (47) and included in the domain of asymptotic stability of $\{O^*\}$.

Then:

C1) the origin $\{w = 0\}$ is locally asymptotically stable for system (1), C2) S is an estimation of its domain of asymptotic stability. ■

Proof see appendix.

Remarks

1) Obviously, H1 and H3 can be obtained using the preceeding results.
2) H2 can be evaluated by using the dynamics of the s variable. This can be achieved using for example a special Lyapunov function $V_1(s) = |s|$ and theorem 3, 4 or 5 to overvalue the time evolution of the s variable. One can notice that H2) implies H3) only for points in the set reachable from S and included in D (and not for the whole set D), thus H3) is not an extra hypothesis.
3) System (47) represents the evolution of the system on the sliding surface. Thus classical results based on Lyapunov functions are available to test asymptotic stability of the origin {O*} (see [6, 7]).
4) If one uses the usual decomposition of u into a continuous control, called "equivalent control" (u_{eq}) and a discontinuous one [9, 10], theorem 6 has a direct application. Moreover, usually u_{eq} is unbounded, and thus $u = u_{eq} - k\,\mathrm{Sign}(s)$ is an unbounded discontinuous control. Notice that Slotine propose to smooth the control, changing $-k\,\mathrm{sign}(s)$ into a saturation function [8], which avoids chattering. Practically, the global control (unbounded) is not realistic, and thus using theorem 6 (and the previous ones) in order to obtain D_s of H1 and S_{12} of H2 in theorem 6, a bounded control can be designed.

Example 5 Selecting an unbounded control given by (36) for system (30) of example 2, the use of Theorem 2 leads to $D_s = \{w \in \mathbb{R}^n \colon s = 0\}$ and $S_{12} = \mathbb{R}^n$.

First, as $\frac{ds}{dt} = -k$, $s(t)$ reaches D_s in finite time $t_0 + s(t_0) / k$. Secondly, obviously the analogous of (47) for system (30) is reduced to $\frac{ds^*}{dt} = -s^*$, for which the origin is globally asymptotically stable. Thus Theorem 6 leads to the conclusion that for all initial conditions, the solutions $\underline{x}(t)$ tends to zero and thus the output variable (y) too.

The last result is stated in a practical form: to check asymptotic stability of the origin and an estimation of its corresponding domain we need two Lyapunov functions and that the sliding domain be included in the domain of asymptotic stability of the origin relatively to the evolution on the sliding surface.

THEOREM 7 *Let us suppose that:*

H1) there exist two continuous functions $V_1(s)$ and $V_2(w)$, and there exist α_1 (finite or infinite) and α_2 such that H1) and H2) of theorem 1 are satisfied.

H2) the origin $\{O^*\} \in \{w \in \mathbb{R}^n: s = 0\} \subset \mathbb{R}^{n-1}$, is locally asymptotically stable for the following system:

$$\frac{ds^*}{dt} = \begin{bmatrix} 0 \\ N \end{bmatrix} [f(t, F^{-1} \begin{bmatrix} 0 \\ s^* \end{bmatrix}) + g(t, F^{-1} \begin{bmatrix} 0 \\ s^* \end{bmatrix}) u_{eq}], \tag{48}$$

with D_s positively invariant w.t. (48) and included in the domain of asymptotic stability of $\{O^*\}$.

Then:

C1) obviously the conclusion of theorem 1 concerning D_s holds,

C2) the origin $\{w = 0\}$ is locally asymptotically stable for system (1),

C3) $S_{12}(\alpha_1; \alpha_2)$ is an estimate of its domain of asymptotic stability of the origin $\{w = 0\}$ for system (1). ■

Proof see appendix.

Example 6 Let us consider system (30) with a bounded control given by (22). Using $V_1(s) = \frac{1}{2} s^2$, $V_2(s^*) = \frac{1}{2} s^{*2}$, leads to the conclusion that $D_s = \{s = 0 \text{ and } |s^*| < k\}$ and $S = \{|s| < k \text{ and } |s^*| < k\}$. Now, obviously the analogous of (48) for system (30) is reduced to $\frac{ds^*}{dt} = -s^*$, for which the origin is globally asymptotically stable. Thus Theorem 7 leads to the conclusion that S is an estimate of the domain of asymptotic stability of the origin for system (30): any solution $\underline{x}(t)$ originating in S tends to zero and the output variable (y) too.

6. EXAMPLES: DESIGN OF BOUNDED CONTROL LAW

When dealing with sliding mode approach, it is classical to select the following unbounded control $u = u_{\text{eq}} - k \,\text{Sgn}(s)$ (Theorem 2 implies that the entire sliding surface is a sliding domain or [9, 10]). This leads to select the parameters defining the sliding surface (c_i) in such a way that the origin O* is asymptotically stable: one can design the "sliding modes" via the c_i.

Obviously, such a control is not realistic because it is unbounded, so let us design a bounded control in the following way:

1st Step) Compute the equivalent control: u_{eq} is such that $\frac{\mathrm{d}s}{\mathrm{d}t}=0$ or see equation (4).

2nd Step) Select the following bounded control:

$$u = \text{Sat}_{k_2}(u_{\text{eq}}) - k_1 \,\text{Sgn}(s) \tag{49}$$

The saturation function is continuous and defined as follows:

$$\text{Sat}_{k_2}(u_{\text{eq}}) = \begin{cases} u_{\text{eq}} \text{ if } |u_{\text{eq}}| \leq k_2, \\ k_2 \,\text{Sgn}(u_{\text{eq}}) \text{ if } |u_{\text{eq}}| \geq k_2 \end{cases}. \tag{50}$$

3rd Step) Use the previous results in order to prove the existence of a sliding domain and to obtain an estimation of the domain of asymptotic stability of the origin for the closed loop system.

Example 7 Consider a second order SISO system described by the following transfer function:

$$\frac{Y(p)}{U(p)} = F(p) = \frac{p+1}{p^2 - p + 1}. \tag{51}$$

The control u is constrained by:

$$|u| \leq k. \tag{52}$$

Using a companion form and the transformation $w = Fx = \begin{bmatrix} s \\ s^* \end{bmatrix}$, with $F = \begin{bmatrix} c_1 & 1 \\ 1 & 0 \end{bmatrix}$, leads to:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = (c_1 + 1)\, s + (-1 - c_1\,(1 + c_1))\, s^* + u, \tag{53a}$$

$$\frac{ds^*}{dt} = (s - c_1 s^*), \tag{53b}$$

$$y = s + (1 - c_1) s^*, \tag{53c}$$

The design of a bounded control can be achieved with the following steps:

1st Step) from (53 a), compute the equivalent control (u_{eq} is such that $\frac{ds}{dt} = 0$):

$$u_{eq} = -[(c_1 + 1) s + (-1 - c_1 (1 + c_1)) s^*]. \tag{54}$$

2nd Step) Let us define the control as:

$$u = -k_1 \operatorname{Sgn}(s) + Sat_{k_2}(u_{eq}), \text{ with } k_1 > 0, \text{ and } k_2 > 0. \tag{55}$$

Obviously the control satisfies (52) if:

$$k_1 + k_2 < k. \tag{56}$$

3rd Step) from (53 b), if c_1 is positive, the sliding motion converge to zero: it is to say that H2 of Theorem 7 is satisfied: $c_1 = 1$ (thus $s = y$, in that case $s = 0$ is sufficient to make the output cancel). So $u_{eq} = 3s^* - 2s$.

3.1) Obviously, when $|u_{eq}| \leq k_2$, function $V_1(s) = \frac{1}{2} s^2$ is decreasing and satisfies H1 of theorem 1.

3.2) The function $V_2(w) = |u_{eq}| = |3s^* - 2s|$, with $\alpha_2 = k_2$, satisfies H2 of Theorem 1 if:

$$|s| < k_2 - 2 k_1. \tag{57}$$

Let $k_2 > 2 k_1$ such that (56) is satisfied, then Theorem 7 leads to the conclusion that the origin is locally asymptotically stable and an estimation of its domain of asymptotic stability is given by:

$$S_{12} = \{w \in \mathbb{R}^2 : |3s^* - 2s| \leq k_2 \text{ and } |s| < k_2 - 2 k_1\}. \tag{58}$$

The sliding domain is:

$$D_s = \{|3s^*| \leq k_2 \text{ and } s = 0\}. \tag{59}$$

Thus, on one hand, an interesting choice is to take k_1 very small because for this choice: k_2 is close to k and the sets S_{12} and D_s are the largest ones (due to (58) and (59)). And on the other, the higher the parameter k_1 is, the faster the convergence onto the sliding surface is. Finally, in order to choose the two parameters (k_1 and k_2), one has to chose a compromise solution according to the above related phenomena.

This method can be extended to nonlinear systems for example if the system is linearizable, or directly as in the following example.

Example 8 Consider a second order nonlinear system described by the following ordinary differential equation:

$$\frac{dx}{dt} = u, \tag{60a}$$

$$\frac{dy}{dt} = y^2 x, \tag{60b}$$

$$y \text{ is the output.} \tag{60c}$$

The control u is constrained by:

$$|u| \leq k. \tag{61}$$

Using the transformation $w = F\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} s \\ s^* \end{bmatrix}$, with $F = \begin{bmatrix} c_1 & 1 \\ 1 & 0 \end{bmatrix}$, leads to:

$$\frac{ds}{dt} = Cs^{*2}(s - c_1 s^*) + u, \tag{62a}$$

$$\frac{ds^*}{dt} = Cs^{*2}(s - c_1 s^*), \tag{62b}$$

$$y = s^*. \tag{62c}$$

The design of a bounded control can be achieved with the following procedure:

1rst Step) from (62 a), compute the equivalent control (u_{eq} is such that $\frac{\mathrm{d}s}{\mathrm{d}t} = 0$):

$$u_{\mathrm{eq}} = -C\,[s^{*2}\,(s - c_1\, s^*)]. \tag{63}$$

2nd Step) Let us define the control as:

$$u = -\,k_1\,\mathrm{Sgn}(s) + Sat_{k_2}\,(u_{\mathrm{eq}}),\ \text{with}\ k_1 > 0,\ \text{and}\ k_2 > 0. \tag{64}$$

Obviously the control satisfies (61) if:

$$k_1 + k_2 < k. \tag{65}$$

3rd Step) from (62 b), if c_1 is positive, the sliding motion converges to zero: it is to say that H2 of Theorem 7 is satisfied: $c_1 = 1$. So $u_{\mathrm{eq}} = -\,[s^{*2}\,(s - s^*)]$.

3.1) If $|u_{\mathrm{eq}}| \leq k_2$ then the function $V_1(s) = \frac{1}{2}\,s^2$ is decreasing and satisfies H1 of Theorem 1.

Note that $\partial\,\{w \in \mathbb{R}^2\text{: } s^{*2}\,(s^* - s) \leq k_2\}$ is defined by the curves $s = s^* \pm \frac{k_2}{s^{*2}}$.

3.2) If one select the function $V_2(w) = |u_{\mathrm{eq}}| = |s^{*2}\,(s^* - s)|$ and $\alpha_2 = k_2$, then hypothesis H2 of Theorem 1 is satisfied if:

$$\frac{\mathrm{d}V_2(w)}{\mathrm{d}t}\Big|_{|u_{\mathrm{eq}}|=k_2} = \mathrm{Sgn}(u_{\mathrm{eq}})\,[-\,2\,s^{*3}\,(s^* - s)^2 - u\,s^{*2}]\Big|_{|u_{\mathrm{eq}}|\,=\,k_2} < 0. \tag{66}$$

(64) and (66) leads to:

$$\frac{\mathrm{d}V_2(w)}{\mathrm{d}t}\Big|_{|u_{\mathrm{eq}}|=k_2} = -\,s^{*2}\left[k_2\,(\frac{2\,k_2}{s^{*3}} + 1) - k_1\,\mathrm{Sgn}(u_{\mathrm{eq}})\,Sgn(s)\right]\Big|_{|u_{\mathrm{eq}}|\,=\,k_2}. \tag{67}$$

$$\frac{dV_2(w)}{dt}\Big|_{|u_{eq}| = k_2} < 0, \text{ if } k_2 > k_1. \tag{68}$$

Let $k_2 > k_1$ such that (65) is satisfied, then Theorem 7 shows that the origin is locally asymptotically stable, and an estimate of its domain of asymptotic stability is given by:

$$S_{12} = \{w \in \mathbb{R}^2 \colon |s^{*2}(s^* - s)| \leq k_2 \text{ and } |s| < \frac{1 + 2k_2^2}{(2k_2)^{\frac{1}{3}}}\}. \tag{69}$$

The sliding domain is:

$$D_s = \{|s^{*3}| \leq k_2 \text{ and } s = 0\}. \tag{70}$$

As in the previous example 7, in order to choose the two parameters (k_1 and k_2), one has to chose a compromise solution.

The following figures shows two simulations for the closed loop system ($k_1 = 1$ and $k_2 = 9$ for $k = 10$):

–one with initial value (0.5, 0.5) for which '*x*' and '*y*' are indexed by '1',

–the other with initial value (2, 2) for which '*x*' and '*y*' are indexed by '2'.

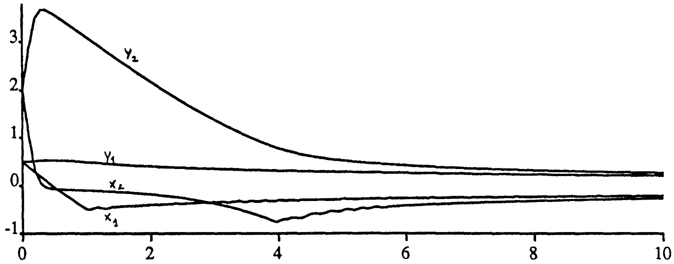

FIGURE 2 *x* and *y* versus time.

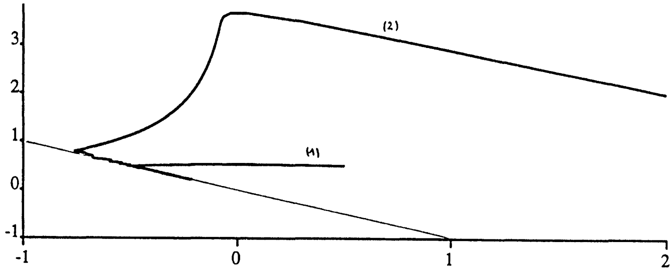

FIGURE 3 y versus x.

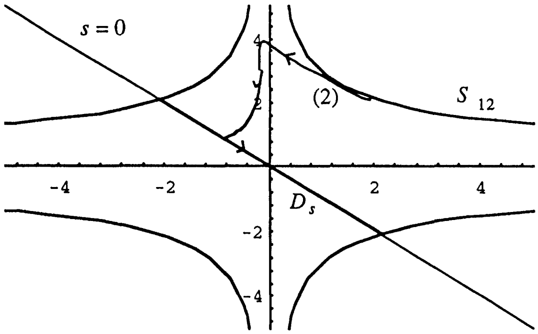

FIGURE 4 S_{12}, D_s and simulation.

References

[1] A.A. Andronov, E.A. Vitt, S.E. Khaiken, 1959 "Theory of Oscillators", Pergamon Press: Oxford (1966) [translated from russian].

[2] A.G. Filipov, 1960 "Application of the Theory of Differential Equations with Discontinuous Right-Hand sides to Non-Linear Problems in Automatic Control", in Proc. 1st IFAC Congres (1960), p. 923–927.

[3] A.G. Filipov, 1979 "Differential Equations with Second Members Discontinuous on Intersecting Surfaces", Diff. Urav., vol. **15** (1979) No. 10, p. 1814–1832.

[4] LJ.T. Grujiĉ, 1975, "Novel Development of Lyapunov Stability of Motion", Int. J. Control, Vol. **22** (1975) No. 4, p. 525–549.

[5] V. Lakshmikantham, S. Leela, 1969 "Differential and Integral Inequalities", Vol. 1 (1969), Academic Press, New York.

[6] W. Perruquetti, J.P. Richard, P. Borne, 1995 "Vector Lyapunov Functions: recent developments for Stability, Robustness, Practical Stability, and Constrained Control", to appear in Non Linear Times and Digest, (1995).

[7] W. Perruquetti, J.P. Richard, LJ. T. Grujiĉ, P. Borne, 1995, "On Practical Stability with the Settling Time via Vector Norms", Int. J. Control Vol. **62** (1995) No. 1, p. 173–189.

[8] J.J.E. Slotine, 1984 "Sliding controller design for non-linear systems", Int. J. Control, Vol. **40** (1984) No. 2, p. 421–434.

[9] V.I. Utkin, 1977 "Variable Structure Systems with Sliding Modes", I.E.E.E. TAC, Vol. **AC-22** (1977) No. 2, p. 212–222.

[10] V.I. Utkin, 1981 "Sliding Modes in Control Optimization", Springer-Verlag, (1992).

APPENDIX

Proof of Theorem 1 Let us define $\Delta(\varepsilon) = \{w \in \mathbb{R}^n\colon |s| \leq \varepsilon\}$, and $D(\varepsilon)$ the set of strictly positive numbers such that: $S_{12}(\alpha_1; \alpha_2) \cap \mathring{\Delta}(\varepsilon) \neq \emptyset$.

$D(\varepsilon)$ is non empty and contains a subset $]0, \varepsilon_{\sup}[$: this comes directly from the definition of $S_{12}(\alpha_1; \alpha_2)$, H2 (9):

$$\mathring{S}_2(\alpha_2) \cap \{w \in \mathbb{R}^n\colon s = 0\} \neq \emptyset,$$

and H2 (12):

$S_1(\alpha)$ contains a neighborhood of $s = 0$.

Let $w_0 \in S_{12}(\alpha_1; \alpha_2)$, thus under the hypothesis on (f, g, u) the solution $\underline{x}(t)$ exists is continuous w.r.t time and defined for $t \geq t_0$. And so is $\underline{w}(t)$.

P1) First let us prove that:

$\forall \varepsilon \in D(\varepsilon)$, $\forall w_0 \in (S_{12}(\alpha_1; \alpha_2) - \mathring{\Delta}(\varepsilon))$, the solution $\underline{x}(t)$ of (1) can only leaves the set $(S_{12}(\alpha_1; \alpha_2) - \mathring{\Delta}(\varepsilon))$ through $\partial\, \Delta(\varepsilon)$.

If this assumption is false, then, as $\underline{x}(t)$ is continuous:

$\exists \mathrm{T} \geq t_0$ such that $\underline{x}(\mathrm{T}) \in \partial\, (S_{12}(\alpha_1; \alpha_2) - \mathring{\Delta}(\varepsilon)) - \partial\, \Delta(\varepsilon)$, and for $\mathrm{T} \geq t \geq t_0$, we have $\underline{x}(t) \in \mathring{S}_{12}(\alpha_1; \alpha_2) - \Delta(\varepsilon)$, thus:

$$V_1(C^{\mathrm{T}} \underline{x}(t)) \leq V_1(\alpha_1), \text{ and} \tag{A.1}$$

$$V_2(\underline{x}(t)) \leq V_2(\alpha_2). \tag{A.2}$$

But, as $\underline{x}(\mathrm{T}) \in \partial\, (S_{12}(\alpha_1; \alpha_2) - \mathring{\Delta}(\varepsilon)) - \partial\, \Delta(\varepsilon)$, we have either:

P1*i*) $\underline{x}(T) \in (\partial\, S_1(\alpha_1)) \cap S_2(\alpha_2)$ or

P1*ii*) $\underline{x}(T) \in (\partial\, S_2(\alpha_2)) \cap S_1(\alpha_1)$.

Suppose we have P1i) then (A1) yields:

$$\frac{V_1(C^T \underline{x}(\mathrm{T} + \theta)) - V_1(C^{\mathrm{T}} \underline{x}(\mathrm{T}))}{\theta} > 0, \text{ for } 0 > \theta > t_0 - \mathrm{T}, \tag{A.3}$$

thus letting $\theta \to 0^-$ in (A3),

$$D_t\, V_1(C^{\mathrm{T}} \underline{x}(\mathrm{T})) > 0, \tag{A.4}$$

which contradicts hypothesis H2 (16).

Suppose we have P1ii) then a similar reasoning shows that:

$$D_t V_2(\underline{x}(T)) > 0. \tag{A.5}$$

But hypothesis (14), (15) and (16) in H2 imply that:

$$D_t V_2(\underline{x}(T)) = \nabla(V_2; s^*)^T \cdot \frac{ds^*}{dt} + \nabla(V_2 ; s) \cdot \frac{ds}{dt} < 0. \tag{A.6}$$

Thus (A5) and (A6) are contradictory.

P2) $\forall \varepsilon \in D(\varepsilon)$, $\forall w_0 \in (S_{12}(\alpha_1; \alpha_2) - \mathring{\Delta}(\varepsilon))$, let us define $T_{\mathring{\sup}}(\varepsilon; w_0) \geq t_0$, the supremum time for which the solution $\underline{w}(t)$ is out of $\Delta(\varepsilon)$. Let us prove that $T_{\sup}(\varepsilon; w_0)$ exists and is finite.

As $w_0 \in (S_{12}(\alpha_1; \alpha_2) - \mathring{\Delta}(\varepsilon))$ and as the solution $\underline{w}(t)$ is time continuous and defined for $t \geq t_0$, then for $t \geq t_0$ sufficiently closed to t_0, the solution $\underline{w}(t)$ is out of $\mathring{\Delta}(\varepsilon)$.

If $T_{\sup}(\varepsilon; w_0)$ is infinite, then using function $V1$, one can construct a strictly decreasing sequence (use a projection on line spanned by vector c and H2 (16)) on a compact set ($\{s$: $V_1(s) \leq \alpha_1\} -]-\varepsilon, \varepsilon[$) of $\mathbb{R}$ with no limit, which is impossible.

P3) As, $S_1(\alpha_1)$ contains a neighbourhood of $s = 0$, and as $\mathring{S}_2(\alpha_2) \cap \{w \in \mathbb{R}^n$: $s = 0\} \neq \emptyset$, then making $\varepsilon \to 0$, the conclusions C1) and C2) follows from P2 and P1.

Proof of Theorem 2 As in the proof of theorem 1, let us define $\Delta(\varepsilon)$ and $D(\varepsilon)$. Let $w_0 \in S_1(\alpha_1)$, thus under the hypothesis on (f, g, u) the solution $\underline{x}(t)$ exists is continuous w.r.t time and defined for $t \geq t_0$. And so is $\underline{w}(t)$.

$\forall \varepsilon \in D(\varepsilon)$, $\forall w_0 \in (S_1(\alpha_1) - \mathring{\Delta}(\varepsilon))$, let us define $T_{\sup}(\varepsilon; w_0) \geq t_0$, the supremum time for which the solution $\underline{w}(t)$ is out of $\mathring{\Delta}(\varepsilon)$. A similar reasoning to the proof of theorem 1 proves that $T_{\sup}(\varepsilon; w_0)$ exists and is finite. Thus, as, for $0 < \alpha \leq \alpha 1$, $S_1(\alpha)$ contains a neighbourhood of $s = 0$, then letting $\varepsilon \to 0$, the conclusions C1) and C2) follows.

Proof of Theorem 3 and 4 The proof is essentially the same as for Theorem 1 (or Theorem 2) except that time-derivatives must be replaced either by right or left time Dini derivatives, gradients must be replaced either by right or left gradients. Moreover one must be aware of the use of Zygmund's lemma (see [5] p.9).

Proof of Theorem 5 As in the proof of theorem 1, let us define $\Delta(\varepsilon)$ and $D(\varepsilon)$.

Let $w_0 \in (S_{12}(\alpha_1; \alpha_2) - \mathring{\Delta}(\varepsilon))$.

P1) Conditions on V_2 show that the solution cannot cross $S_2(\alpha_2)$ (similar reasoning as in the proof of theorem 1).

P2) Using results in [5, 6], as D is positively invariant for system (38) then it is also positively invariant for (37) and for $0 < |s_0| \leq z_0 \leq \alpha 1$, we have: $0 \leq |\underline{s}(t)| \leq \underline{z}(t) \leq \alpha_1$. Moreover condition (39) means that for arbitrary ε, there exists a time $\mathrm{T}(\varepsilon)$ such that $|\underline{z}(t)| \leq \varepsilon$ for $t \geq t_0 + \mathrm{T}(\varepsilon)$. Consequently, $\underline{s}(t) \in \Delta(\varepsilon)$ for $t \geq t_0 + \mathrm{T}(\varepsilon)$.

P3) As $\mathring{\bar{S}}_2(\alpha_2) \cap \{w \in \mathbb{R}^n: s = 0\} \neq \emptyset$, then making $\varepsilon \to 0$, the conclusions C1) and C2) follows from P2 and P1.

Proof of Theorem 6 Hypothesis H1 to H3 and lemma 2 of [3] imply that the solution is on D and can be continued to infinity or to the time it reaches ∂D: the solution is thus described by (47). But, from H4, D is positively invariant with respect to (47) and is included in the domain of asymptotic stability of {O*}; therefore the solution tends to the origin.

Proof of Theorem 7 Notice that the notion of D_s implies that the solution in Fillipov' sense stays on the sliding surface after hitting it (until the solution reach the boundaries of D_s): if not there exists a time (T) such that the solution is on D_s before T and out of D_s at time T. Thus using a similar reasoning as in the proof of theorem 1 one can obtain $D_t\, V_1(C^{\mathrm{T}}\, \underline{x}(\mathrm{T})) > 0$ or $D_t\, V_2(C^{\mathrm{T}}\, \underline{x}(\mathrm{T})) > 0$ which are both in contradiction with Hypothesis H1.

Let $w_0 \in \bar{S}(\alpha)$, either:

the solution reaches D_s which is included in the domain of asymptotic stability of {O*} and thus tends to the origin,

or:

the solution never reaches D_s and thus classical results on Lyapunov's functions leads to the conclusion.