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Variational Lyapunov Method and Stability Theory

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By unifying the method of variation of parameters and Lyapunov's second method, we develop a fruitful technique which we call variational Lyapunov method. We then consider the stability theory in this new framework showing the advantage of this unification.

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1. INTRODUCTION

The method of variation of parameters has been a well known and useful tool in the investigation of the properties of solutions of differential equations [1,3,4]. Since one can employ this method to nonlinear differential equations whose unperturbed parts are linear or smooth nonlinear, the method of variation of parameters has gained importance.

It is also very well known that Lyapunov's second method is an important and fruitful technique that has gained increasing significance and has given a decisive impetus for modern development of stability analysis of nonlinear dynamic systems. It is now well recognized that

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the concept of Lyapunov-like functions and the theory of differential inequalities can be utilized to investigate qualitative properties of a variety of dynamic systems [3,4,6,11,12].

Since the method of variation of parameters and the comparison method via Lyapunov-like functions are both extremely useful and effective techniques in the investigation of nonlinear problems, it is natural to combine these two approaches in order to utilize the benefits of the two important methods. Such a unification has recently been achieved and awaits further development [2,4,6,8–10].

In this paper, we shall exploit the ideas introduced in [2] to refine the unification from a different perspective and develop systematically the variational Lyapunov method. Even the method of variation of parameters is looked from a more general point of view so that known results become special cases. Moreover, the variational comparison result is considered via vector Lyapunov-like functions in a more general set up than the corresponding result considered in [2,4].

Since Lyapunov stability does not rule out the possibility of asymptotic stability, nor asymptotic stability guarantees any information about the rate of decay of solutions; the concept of strict stability was introduced in [3] which offers an interesting refinement of Lyapunov stability notions. Such a concept becomes necessary to prove stability results parallel to Lyapunov's original theorems in the present framework of variational Lyapunov method. However, we need to modify suitably the existing concepts of strict stability slightly in order to prove stability criteria in terms of two different measures [5]. Finally, we point out how the variational comparison result in terms of several Lyapunov-like functions would be useful in large scale dynamic systems to bring out the good properties of interconnections, if any. The present approach, of course, improves the properties of unperturbed systems by utilizing the good character of perturbation compared to the usual Lyapunov method.

2. NONLINEAR VARIATION OF PARAMETERS

Consider the two differential systems

$$y' = F(t,y), \quad y(t_0) = x_0, \tag{2.1}$$

$$x' = f(t,x), \quad x(t_0) = x_0, \tag{2.2}$$

where $F, f \in C[R_+ \times R^n, R^n]$. Let $y(t, t_0, x_0)$ be the solution of (2.1) such that $\partial y(t, t_0, x_0)/\partial t_0$, $\partial y(t, t_0, x_0)/\partial x_0$ exist and are continuous on $R_+ \times R^n$. Then setting

$$p(s) = y(t, s, x(s)), \quad t_0 \leq s \leq t,$$

where $x(s) = x(s, t_0, x_0)$ is any solution of (2.2) for $s \geq t_0$, we have

$$\begin{aligned} p'(s) &= \frac{\partial y}{\partial t_0}(t, s, x(s)) + \frac{\partial y}{\partial x_0}(t, s, x(s)) f(s, x(s)) \\ &\equiv G(t, s, x(s)). \end{aligned}$$

Integrating both sides from t_0 to t, we get

$$p(t) - p(t_0) = \int_{t_0}^{t} G(t, s, x(s))\, \mathrm{d}s,$$

which implies

$$x(t) = y(t) + \int_{t_0}^{t} G(t, s, x(s))\, \mathrm{d}s, \quad t \geq t_0. \tag{2.3}$$

This is the nonlinear variation of parameters formula. If, on the other hand, setting

$$p(s) = |y(t, s, x(s))|^2$$

we arrive at

$$p'(s) = 2y(t, s, x(s)) G(t, s, x(s)).$$

Consequently, proceeding as before, we get

$$|x(t)|^2 = |y(t)|^2 + \int_{t_0}^{t} 2y(t, s, x(s)) G(t, s, x(s))\, \mathrm{d}s. \tag{2.4}$$

This suggests that one can obtain in general by setting

$$p(s) = V(s, y(t, s, x(s))),$$

where $V \in C^1[R_+ \times R^n, R_+]$, to get

$$V(t, x(t)) = V(t_0, y(t)) + \int_{t_0}^{t} \frac{\mathrm{d}V}{\mathrm{d}s}(s, y(t, s, x(s)))\,\mathrm{d}s, \tag{2.5}$$

where

$$\frac{\mathrm{d}V}{\mathrm{d}s}(s, y(t, s, x(s))) \equiv \left(\frac{\partial V}{\partial t}(s, y(t, s, x(s))) + \frac{\partial V}{\partial x}(s, y(t, s, x(s)))\right) G(t, s, x(s)).$$

The relation (2.5) can be considered as the generalized variation of parameters formula which brings out the advantage of unifying the methods of nonlinear variation of parameters and Lyapunov-like functions.

If we assume that $\partial F(t, y)/\partial y$ exists and is continuous on $R_+ \times R^n$, then we know [3] that $\partial y(t, t_0, x_0)/\partial t_0$ and $\partial y(t, t_0, x_0)/\partial x_0$ are solutions of the variational system

$$Z' = F_y(t, y(t, t_0, x_0))Z$$

satisfying the initial conditions

$$\frac{\partial y(t_0, t_0, x_0)}{\partial t_0} = -F(t_0, x_0), \ \frac{\partial y}{\partial x_0}(t_0, t_0, x_0) = I \quad \text{(Identity matrix)}$$

and the identity

$$\frac{\partial y}{\partial t_0}(t, t_0, x_0) + \frac{\partial y}{\partial x_0}(t, t_0, x_0)F(t_0, x_0) \equiv 0, \tag{2.6}$$

where $y(t, t_0, x_0)$ is the solution of (2.1). See [3,4,6]. Hence, when $f(t, x) = F(t, x) + R(t, x)$, $R(t, x)$ being treated as the perturbation, the relations (2.3) and (2.6) yield

$$\begin{aligned} x(t) &= y(t) + \int_{t_0}^{t} G(t, s, x(s))\,\mathrm{d}s \\ &= y(t) + \int_{t_0}^{t} \frac{\partial y}{\partial x_0}(t, s, x(s))R(s, x(s))\,\mathrm{d}s, \end{aligned} \tag{2.7}$$

which is Alekseev's nonlinear variation of parameters formula [3]. Using (2.7) to estimate $|x(t)|$ leads to evaluating $|\partial y(t,s,x(s))/\partial x_0|$ and $|R(s,x(s))|$, which destroys good behavior of $R(t,x)$, if any. As a result, one can at best only preserve the properties of the unperturbed system (2.1). If, on the other hand, we utilize the relation (2.4), it is easy to see that the expression $(\partial y(t,s,x(s))/\partial x_0)y(t,s,x(s))R(s,x(s))$ need not be nonnegative and therefore the behavior of R plays an important role in improving the properties of the unperturbed system. Similarly, one can employ (2.5) noting that $\mathrm{d}V/\mathrm{d}s$ need not be nonnegative. In fact, if $\mathrm{d}V/\mathrm{d}s \le -\alpha V$, $\alpha > 0$ for example, we can obtain

$$V(t,x(t)) \le V(t_0,y(t))\exp(-\alpha(t-t_0)), \quad t \ge t_0,$$

which exhibits the beneficial role of $R(t,x)$.

3. VARIATIONAL COMPARISON RESULT

In general, we can employ a vector Lyapunov-like functions and prove the following comparison result in terms of vector Lyapunov-like functions. As usual, inequalities between vectors are componentwise.

THEOREM 3.1 *Assume*

(A_1) $V \in C[R_+ \times R^n, R_+^N]$ $V(t,x)$ and $|y(t,s,x)|$ *are locally Lipschitzian in* x *for each* (t,s);

(A_2) *for* $t_0 \le s \le t$,

$$\begin{aligned} D_-V(t,s,x) \equiv \liminf_{h\to 0^-} \frac{1}{h}[&V(s+h, y(t,s+h,x+hf(s,x))) \\ &- V(s,y(t,s,x))] \\ \le g(t,s,&V(t,s,y(t,s,x))); \end{aligned} \tag{3.1}$$

(A_1) $g \in C[R_+^2 \times R_+^N, R^N]$, $g(t,s,u)$ *is quasimonotone nondecreasing in* u *for each* (t,s) *and* $r(t,s,t_0,u_0)$ *is the maximal solution of*

$$\frac{\mathrm{d}u(s)}{\mathrm{d}s} = g(t,s,u(s)), u(t_0) = u_0 \ge 0 \tag{3.2}$$

existing for $t_0 \le s \le t < \infty$.

Then, $V(t_0, y(t,t_0,x_0)) = u_0$ *implies*

$$V(t,x(t,t_0,x_0)) \le r_0(t,t_0,V(t_0,y(t,t_0,x_0))), \tag{3.3}$$

where $r_0(t, t_0, u_0) \equiv r(t, t, t_0, u_0)$, $y(t, t_0, x_0)$ *is the solution of* (2.1) *and* $x(t, t_0, x_0)$ *is any solution of* (2.2).

Proof Set $m(t,s) = V(s, y(t, s, x(s)))$. Using ($A_1$), ($A_2$), with standard computation, we get the differential inequality

$$D_{-}m(t,s) \leq g(t,s,m(t,s)) \quad \text{for } t_0 \leq s \leq t.$$

This yields by comparison theorem [3, Theorem 1.7.1],

$$m(t,s) \leq r(t,s,t_0, V(t_0, y(t, t_0, x_0))), \quad t_0 \leq s \leq t,$$

which implies the desired estimate (3.3) for $s = t$.

We shall refer to Theorem 3.1 as the variational comparison theorem and many interesting remarks can be made as special cases of this result.

Remarks (1) The estimate (3.3) emphasizes the interplay between solutions of three differential systems (2.1), (2.2) and (3.2).

(2) If $F(t,y) \equiv 0$ so that $y(t, t_0, x_0) \equiv x_0$, Theorem 3.1 reduces to the comparison theorem in terms of vector Lyapunov functions [3,5] which yields

$$V(t, x(t)) \leq r(t, t_0, V(t_0, x_0)), \quad t \geq t_0,$$

since

$$\begin{aligned} D_{-}V(t,x) &\equiv \liminf_{h \to 0^-} \frac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)] \\ &\leq g(t, V(t,x)) \end{aligned}$$

and $V(t_0, x_0) = u_0$.

The case $N = 1$ corresponds to the well known comparison theorem in terms of a single Lyapunov function [3–5] which is more often used.

COROLLARY 3.1 *Under the assumptions of Theorem* 3.1 *with* $N = 1$, $g(t,s,u) \equiv 0$ *we have*

$$V(t, x(t, t_0, x_0)) \leq V(t_0, y(t, t_0, x_0)), \quad t \geq t_0.$$

COROLLARY 3.2 *Assume* (A_1) *by Theorem* 3.1 *and* $N=1$. *Suppose that*

$$D_-V(t,s,x) \leq -c(h_1(s,y(t,s,x))), \quad t_0 \leq s \leq t < \infty,$$

where $c \in \mathcal{K} = \{\phi \in C[R_+, R_+] : \phi(0)=0, \ \phi(s) \text{ increasing in } s\}$ *and* $h_1 \in C[R_+ \times R^n, R_+]$.

Then, for $t \geq t_0$,

$$V(t,x(t,t_0,x_0)) \leq V(t_0,y(t,t_0,x_0)) - \int_{t_0}^{t} c(h_1(s,y(t,s,x(s))))\,\mathrm{d}s, \quad (3.4)$$

where $x(s)=x(s,t_0,x_0)$ *is any solution of* (2.2), $s \geq t_0$.

Proof Set

$$W(s,y(t,s,x(s))) \equiv V(s,y(t,s,x(s))) + \int_{t_0}^{s} c(h_1(\sigma,y(t,\sigma,x(\sigma))))\,\mathrm{d}\sigma.$$

Then, it is easy to compute

$$D_-W(t,s,x(s)) \leq D_-V(t,s,x(s)) + c(h_1(s,y(t,s,x(s)))) \leq 0.$$

By Corollary 3.1, we have

$$W(t,x(t)) \leq W(t_0,y(t)), \quad t \geq t_0$$

which implies, by definition of W,

$$V(t,x(t)) \leq V(t_0,y(t)) - \int_{t_0}^{t} c(h_1(s,y(t,s,x(s))))\,\mathrm{d}s$$

for $t \geq t_0$, where $x(t)=x(t,t_0,x_0)$ and $y(t)=y(t,t_0,x_0)$.

4. STABILITY IN TERMS OF TWO MEASURES

Let us begin by introducing the following classes of functions:

$$C\mathcal{K} = \{a \in C[R_+^2, R_+] : a(t,u) \in \mathcal{K} \text{ for each } t \in R_+\},$$

$$\Gamma = \left\{h \in C[R_+ \times R^n, R_+] : \inf_{(t,x)} h(t,x) = 0\right\},$$

$$\Gamma_0 = \left\{h \in \Gamma : \sup_{R_+} h(t,x) \text{ exists for } x \in R^n\right\}.$$

In order to discuss stability results corresponding to Lyapunov's original theorems in the framework of variational Lyapunov method, we need the notion of strict stability of the system (2.1). Such a notion was introduced in [3] and was also called stability in tube-like domains. For our present purpose, the strict stability definition given in [3, p. 293] is very flexible and therefore, we shall given below a refinement of that concept.

DEFINITION 4.1 The trivial solution of (2.1) is said to be:

(1) *strictly stable* if given $\epsilon_1 > 0$ and $t_0 \in R_+$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that $|x_0| \leq \delta_1$ implies $|y(t, t_0, x_0)| < \epsilon_1$ for $t \geq t_0$ and for every $\delta_2 \leq \delta_1$, there exists an $\epsilon_2 < \delta_2$ such that $\delta_2 \leq |x_0|$ implies $\epsilon_2 < |y(t, t_0, x_0)|$, $t \geq t_0$;
(2) *strictly uniformly stable*, if δ_1, δ_2 and ϵ_2 are independent of t_0
(3) *strictly uniformly stable*, if given $\delta_1 > 0$, $\epsilon_1 > 0$ and $t_0 \in R_+$, for ever $\alpha_2 \leq \alpha_1$, there exists and $\epsilon_2 < \epsilon_1$ and $T_1, T_2 > 0$ (depending on t_0) such that $\alpha_2 \leq |x_0| \leq \alpha_1$ impliesl $\epsilon_2 < |y|y(t, t_0, x_0)| < \epsilon_1$ for $t_0 + T_1 \leq t \leq t_0 + T_2$
(4) *strictly uniformly attractive* if T_1, T_2 are independent of t_0 in (3).

We wish to consider the stability properties of the system (2.2) in terms of two measures so that several concepts can be unified in one result. We shall define the notion of such stability. Let $h_0, h \in \Gamma$.

DEFINITION 4.2 The system (2.2) is said to be (h_0, h)-*stable* if given $\epsilon > 0$ and $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $h_0(t_0, x_0) \leq \delta$ implies $h(t, x(t, t_0, x_0)) < \epsilon$, $t \geq t_0$.

Based on this definition, other concepts can be formulated. See [5] for other definitions and the discussion of generality of the concept of two measures.

If we wish to prove the results of (h_0, h)-stability for the system (2.2), we require the concept of strict stability of the system (2.1). However, the extension of the notion of strict stability given in Definition 4.1 will be meaningless when two different measures are used (see remarks following Theorem 4.2). We therefore define a suitable concept which makes sense and reduces to Definition 4.1 when $h_0(t, x) = h(t, x) = |x|$.

DEFINITION 4.3 Let $h_0, h_1 \in \Gamma$ and $\tilde{h}_0(x) = \sup_{R^+} h_0(t_0, x), \tilde{h}_1(x) = \sup_{R_+} h_1(t_0, x)$. Then we say that the system (2.1) is $(\tilde{h}_0, \tilde{h}_1)$-*strictly* *stable* if given $\epsilon_1 > 0$ and $t_0 \in R_+$, there exists a $\delta_1 = \delta_1(\epsilon_1)$ such that $\tilde{h}_0(x_0) \leq \delta_1$ implies $\tilde{h}_0(y(t, t_0, x_0)) < \epsilon_1$, $t \geq t_0$ and for every $\delta_2 \leq \delta_1$, there exists an $\epsilon_2 \leq \delta_2$ such that $\delta_2 \leq h_1(t_0, x_0)$ implies $\epsilon_2 < h_1(t, y(t, t_0, x_0))$, $t \geq t_0$.

We note that when $\tilde{h}_0(x) = \tilde{h}_1(x) = |x|$, Definition 4.1 coincides with Definition 4.3.

We are now in a position to prove a result parallel to Lyapunov's first theorem.

THEOREM 4.1 *Assume that*

(C_1) $V \in C[R_+ \times R^n, R_+]$, $V(t, x)$ *and* $|y(y, s, x)|$ *are locally Lipschitzian in* x *for each* (t, s)

(C_2) $D_-V(t, s, x) \leq 0$ *in* $S(h, \rho) = \{(t, x)\colon h(t, x) < \rho\}$ *for some* $h \in \Gamma$ *and* $\rho > 0$;

(C_3) $b(h(t, x)) \leq V(t, x)$ *in* $S(h, \rho)$ *and*

$$V(t, x) \leq a_1(t, h_1(t, x)) + a_0(t, h_0(t, x)) \quad in\ S(h_1, \rho),$$

where $b \in \mathcal{K}$ *and* $a_1, a_0 \in C\mathcal{K}$;

(C_4) h_0 *is finer than* h_1 *and* h, *that is, there exist functions* $\phi, \psi \in \mathcal{K}$ *such that* $h_1(t, x) \leq \phi(h_0(t, x))$ *and* $h(t, x) \leq \psi(h_0(t, x))$ *whenever* $h_0(t, x) \leq \rho_0$, *for some* ρ_0 *with* $\phi(\rho_0)$, $\psi(\rho_0) \leq \rho$;

(C_5) *the system* (2.1) *is* $(\tilde{h}_0, \tilde{h}_0)$,*-stable.*

Then the system (2.2) *is* (h_0, h)*-stable.*

Proof Given $0 < \epsilon < \rho$, let ρ_0 be such that $\phi(\rho_0), \psi(\rho_0) \leq \rho$. Choose $\eta > 0$ such that (η depends on t_0 and ϵ) for $t \geq t_0$,

$$a_0(t_0, h_0(t_0, y(t))) < \frac{b(\epsilon)}{2} \quad \text{whenever } h_0(t_0, y(t)) < \eta.$$

Then given $\eta > 0$, by hypothesis (C_5), there exists a $\delta_1 = \delta_1(t_0, \epsilon)$ such that

$$h_0(t_0, y(t)) < \eta \quad \text{provided } h_0(t_0, x_0) < \delta_1.$$

Thus, for $t \geq t_0$,

$$a_0(t_0, h_0(t_0, y(t))) < \frac{b(\epsilon)}{2} \quad \text{whenever } h_0(t_0, x_0) < \delta_1. \tag{4.1}$$

This is possible since $a_0 \in C\mathcal{K}$. Similarly choose $\sigma > 0$ such that $h_1(t_0, y(t)) < \sigma$ implies $a_1(t_0, h_1(t_0, y(t))) < b(\epsilon)/2$. Since h_0 is finer than h_1, $h_1(t_0, y(t)) \leq \phi(h_0(t_0, y(t)))$. Let $\delta = \min(\delta_1, \phi(\eta), \phi(\delta_1), \sigma)$ and $h_0(t_0, x_0) < \delta$. Then we get

$$a_1(t_0, h_1(t_0, y(t))) < \frac{b(\epsilon)}{2} \quad \text{whenever } h_0(t_0, x_0) < \delta. \tag{4.2}$$

We claim that system (2.2) is (h_0, h)-stable, that is,

$$h(t, x(t)) < \epsilon, \ t \geq t_0 \quad \text{whenever } h_0(t_0, x_0) < \delta. \tag{4.3}$$

Note that, in view of (4.1), (4.2) and the choice of δ,

$$\begin{aligned} b(h(t_0, x(t_0))) &\leq V(t_0, x_0) \leq a_1(t_0, h_1(t_0, x_0)) + a_0(t_0, h_0(t_0, x_0)) \\ &\leq a_1(t_0, \phi(\delta)) + a_0(t_0, \delta) < b(\epsilon) \end{aligned}$$

implying $h(t_0, x(t_0)) < \epsilon$. If the claim (4.3) is not true, then there exists a solution $x(t) = x(t, t_0, x_0)$ of (2.2) and $t_1 > t_0$ such that

$$h_0(t_0, x_0) < \delta, \ \ h(t_1, x(t_1)) = \epsilon \ \text{and} \ h(t, x(t)) \leq \epsilon, \quad t \in [t_0, t_1].$$

We can apply Corollary 3.1(Theorem 3.1 with $N = 1$, $g \equiv 0$) to get

$$V(t, x(t)) \leq V(t_0, y(t)), \quad t \in [t_0, t_1],$$

where $y(t) = y(t, t_0, x_0)$ is the solution of (2.1). At $t = t_1$,

$$\begin{aligned} b(\epsilon) &\leq V(t_1, x(t_1)) \leq V(t_0, y(t_1, t_0, x_0)) \\ &\leq a_1(t_0, h_1(t_0, y(t_1, t_0, x_0))) + a_0(t_0, h_0(t_0, y(t_1, t_0, x_0))) \\ &< b(\epsilon) \end{aligned}$$

in virtue of (C_3), (4.1) and (4.2). This contradiction proves the claim that the system (2.2) is (h_0, h)-stable.

THEOREM 4.2 *Assume that*:

(i) *the hypotheses* (C_1), (C_3) *and* (C_4) *of Theorem* 4.1 *hold*;
(ii) $D_- V(t, s, x) \leq -c(h_1(s, y(t, s, x)))$ *in* $S(h, \rho)$ *and* $c \in \mathcal{K}$;
(iii) *the system* (2.1) *is* $(\tilde{h}_0, \tilde{h}_1)$*-strictly uniformly stable*
(iv) $h_1 \in C^1[R_+ \times R^n, R_+]$ *and* $\mathrm{d}h_1(t, x)/\mathrm{d}t(\equiv \partial h/\partial t + \partial h/\partial x(f))$ *is bounded from above or below in* $S(h, \rho)$.

Then, the system (2.2) *is* (h_0, h)*-stable and* (h_0, h_1)*-attractive.*

Proof Let $\epsilon = \rho$ and $\delta(\rho) = \delta_0$. Thus, (i)–(iii) and Theorem 4.1 yields that

$$h(t, x(t)) < \rho, \ t \geq t_0, \quad \text{whenever } h_0(t_0, x_0) \leq \delta_0.$$

If $h_0(t_0, x_0) \leq \delta_0$, we claim that $\lim_{t\to\infty} h_1(t, x(t)) = 0$. If this claim is not true, let $\liminf_{t\to\infty} h_1(t, x(t)) \neq 0$ which implies that there exists $\eta > 0$ and $T \geq t_0$ such that

$$h_1(t, x(t)) \geq \eta \quad \text{for } t \geq T.$$

Assumption (iii) implies that whenever

$$h_1(s, x(s)) \geq \eta$$

we have

$$h_1(s, y(t, s, x(s))) > \epsilon_2, \quad t \geq s \geq T,$$

in view of Definition 4.3 with $\eta = \delta_2$. Using (C_3) and Corollary 3.2 we get

$$\begin{aligned} 0 \leq V(t, x(t)) &\leq V(t_0, y(t)) - \int_{t_0}^{t} c(h_1(s, y(t, s, x(s))))\,\mathrm{d}s \\ &\leq a_1(t_0, h_1(t_0, y(t))) + a_0(t_0, h_0(t_0, y(t))) \\ &\quad - \int_{T}^{t} c(h_1(s, y(t, s, x(s))))\,\mathrm{d}s \\ &\leq a_1(t_0, \phi(\delta_0)) + a_0(t_0, \delta_0) - c(\epsilon_2)(t - T) \end{aligned}$$

which leads to a contradiction for large t. Hence $\liminf_{t\to\infty} h_1 \cdot (t, x(t)) = 0$.

If $\limsup_{t\to\infty} h_1(t, x(t)) \neq 0$, then there exists a sequence $\{t_n\}$, $t_n \to \infty$ and $\eta > 0$ such that

$$h_1(t_n, x(t_n)) \geq \eta.$$

This implies that there exist nonoverlapping intervals $[t_n', t_n'']$ that can be chosen such that either

$$h_1(t_n', x(t_n')) = \eta/2, \;\; h_1(t_n'', x(t_n'')) = \eta \text{ and } h_1(t, x(t)) \in [\eta/2, \eta]$$

or

$$h_1(t_n'', x(t_n'')) = \eta, \;\; h_1(t_n', x(t_n')) = \eta/2 \text{ and } h_1(t, x(t)) \in [\eta/2, \eta].$$

Thus follows from assumption (iii) and Definition 4.3 with $\eta/2 = \delta_2$ that there exists $\epsilon_2 > 0$ such that

$$h_1(s, y(t, s, x(s))) \geq \epsilon_2, \quad t \geq s, \;\; s \in [t_n', t_n''], \;\; n = 1, 2, \ldots$$

By assumption (iv), we can assume without loss of generality that $h_1'(t, x) = M$, which yields

$$\begin{aligned} \eta/2 = h_1(t_n'', x(t_n'')) - h_1(t_n', x(t_n')) = \int_{t_n'}^{t_n''} h_1'(t, x(t))\,\mathrm{d}t \\ \leq M(t_n'' - t_n'). \end{aligned}$$

By Corollary 3.2, we have the estimate (3.4) which gives

$$\begin{aligned} 0 \leq V(t, x(t)) \leq V(t_0, y(t)) - \sum_1^m \int_{t_n'}^{t_n''} c(h_1(s, y(t, s, x(s))))\,\mathrm{d}s \\ \leq a_1(t_0, \phi(\delta_0)) + a_0(t_0, \delta_0) - c(\epsilon_2)\sigma m, \end{aligned}$$

where $\sigma = \eta/2M$ and $t_n'' - t_n' \geq \sigma$. As $m \to \infty$, this leads to a contradiction. Hence $\limsup_{t\to\infty} h_1(t, x(t)) = 0$. The proof is complete.

Remarks (1) If $h_1 = h$ in Theorem 4.2, we obtain (h_0, h)-asymptotic stability of (2.2). The same conclusion is valid when h_1 is finer than h.

(2) If $h_1 = h_0$ in Theorem 4.2, we get (h_0, h)-stability and (h_0, h_0)-attractivity for (2.2). One should note that in proving a converse theorem in the framework of two measures, parallel to Massera's converse theorem for uniform asymptotic stability such that a (h_0, h_0)-attractivity concept is required. See [7] for details.

(3) If $h_1 = h_0$ and $a_0, a_1 \in \mathcal{K}$ we have

$$V(t, x(t)) \leq a_1(h_0) + a_0(h_0) \equiv a(h_0), \quad a \in \mathcal{K}$$

and consequently assumption (ii) reduces to

$$\text{(ii*)} \qquad D_- V(t, s, x) \leq -\gamma(V(s, y(t, s, x))),$$

where $\gamma = ca^{-1}$ and $\gamma \in \mathcal{K}$. This gives the comparison equation

$$u' = -\gamma(u)$$

the zero solution of which is uniformly asymptotically stable and therefore the assumption (iii) and (iv) become redundant (we need only (C_5) instead of (iii)). This fact can be observed in the proof of the general comparison result with $g(t, s, u)$.

(4) Finally, note that if $h_0 = h_1 = |x|$, assumption (iii) reduces to the strict stability of the zero solution of (2.1), according to Definition 4.1.

THEOREM 4.3 *Assume that*

(1^0) *the hypotheses* (C_1), (C_3) *and* (C_4) *of Theorem* (4.1) *hold*;
(2^0) $D_- V(t, s, x) \leq g(t, s, V(s, y(t, s, x)))$ *on* $S(h, \rho)$ *where* $g \in C[R_+^3, R]$
(3^0) *the system* (2.1) *is* $(\tilde{h}_0, \tilde{h}_0)$*-stable.*

Then the stability properties of the trivial solution of (3.2) *imply the corresponding* (h_0, h)*-nonuniform stability properties of* (2.2).

Note When we say that the trivial solution of (3.2) is stable, we mean the following: given $\epsilon > 0$ and $t_0 \in R_+$ there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that

$$0 \leq u_0 \leq \delta \text{ implies } u_0(t, t_0, u_0) < \epsilon, \quad t \geq t_0,$$

where $u_0(t, t_0, u_0) \equiv u(t, t, t_0, u_0)$ and $u(t, s, t_0, u_0)$ is any solution of (3.2).

Proof Suppose that the trivial solution of (3.2) is stable. Let $0 < \epsilon < \rho$ and $t_0 \in R_+$ be given. Then given $b(\epsilon) > 0$ there exists a $\delta^* > 0$ such that

$$u(t, t_0, u_0) < b(\epsilon) \;\; t \geq t_0 \quad \text{whenever } 0 \leq u_0 \leq \delta^*.$$

Using this δ^* in place of $b(\epsilon)$ in the proof of Theorem 4.1, we can find a $\delta > 0$ as before (see proof of Theorem 4.1). Then we claim that

$h(t, x(t)) < \epsilon$, $t \geq t_0$ whenever $h_0(t_0, x_0) \leq \delta$. Note that $h_0(t_0, x_0) < \epsilon$ since

$$b(h(t_0, x_0)) \leq V(t_0, x_0) \leq a_1(t_0, h_1(t_0, x_0)) + a_0(t_0, h_0(t_0, x_0))$$
$$\leq a_1(t_0, \phi(\delta)) + a_0(t_0, \delta) < \delta^* < b(\epsilon).$$

If our claim is not true, then there exists a solution $x(t)$ of (2.2) and $t_1 > t_0$ such that

$$h_0(t_0, x_0) < \delta, \qquad h(t_1, x(t_1)) = \epsilon \quad \text{and} \quad h(t, x(t)) \leq \epsilon, \;\; t \in [t_0, t_1].$$

By Theorem 3.1, we have

$$V(t, x(t)) \leq r_0(t, t_0, V(t_0, y(t))), \quad t \in [t_0, t_1].$$

At $t = t_1$, using the assumptions (1^0), (2^0) and (3^0) we get

$$b(\epsilon) = b(h(t_1, x(t_1))) \leq V(t_1, x(t_1))$$
$$\leq r_0(t_1, t_0, a_1(t_0, h_1(t_0, y(t_1))) + a_0(t_0, h_0(t_0, y(t_1)))).$$

But $a_1(t_0, h_1(t_0, y(t_1))) + a_0(t_0, h_0(t_0, y(t_1))) < \delta^*$ whenever $h_0(t_0, x_0) < \delta$. Consequently, we have

$$r_0(t_1, t_0, \delta^*) < b(\epsilon),$$

which is a contradiction. Hence the system (2.2) is (h_0, h)-stable.

Next, suppose that the trivial solution of (3.2) is asymptotically stable. This implies stability and attractivity. Taking $\epsilon = \rho$, we get $\delta_0(t_0) = \delta(\rho, t_0)$. Also corresponding to $b(\rho)$, we have $\delta_0^*(t_0) = \delta^*(\rho, t_0)$. Also corresponding to $b(\rho)$, we have $\delta_0^*(t_0) = \delta^*(\rho, t_0)$ and given $\epsilon > 0$, there exists a $T > 0$ such that

$$0 \leq u_0 \leq \delta_0^*(t_0) \text{ implies } u_0(t, t_0, u_0) < b(\epsilon), \quad t \geq t_0 + T.$$

Moreover,

$$h_0(t_0, x_0) < \delta_0 \text{ implies } h(t, x(t))$$

from (h_0, h)-stability of (2.2). Thus, taking $\tilde{\delta}_0 = \min(\delta_0^*, \delta_0)$, we assert that $h(t, x(t)) < \epsilon$, $t \geq t_0 + T$ whenever $h_0(t_0, x_0) < \tilde{\delta}_0$. If this is not true, then there would exist a solution $x(t)$ of (2.2) and a divergent sequence $\{t_n\}$, $t_n \geq t_0 + T$, with the property

$$h(t_n, x(t_n)) \geq \epsilon.$$

Since we have in $S(h, \rho)$

$$V(t, x(t)) \leq r_0(t, t_0, V(t_0, y(t))), \quad t \geq t_0,$$

it follows that

$$\begin{aligned} b(\epsilon) &\leq b(h(t_n, x(t_n))) \leq V(t_n, x(t_n)) \leq r_0(t_n, t_0, V(t_0, y(t_n))) \\ &\leq r_0(t_n, t_0, \delta_0^*(t_0)) < b(\epsilon), \quad t_n \geq t_0 + T, \end{aligned}$$

which is a contradiction. Hence the system (2.2) is (h_0, h)-attractive. Thus we get (h_0, h)-asymptotic stability of (2.2). The proof is complete.

We note that in view of the fact that $a_0, a_1 \in C\mathcal{K}$, we can conclude only nonuniform (h_0, h)-stability properties of (2.2) even when we assume uniform stability properties for (3.2) as well as (2.1). If $a_0, a_1 \in \mathcal{K}$, then we get uniform (h_0, h)-stability properties whenever we assume the corresponding notions for (3.2) and (2.1).

Also, note that whenever $D_-V(t, s, x)$ is estimated in terms of a comparison function $g(t, s, v)$, we do not require strict stability properties of (2.1) where as in the other case (Theorem 4.2) we do need strict stability for (2.1).

5. STABILITY OF LARGE SCALE SYSTEMS

Recall that the general comparison result (Theorem 3.1) is in terms of vector Lyapunov functions and in Section 4, we have employed only a single Lyapunov function. One could have, of course, used the method of vector Lyapunov functions, see [3,6], in proving the (h_0, h)-stability results of Section 4. However, vector Lyapunov functions are more useful and important when we consider large scale systems via the method of aggregation and decomposition [11,12]. In the present framework, one can get better results by applying Theorem 4.1 in the following modified manner.

Consider the N systems of differential equations

$$y_i' = F_i(t, y_i), \quad y_i(t_0) = x_{0i}, \tag{5.1}$$

where $i = 1, 2, \ldots, N$, each $y_i \in R^{n_i}$ and $F_i \in C[R_+ \times R^{n_i}, R^{n_i}]$. Let $y_i(t, t_0, x_{0i})$ be the solution of (5.1) for each i, such that $|y_i(t, t_0, x_{0i})|$ is locally Lipschitzian in x_{0i} for each t. Suppose that the system (2.2) is of large scale and we decompose $x = (x_{n_1}, x_{n_2}, \ldots, x_{n_N})$ with $n_1 + n_2 + \cdots + n_N = n$. For $V_i(t, x_i)$, $x_i \in R^{n_i}$, we define

$$\begin{aligned} D_- V_i(t, s, x_i) \equiv \liminf_{h \to 0^-} \frac{1}{h} [V_i(s + h, y_i(t, s + h, x_i + hf_i(s, x))) \\ - V_i(s, y_i(t, s, x_i))] \end{aligned}$$

and find the estimate

$$\begin{aligned} D_- V_i(t, s, x_i) \le g_i(t, s, V_1(s, y_1(t, s, x_1)), V_2(s, y_2(t, s, x_2)), \ldots, \\ V_N(s, y_N(t, s, x_N))). \end{aligned} \tag{5.2}$$

(Here each x_i, $i = 1,2,\ldots,N$ is an element R^{n_i}.)

We see that this fits into the mode of Theorem 3.1 to yield the estimate (3.3) which is

$$V(t, x(t)) \le r_0(t, t_0, V(t_0, y(t))), \quad t \ge t_0,$$

where the vector $V = (V_1, V_2, \ldots, V_N)$. As we stressed earlier, if the large scale system (2.2) is decomposable into

$$x_i' = F_i(t, x_i) + R_i(t, x_1, x_2, \ldots, x_N), \tag{5.3}$$

where $R_i(t, x_1, x_2, \ldots, x_N)$ are the interconnections and $y_i' = F_i(t, y_i)$ are subsystems, we can extract good behavior of the interconnections R_i by employing our approach of variational Lyapunov method and therefore need not depend on preserving the good properties of subsystems only. The subsystems are useful though, to construct suitable Lyapunov functions even when they do not possess the stability properties of the entire large scale system. This advantage needs to be exploited in applications.

Finally, we wish to remark that it is not necessary for the development of the variational Lyapunov method to find suitable known system (2.1) or (5.1). It would be sufficient to choose any function $y \in C[R_+^2 R^n, R^n]$ satisfying (i) $|y(t,s,x)|$ is locally Lipschitzian in x for each (t,s), (ii) $y(t,t,x)=x$ and (iii) $y(t,t_0,x_0)$ has required stability properties.

References

[1] Deo, S.G. and Lakshmikantham, V., *Method of Variation of Parameters for Dynamic Systems*, Gordon and Breach Science Publishers, London, 1997.

[2] Ladde, G.S., Lakshmikantham, V. and Leela, S., A new technique in perturbation theory, *Rocky Mountain Jour.* **6** (1977), 133–140.

[3] Lakshmikantham, V. and Leela, S., *Differential and Integral Inequalities*, Vol. I, Academic Press, New York, 1969.

[4] Lakshmikantham, V., Leela, S. and Martynyuk, A.A., *Stability Properties of Nonlinear Systems*, Marcel Dekker, New York, 1989.

[5] Lakshmikantham, V. and Liu, X., *Stability Analysis in Terms of Two Measures*, World Scientific, Singapore, 1993.

[6] Lakshmikantham, V., Matrosov, V.M. and Sivasundaram, S., *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*, Kluwer Academic Publishers, 1991.

[7] Lakshmikantham, V. and Salvadori, L., On Massera type converse theorem in terms of two different measures, *Bull. U.M.I.* **13** (1976), 293–301.

[8] Liu, X. and Vatsala, A.S., Variations of parameters in terms of Lyapunov-like functions and stability of perturbed systems, to appear.

[9] Liu, X. and Sivasundaram, S., Stability of nonlinear systems under constantly acting perturbations, *International J. Math. and Math. Sci.* **18** (1995), 273–278.

[10] Rajalakshmi, S. and Sivasundaram, S., Variational Lyapunov second method, *Dyn. Sys. and Appl.* **2** (1993), 485–490.

[11] Siljak, D.D., *Large Scale Dynamic Systems*, North-Holland, New York, 1978.

[12] Siljak, D.D., *Decentralized Control of Complex Systems*, Academic Press, New York, 1991.