Mathematical Problems in Engineering
Volume 4, pp. 423–435
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On Absolute Stability of Nonlinear Systems with Small Delays*

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(Received 17 February 1998; Revised 31 March 1998)

Nonlinear nonautonomous retarded systems with separated autonomous linear parts and continuous nonlinear ones are considered. It is assumed that deviations of the argument are sufficiently small. Absolute stability conditions are derived. They are formulated in terms of eigenvalues of auxiliary matrices.

Keywords: Nonlinear differential–difference systems; Absolute stability

AMS (MOS) Subject Classification: 34 K 20

1 INTRODUCTION AND STATEMENT OF THE RESULT

Stability of nonlinear differential–difference systems has been discussed by many authors (see [1–3], and references given therein). The basic method for the stability analysis is the Lyapunov functionals one. By this method many very strong results are obtained. But finding Lyapunov's functionals is usually difficult. In [4] (see also [2, p. 111]) an explicit absolute stability criterion for a class of retarded systems was derived. It is formulated in the terms of the eigenvalues of the characteristic matrix-valued functions. That result was further developed in [5].

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The present paper is devoted to a class of nonlinear nonautonomous differential–difference systems having sufficiently small deviations of the argument. We will obtain absolute exponential stability conditions in terms of eigenvalues of auxiliary matrices. These conditions allow us to avoid the analysis of quasipolynomials. There have been many stability conditions for nonlinear systems with sufficiently small delays in recent years usually in the category of "robust stability" for time-delay systems (e.g. [6,7], and references therein). Besides, only systems with constant discrete delays are considered. Below we will consider systems with arbitrary (time-dependent and discrete, in general) delays.

Let $\mathbf{C}^n$ be a Euclidean space with the Euclidean norm $\|.\|_{C^n}$. Moreover, $C([a,b],\mathbf{C}^n)$ is the space of continuous functions defined on the real segment $[a,b]$ with values in $\mathbf{C}^n$ and equipped with the norm

$$\|v\|_{C([a,b],\mathbf{C}^n)} := \sup_{a\le t\le b} \|v(t)\|_{C^n} \quad (v \in C([a,b],\mathbf{C}^n).$$

In addition, let $L^2([a,b],\mathbf{C}^n)$ be the space of n-vector-valued functions defined on $[a,b]$ with values in $\mathbf{C}^n$ and equipped with the norm

$$\|v\|_{L^2([a,b],\mathbf{C}^n)} := \left[\int_a^b \|v(t)\|^2_{C^n}\,\mathrm{d}t\right]^{1/2} \quad (v \in L^2([a,b],\mathbf{C}^n)).$$

Let $\eta < \infty$ be a positive constant. As usually, for all $t \ge 0$ and $x \in C([0,\infty),\mathbf{C}^n)$, x_t is defined by the relation

$$x_t(\theta) \equiv x(\theta + t), \quad -\eta \le \theta \le 0,$$

cf. [1, p. 36], [2, p. 19]. That is, for any $x \in C([0,\infty), \mathbf{C}^n)$, x_t is some function from $C([-\eta,\infty),\mathbf{C}^n)$.

Consider in $\mathbf{C}^n$ the equation

$$\dot{x}(t) = \sum_{k=0}^{m} A_k x(t - h_k) + F(t, x_t) \quad (t \ge 0,\ \dot{x} = \mathrm{d}x/\mathrm{d}t), \tag{1.1}$$

where A_k $(k = 0,\dots,m)$ are constant matrices,

$$0 = h_0 < h_1 < \cdots < h_m = \eta$$

are numbers, and F continuously maps $[0,\infty) \times C([-\eta,0],\mathbf{C}^n)$ into $\mathbf{C}^n$.

It is assumed that for every $u \in L^2([-\eta,\infty),\mathbf{C}^n)\cap C([-\eta,\infty),\mathbf{C}^n)$, the inequality

$$\|F(\cdot,u_t)\|_{L^2([0,\infty),\mathbf{C}^n)} \equiv \left[\int_0^\infty \|F(t,u_t(t))\|^2_{C^n}\,\mathrm{d}t\right]^{1/2} \leq q\|u\|_{L^2([-\eta,\infty),\mathbf{C}^n)} \tag{1.2}$$

is fulfilled. For instance, let there exist constants $b_j > 0$ and $q_j \geq 0$ ($j = 1,\ldots,m < \infty$), such that the relation

$$\|F(t,u_t)\|_{\mathbf{C}^n} \leq \sum_{k=1}^{m} q_j\|u(t-h_j(t))\|_{C^n} \quad \text{for all } u \in L^2([-\eta,\infty),\mathbf{C}^n) \tag{1.3}$$

holds, where $h_j(t)$ ($j = 1,\ldots,m$) are differentiable scalar-valued functions with the properties

$$1-\dot{h}_j(t) \geq b_j > 0 \quad \text{and} \quad 0 \leq h_j(t) \leq \eta \ \ (t \geq 0, j = 1,\ldots,m). \tag{1.4}$$

Then by Proposition 8.1.2 of [8], condition (1.2) holds with

$$q = \sum_{j=1}^{m} q_j b_j^{-1/2}. \tag{1.5}$$

Take the initial condition

$$x(t) = \Phi(t) \quad \text{for } -\eta \leq t \leq 0, \tag{1.6}$$

where $\Phi(t)$ is a given continuous vector-valued function defined on $[-\eta,0]$. *A solution of* (1.1) is an absolutely continuous vector-valued function $x:[-\eta,\infty)\to\mathbf{C}^n$ which satisfies that equation over $R_+ := [0,\infty)$ almost everywhere with a given initial condition (1.6). The solution existence is assumed.

DEFINITION 1.1 We will say that the zero solution of (1.1) is absolutely exponentially stable in the class of nonlinearities (1.2) if there exist positive constants N and ϵ independent of the specific form of the function F (but dependent on q) such that for any solution $x(t)$

of (1.1) with an initial function Φ, the inequality

$$\|x(t)\|_{C^n} \leq N\exp[-\epsilon t]\|\Phi\|_{C([-\eta,0],C^n)} \quad (t \geq 0)$$

is fulfilled.

Let A be an $n \times n$-matrix, and let $\lambda_k(A)$ $(k=1,\dots,n)$ denote the eigenvalues of A including with their multiplicities. The following quantity plays an essential role hereafter:

$$g(A) = (N^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2)^{1/2},$$

where $N(A)$ is the Hilbert–Schmidt (Frobenius) norm of A, i.e. $N^2(A) = \text{Trace}(AA^*)$. The following properties of $g(A)$ are proved in [8, Sections 1.1 and 1.3]: $g^2(A) \leq N^2(A) - |\text{Trace}\, A^2|$,

$$g(A) \leq \sqrt{1/2}N(A^* - A) \quad \text{and} \quad g(Ae^{i\tau} + zI) = g(A) \text{ for every } \tau \in \mathbf{R},\ z \in \mathbf{C}. \tag{1.7}$$

Here and below, I is the unit matrix. In [9, p. 185], $g(A)$ is called the deviation from normality of A, since $g(A) = 0$ if A is a normal matrix, i.e., if $AA^* = A^*A$. Further, set

$$\tilde{A} = \sum_{k=0}^{m} A_k \quad \text{and} \quad V_2 = \sum_{k=1}^{m} h_k \|A_k\|_{C^n}.$$

Everywhere below it is assumed that $\tilde{A}$ is a Hurwitz matrix. Denote

$$\alpha(\tilde{A}) = \max_k \text{Re}\lambda_k(\tilde{A}), \qquad \omega(\tilde{A}) = \max_k |\text{Im}\lambda_k(\tilde{A})|,$$

and

$$\Gamma(\tilde{A}) = \sum_{k=1}^{n-1} \frac{g^k(\tilde{A})}{(k!)^{1/2}|\alpha(\tilde{A})|^{k+1}}.$$

The aim of this paper is to prove the following.

THEOREM 1.2 *Let the condition*

$$[(\alpha^2(\tilde{A}) + \omega^2(\tilde{A}))^{1/2} V_2 + q]\Gamma(\tilde{A}) < 1 \qquad (1.8)$$

hold. Then the zero solution of Eq. (1.1) *is absolutely exponentially stable in the class of nonlinearities* (1.2).

Remark 1.3 Since $V_2 = h_1\|A_1\|_{\mathbf{C}^n} + \cdots + h_m\|A_m\|_{\mathbf{C}^n}$, if $q\Gamma(\tilde{A}) < 1$, then sufficently small delays provide the stability.

If $\tilde{A}$ is a normal (in particular, Hermitian) matrix. Then $g(\tilde{A}) = 0$, $\Gamma(\tilde{A}) = |\alpha(\tilde{A})|^{-1}$, and condition (1.8) takes the form

$$[(\alpha^2(\tilde{A}) + \omega^2(\tilde{A}))^{1/2} V_2 + q]|\alpha(\tilde{A})|^{-1} < 1 \qquad (1.9)$$

Remark 1.4 Theorem 1.2 is exact, and condition (1.8) cannot be improved without additional restrictions.

In fact, let us consider the system

$$\dot{x}(t) = Ax(t) + F(t, x(t)), \qquad (1.10)$$

where A is a constant Hurwitz matrix, and F continuously maps $[0, \infty) \times \mathbf{C}^n$ into $\mathbf{C}^n$ with the property

$$\|F(t, h)\|_{C^n} \le q\|h\|_{C^n} \quad (h \in \mathbf{C}^n;\ t \ge 0). \qquad (1.11)$$

Under consideration $V_2 = 0$, and (1.8) takes the form $q\Gamma(A) < 1$. If A is normal, then the inequality

$$q|\alpha(A)|^{-1} < 1 \qquad (1.12)$$

gives the absolute stability conditions of the zero solution of Eq. (1.10) in the class of nonlinearities (1.11). Take $F(t, h) = qh$ ($h \in \mathbf{C}$). Then for the asymptotic stability of Eq. (1.10) it is necessary and sufficient that $\alpha(A) + q < 0$. But this condition coincides with (1.12).

2 PROOF OF THEOREM 1.2

For brevity put $\|.\|_{L^2([0,\infty),\mathbf{C}^n)} = \|.\|_L$, and

$$\|y(\mathrm{i}\omega)\|_{\tilde{L}} \equiv \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \|y(\mathrm{i}\omega)\|_{C^n}^2 \,\mathrm{d}\omega\right]^{1/2} \quad (y \in L^2(\mathbf{R}, \mathbf{C}^n)).$$

Denote

$$c_0(\tilde{A}) = \sup_{\omega \in \mathbf{R}} \|(\mathrm{i}\omega I - \tilde{A})^{-1}\|_{C^n} \quad \text{and} \quad c_1(\tilde{A}) = \sup_{\omega \in \mathbf{R}} \|\omega(\mathrm{i}\omega I - \tilde{A})^{-1}\|_{C^n}.$$

Let us consider the equation

$$\dot{x}(t) = \sum_{k=0}^{m} A_k x(t - h_k) + f(t) \quad (t \geq 0). \tag{2.1}$$

LEMMA 2.1 *Under the condition* $c_1(\tilde{A})V_2 < 1$, *let* $f \in L^2([0,\infty), \mathbf{C}^n)$. *Then the bound*

$$\|x\|_L \leq [1 - c_1(\tilde{A})V_2]^{-1} c_0(\tilde{A}) \|f\|_L + m_1 \|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \tag{2.2}$$

is valid with a constant m_1 *independent of* Φ *for any solution* $x(t)$ *of* (2.1) *with a continuous initial function* Φ.

Proof Rewrite (2.1) in the form

$$\dot{x}(t) = \tilde{A}x(t) + \sum_{k=1}^{m} A_k(x(t - h_k) - x(t)) + f(t) \quad (t \geq 0). \tag{2.3}$$

The Laplace transform of $x(t - h_k)$ is $\tilde{x}(\lambda)\mathrm{e}^{-h_k\lambda} + v_k(\lambda)$ where $\tilde{x}(\lambda)$ is the Laplace transform of $x(t)$, and

$$v_k(\lambda) = \int_0^{h_k} \mathrm{e}^{-t\lambda} x(t - h_k)\,\mathrm{d}t = \int_0^{h_k} \mathrm{e}^{-t\lambda} \Phi(t - h_k)\,\mathrm{d}t.$$

Thus the application of the Laplace transformation to Eq. (2.3) gives

$$\lambda\tilde{x}(\lambda) - x(0) = \tilde{A}\tilde{x}(\lambda) + \sum_{k=1}^{m} A_k[\tilde{x}(\lambda)(\mathrm{e}^{-h_k\lambda} - 1) + v_k(\lambda)] + \tilde{F}(\lambda) \quad (\lambda \in \mathbf{C}),$$

where $\tilde{F}(\lambda)$ is the Laplace transform of the function $f(t)$. Hence,

$$\tilde{x}(\lambda) = (\lambda I - \tilde{A})^{-1} \left[\sum_{k=1}^{m} A_k(\mathrm{e}^{-h_k\lambda} - 1)x(\lambda) + \tilde{F}(\lambda) \right] + e(\lambda) \quad (\lambda \in \mathbf{C}), \tag{2.4}$$

where

$$e(\lambda) = (\lambda I - \tilde{A})^{-1}\left(x(0) + \sum_{k=1}^{m} A_k v_k(\lambda)\right).$$

Since $|1 - \mathrm{e}^{-h\mathrm{i}\omega}| \leq h|\omega|$ $(\omega \in \mathbf{R}, h > 0)$, we have

$$\begin{aligned}
&\|(\tilde{A} - \mathrm{i}\omega I)^{-1}\sum_{k=1}^{m} A_k(1 - \mathrm{e}^{-h_k\mathrm{i}\omega})\tilde{x}(\mathrm{i}\omega)\|_{\tilde{L}} \\
&\leq \|\omega(\tilde{A} - \mathrm{i}\omega I)^{-1}\|_{C^n}\sum_{k=1}^{m} h_k\|A_k\|_{C^n}\|\tilde{x}(\mathrm{i}\omega)\|_{\tilde{L}} \\
&\leq c_1(\tilde{A})V_2\|\tilde{x}(\mathrm{i}\omega)\|_{\tilde{L}} \quad (\omega \in \mathbf{R}).
\end{aligned} \tag{2.5}$$

Due to (2.4) and (2.5), it can be written

$$\|\tilde{x}(\mathrm{i}\omega)\|_{\tilde{L}} \leq V_2 c_1(\tilde{A})\|\tilde{x}(\mathrm{i}\omega)\|_{\tilde{L}} + c_0(\tilde{A})\|\tilde{F}\|_{\tilde{L}} + \|e(\mathrm{i}\omega)\|_{\tilde{L}}.$$

Hence, taking into account that $c_1(\tilde{A})V_2 < 1$, we get

$$\|\tilde{x}(\mathrm{i}\omega)\|_{\tilde{L}} \leq (1 - V_2 c_1(\tilde{A}))^{-1}(c_0(\tilde{A})\|\tilde{F}\|_{\tilde{L}} + \|e(\mathrm{i}\omega)\|_{\tilde{L}}). \tag{2.6}$$

The simple calculations show that

$$\|v_k(\mathrm{i}\omega)\|_{C^n} \leq h_k\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \quad (\omega \in \mathbf{R})$$

and

$$\begin{aligned}
\|e(\mathrm{i}\omega)\|_{\tilde{L}} &\leq \|(\mathrm{i}\omega I - \tilde{A})^{-1}\|_{\tilde{L}}\left(\|x(0)\|_{C^n} + \sum_{k=1}^{m} h_k\|A_k\|_{C^n}\sup_{\omega\in\mathbf{R}}\|v_k(\mathrm{i}\omega)\|_{C^n}\right) \\
&= \|(\mathrm{i}\omega I - \tilde{A})^{-1}\|_{\tilde{L}}\left(\|\Phi(0)\|_{C^n} + V_2\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}\right).
\end{aligned}$$

Hence,

$$\|e(\mathrm{i}\omega)\|_{\tilde{L}} \leq \|(\mathrm{i}\omega I - \tilde{A})^{-1}\|_{\tilde{L}}(1 + V_2)\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}.$$

Relation (2.6) yields

$$\|\tilde{x}(\mathrm{i}\omega)\|_{\tilde{L}} \leq (1 - V_2 c_1(\tilde{A}))^{-1}c_0(\tilde{A})\|\tilde{F}\|_{\tilde{L}} + m_1\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}.$$

Now due to the Parseval equality, we arrive at the required result.

LEMMA 2.2 *Under condition* (1.2), *let the inequality*

$$c_1(\tilde{A})V_2 + c_0(\tilde{A})q < 1 \tag{2.7}$$

be fulfilled. Then for any solution $x(t)$ *of* (1.1) *with a continuous initial function* $\Phi(t)$, *the bound*

$$\|x\|_L \le m_0[1 - (c_1(\tilde{A})V_2 + qc_0(\tilde{A}))]^{-1}\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}$$

is valid with a constant m_0.

Proof Rewrite (1.1) in the form (2.1) with $f(t) = F(t, x_t(t))$, where x is the solution of (1.1). Combining the previous lemma with condition (1.2), we easily get

$$\begin{aligned}\|x\|_L &\le [1 - c_1(\tilde{A})V_2)]^{-1}c_0(\tilde{A})\|f\|_L + m_1\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}\\ &\le [1 - c_1(\tilde{A})V_2)]^{-1}c_0(\tilde{A})q(\|x\|_L + \|\Phi\|_{L^2([-\eta,0],\mathbf{C}^n)})\\ &\quad + m_1\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}.\end{aligned}$$

Consequently,

$$\|x\|_L \le [1 - c_1(\tilde{A})V_2)]^{-1}c_0(\tilde{A})q\|x\|_L + m_2\|\Phi\|_{C([-\eta,0];\mathbf{C}^n)}$$

with a constant m_2. Condition (2.7) implies

$$[1 - c_1(\tilde{A})V_2)]^{-1}c_0(\tilde{A})q < 1.$$

Thus,

$$\begin{aligned}\|x\|_L &\le (1 - c_0(\tilde{A})q[1 - c_1(\tilde{A})V_2)]^{-1})m_2\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}\\ &= (1 - c_0(\tilde{A})q - c_1(\tilde{A})V_2)^{-1}m_0\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}.\end{aligned}$$

As claimed.

LEMMA 2.3 *The relations* $c_0(\tilde{A}) \le \Gamma(\tilde{A})$ *and*

$$c_1(\tilde{A}) \le \Gamma(\tilde{A})[\alpha^2(\tilde{A}) + \omega^2(\tilde{A})]^{1/2}$$

are true.

Proof By Corollary 1.2.4 of [8], for any $n \times n$-matrix A, we have the inequality

$$\|(A - I\lambda)^{-1}\|_{C^n} \leq \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}\rho^{k+1}(A,\lambda)} \quad \text{for all regular } \lambda, \tag{2.8}$$

where $\rho(A,\lambda)$ is the distance between the spectrum of A and a complex point λ. Since $\tilde{A}$ is a Hurwitz matrix,

$$\rho(\tilde{A}, \mathrm{i}\omega) \geq |\alpha(\tilde{A})| \quad (\omega \in \mathbf{R}), \tag{2.9}$$

and therefore $c_0(\tilde{A}) \leq \Gamma(\tilde{A})$.

Further, we have

$$|\lambda_k(\tilde{A}) - \mathrm{i}\omega|^2 \geq \alpha^2(\tilde{A}) + (\omega_k - \omega)^2$$

for any eigenvalue $\lambda_k(\tilde{A})$ of $\tilde{A}$ with $\mathrm{Im}\lambda_k(\tilde{A}) = \omega_k$. Hence simple calculations show that

$$\max_{\omega \in \mathbf{R}} \omega^2 |\lambda_k(\tilde{A}) - \mathrm{i}\omega|^{-2} \leq \frac{\alpha^2(\tilde{A}) + \omega_k^2}{\alpha^2(\tilde{A})}.$$

Thus

$$\omega^2 \rho^{-2}(\tilde{A}, \omega) \leq \frac{\alpha^2(\tilde{A}) + \omega^2(\tilde{A})}{\alpha^2(\tilde{A})} \quad (\omega \in \mathbf{R}).$$

Therefore according to (2.8) and (2.9),

$$\begin{aligned} \|\omega(\tilde{A} - \mathrm{i}\omega I)^{-1}\| &\leq \sum_{k=0}^{n-1} \frac{|\omega| g^k(\tilde{A})}{\sqrt{k!}\rho^{k+1}(\tilde{A}, \omega)} \\ &\leq (\alpha^2(\tilde{A}) + \omega^2(\tilde{A}))^{1/2} \sum_{k=0}^{n-1} \frac{g^k(\tilde{A})}{\sqrt{k!}|\alpha(\tilde{A})|^{k+1}} \\ &= (\alpha^2(\tilde{A}) + \omega^2(\tilde{A}))^{1/2} \Gamma(\tilde{A}) \quad (\omega \in \mathbf{R}). \end{aligned}$$

As claimed.

Lemmas 2.2 and 2.3 yield

COROLLARY 2.4 *Let conditions* (1.2) *and* (1.8) *hold. Then for any solution* $x(t)$ *of* (1.1) *with an initial function* $\Phi(t)$, *the bound*

$$\|x\|_L \leq m_0[1 - \Gamma(\tilde{A})(V_2(\alpha^2(\tilde{A}) + \omega^2(\tilde{A}))^{1/2} + q)]^{-1}\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}$$
$$(m_0 = \text{const.})$$

is valid.

Proof of Theorem 1.2 From (1.1) it follows the inequality

$$\|\dot{x}\|_L \leq \sum_{k=0}^{m} \|A_k\|_{C^n}\|x(t-h_k)\|_L + \|F\|_L$$
$$\leq V_1(\|x\|_L + \|x\|_{L^2([-\eta,0],\mathbf{C}^n)}) + \|F\|_L,$$

where

$$V_1 = \sum_{k=0}^{m} \|A_k\|_{C^n}.$$

Due to (1.2) and (1.6) this gives

$$\|\dot{x}\|_L \leq (q + V_1)(\|x\|_L + \|\Phi\|_{L^2([-\eta,0],\mathbf{C}^n)}).$$

Now Corollary 2.4 implies the relation

$$\|\dot{x}\|_L \leq N_1\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \quad (N_1 = \text{const.}).$$

By the trivial Lemma 8.4.5 of [8], we get

$$\|x\|_{C([0,\infty],\mathbf{C}^n)} \leq (2\|x\|_L\|\dot{x}\|_L)^{1/2} \leq N_2\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \quad (N_2 = \text{const.}). \tag{2.10}$$

Further, substitute in (1.1) the relation

$$x(t) = x_\epsilon(t)e^{-\epsilon t} \tag{2.11}$$

with a small enough $\epsilon > 0$. Applying our reasoning above, we can assert according to (2.10) that

$$\|x_\epsilon\|_{C([-\eta,\infty],\mathbf{C}^n)} \le N_\epsilon \|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \quad (N_\epsilon = \text{const.}).$$

Hence, (2.11) yields the absolute exponential stability.

3 EXAMPLE

Let us consider in $\mathbf{C}^n$ the equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\eta) + f\left(t, x(t), \int_0^\eta T(t,s) x(t-s)\,\mathrm{d}s\right), \quad (3.1)$$

where the function $f\colon [0,\infty) \times \mathbf{C}^{2n} \to \mathbf{C}^n$ has the property

$$\|f(t,h,w)\|_{\mathbf{C}^n} \le q_0 \|h\|_{C^n} + q_1 \|w\|_{C^n} \quad (h, w \in \mathbf{C}^n;\, t \ge 0), \quad (3.2)$$

and the matrix-valued function $T(t,s)$ defined on $[0,\infty) \times [0,\eta]$ satisfies the inequality

$$m_T \equiv \sup_{t\ge 0} \int_0^\eta \|T(t,s)\|_{C^n}\,\mathrm{d}s < \infty. \quad (3.3)$$

In the considered case

$$F(t, x_t) = f\left(t, x(t), \int_0^\eta T(t,s) x(t-s)\,\mathrm{d}s\right).$$

Taking into account, that

$$\int_0^\infty \left\| \int_0^\eta T(t,s) x(t-s)\,\mathrm{d}s \right\|_{C^n}^2 \mathrm{d}t$$

$$\le \int_0^\infty \left(\int_0^\eta \|T(t,s)\|_{C^n} \|x(t-s)\|_{C^n}\,\mathrm{d}s \right)^2 \mathrm{d}t$$

$$\le \sup_{t\ge 0} \left(\int_0^\eta \|T(t,s)\|_{C^n}\,\mathrm{d}s \right)^2 \int_{-\eta}^\infty \|x(t)\|_{C^n}^2\,\mathrm{d}t = m_T^2 \|x\|_{L^2([-\eta,0],\mathbf{C}^n)}^2,$$

we easily get the inequality

$$\left[\int_0^\infty \left\| f\left(t, u(t), \int_0^\eta T(t,s)u(t-s)\,\mathrm{d}s\right)\right\|_{C^n}^2 \mathrm{d}t\right]^{1/2} \le q_0\|u\|_{L^2([0,\infty];\mathbf{C}^n)} + q_1 m_T\|u\|_{L^2([-\eta,\infty],\mathbf{C}^n)}$$

for any $u \in L^2([-\eta,\infty),\mathbf{C}^n) \cap C([-\eta,\infty),\mathbf{C}^n)$. Hence it follows that (1.2) holds with

$$F(t,u_t) = f\left(t, u(t), \int_0^\eta T(t,s)u(t-s)\,\mathrm{d}s\right) \text{ and } q = q_0 + q_1 m_T. \quad (3.4)$$

Clearly, under consideration

$$\tilde{A} = A_0 + A_1, \quad V_2 = \eta\|A_1\|_{C^n}. \quad (3.5)$$

Now Theorem 1.2 yields the following result. Let $\tilde{A}$, q and V_2 be defined by (3.4) and (3.5). Let condition (1.8) hold. Then the zero solution of Eq. (3.1) is absolutely exponentially stable in the class of nonlinearities satisfying conditions (3.2) and (3.3).

4 CONCLUDING REMARKS

We have derived the sufficient absolute stability conditions for Eq. (1.1) in the class of nonlinearities (1.2) in terms of the eigenvalues of the matrix $\tilde{A}$. These conditions are exact according to Remark 1.4. As it was mentioned, the stability conditions suggested in [6,7] can be applied to systems with constant discrete delays only, while Theorem 1.2 allows us to investigate systems containing time-variant and distributed delays (see relations (1.3) and (3.4) above). Consider now the equation

$$\dot{x} = A_0 x(t) + F(x(t-h)), \quad (4.1)$$

where A_0 is a Hurwitz matrix, $F\colon \mathbf{C}^n \to \mathbf{C}^n$ satisfies the inequality

$$\|F(z)\|_{C^n} \le L\|z\|_{C^n} \quad (L = \text{const.}, z \in \mathbf{C}^n).$$

In the paper [10] among other very interesting results the following stability condition for Eq. (4.1) is established:

$$\Lambda(A_0) + L < 0, \tag{4.2}$$

where

$$\Lambda(A_0) = \lim_{h \to 0+} (\|I + hA_0\| - 1)/h$$

with some norm $\|\cdot\|$ in $\mathbf{C}^n$. But if A_0 is not dissipative with respect to that norm, condition (4.2) cannot be applied. At that same time, as the example shows, Theorem 1.2 gives the stability conditions for (4.1) if L is sufficiently small.

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