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Optimal Hysteretic Control for a *BMAP/SM/1/N* Queue with Two Operation Modes

ALEXANDER N. DUDIN [a,*] and SHOICHI NISHIMURA [b]

[a] *Laboratory of Applied Probabilistic Analysis, Faculty of Applied Mathematics and Computer Sciences, Belarus State University, 4, F. Skorina Ave., Minsk-50, 220050, Belarus;* [b] *Department of Applied Mathematics, Faculty of Science, Science University of Tokyo, Kagurazaka, Tokyo 162, Japan*

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We consider *BMAP/SM/*1 type queueing system with finite buffer of size *N*. The system has two operation modes, which are characterized by the matrix generating function of *BMAP*-input, the kernel of the semi-Markovian service process, and utilization cost. An algorithm for determining the optimal hysteresis strategy is presented.

Keywords: Batch Markovian arrival process; Semi-Markovian service; Optimal hysteresis control

AMS (MOS) Subject Classification: 60K25

1 INTRODUCTION

The single-server queues with controllable modes of operation have many promising applications in modern broadband integrated services digital networks. Since there are many different types of data with different requirements to Quality of Service and different economical values, it is necessary to organize a dynamic schedule of information transmission. Controllable queueing models provide appropriate tools

* Corresponding author. Fax: 375 0172 265548. E-mail: Dudin@fpm.bsu.unibel.by.

for optimizing such dynamic schedules. One of the most interesting queues in this category is a system with two controllable modes of customer processing. One mode is cheaper but slower than the other one. When the quality of the customers being processed is evaluated by some economic criteria, which includes holding, service, and switching costs, the problem of switching the modes in dependence on the current value of queue length is very interesting and complicated. The papers, which are devoted to investigation of such queues, can be divided into two groups. In the first group, the authors deal with the qualitative analysis. As a result, they prove that the optimal strategy possesses some good properties like monotonicity. The number of alternative control designs can be narrowed significantly. Unfortunately, these results have two essential disadvantages. The first one is that until now such results are obtained only for simplest queues under different technical assumptions. The second disadvantage is that these papers do not contain algorithms for synthesis of optimal strategies.

In contrast, the papers from the second group include some class of parametric strategies to be used for control. The algorithms for determination of optimal sets of parameters are elaborated upon.

Our present paper belongs to the second direction in investigation of queues with controllable operation mode. Nevertheless, it is not proved that the optimal homogeneous Markovian strategy belongs to the class of hysteresis strategies; we specify such a class as an appropriate class of potentially optimal strategies and present an algorithm for determining the optimal values for parameters of hysteresis strategy.

There is a rather long history of investigating the analogous systems beginning from Suzuki and Ebe [16]. We mention only several recent papers. Nobel and Tijms [15] investigated an $M^X/G/1$ prototype of our model by using the theory of Markov decision processes. Nobel [14] has exploited a regenerative approach to this model. Nishimura and Jiang [13] investigated an $M/G/1$ prototype by direct approach. Dudin [4] investigated an $M^X/G/1$ prototype also by direct approach, but with exploiting the so-called "Principle of Disregarding." Dudin and Nishimura [5] extended the last result to a $BMAP/G/1$ type system. $BMAP$-inputs were introduced by Lucantoni [9]. Such inputs adequately describe real flows of information in modern telecommunication networks and it motivated paper [5].

The motivation of our present paper is the following. Recently, Lucantoni and Neuts [10] introduced and studied a $BMAP/SM/1$ queue. Machihara [12] obtained the results for an analogous system allowing vacations in the idle state. The finite-buffer variant of this system ($BMAP/SM/1/N$) is not investigated yet. From the past literature on $MAP/G/1/N$ systems (see e.g. Baiocchi [1], Baiocchi and Blefari-Melazzi [2]) and on $BMAP/G/1/N$ systems (see Blondia [3] and Gouweleeuw [6]), the extension of results for the systems with MAP-input and infinite queue to the case of a finite queue is rather nontrivial. In our present paper, we derive results concerning the embedded queueing process for the system $BMAP/SM/1/N$ by exploiting the Principle of Disregarding. It seems that our algorithm in the special case of $BMAP/G/1/N$ system is preferable to the algorithm of Blondia [3]. And our second achievement is that we extended our approach to a controlled variant of the $BMAP/SM/1/N$ system. These results have no analog even for much simpler $M/G/1/N$ controlled systems.

The rest of this paper is organized as follows. In Section 2, we describe our model. In Section 3, we find the stationary distribution for an uncontrolled $BMAP/SM/1/N$ system. In Section 4, we derive the stationary distribution for a $BMAP/SM/1/N$ system with a threshold strategy of control and zero switching time. In Section 5, we consider the general model and derive the stationary distribution of the embedded queueing process for any fixed value of thresholds. In Section 6 we give an expression for the objective function in threshold values. Section 7 contains concluding remarks.

2 THE MODEL

Let us consider a single-server queue with a waiting space of capacity N. Two different modes for processing customers are available. There is an opportunity to switch the mode of operation upon the service completions.

The rth mode of customer processing is described as follows. The input to the system is a $BMAP$ (Batch Markovian Arrival Process). This input is controlled by continuous-time Markov chain ν_t, which is called *a directing process of the BMAP*. The state space of ν_t is

$\{0, 1, \ldots, W\}$. The transitions of process ν_t and arrivals of customers are performed according to a matrix generating function $D^{(r)}(z) = \sum_{k=0}^{\infty} D_k^{(r)} z^k$, $|z| < 1$. A more detailed description of the *BMAP* is given in Lucantoni [9]. Note that we make the same assumptions about functions $D^{(r)}(z)$ as Lucantoni [9] does. Denote by $\vec{\theta}^{(r)}$ the stationary probability row vector of the Markov chain ν_t. It is defined by equations

$$\vec{\theta}^{(r)} D^{(r)}(1) = \vec{0}_{W+1}, \quad \vec{\theta}^{(r)} \mathbf{1}_{W+1} = 1.$$

Here $\vec{0}_{W+1}$ is a null vector of size $W+1$ and $\mathbf{1}_{W+1} = \{1, \ldots, 1\}^T$.

The intensity $\lambda^{(r)}$ of *BMAP*-input (the fundamental rate) is calculated as

$$\lambda^{(r)} = \vec{\theta}^{(r)} \left. \frac{\mathrm{d}D^{(r)}(z)}{\mathrm{d}z} \right|_{z=1} \mathbf{1}_{W+1}.$$

The service of customers is governed by a semi-Markovian process m_t. We refer to it as to a directing process of service. The state space of this process is $\{1, \ldots, M\}$. The sojourn time of m_t in a state m under the condition that the next state is m' is defined by conditional distribution function $B_{m,m'}^{(r)}(t)$, $m, m' = \overline{1, M}$. Denote by $B^{(r)}(t) = (B_{m,m'}^{(r)}(t))_{m,m' \in \{1,\ldots,M\}}$ a matrix of these conditional probabilities. This matrix is called the semi-Markov kernel. Denote by $P^{(r)} = B^{(r)}(\infty) = (p_{m,m'}^{(r)})_{m,m' \in \{1,\ldots,M\}}$ the transition probability matrix of the embedded Markov chain for process m_t.

Let $\vec{\pi}^{(r)}$ be the invariant probability measure of $P^{(r)}$ and let $\alpha^{(r)}$ be the following column vector:

$$\alpha^{(r)} = \left(\sum_{m=1}^{M} \int_0^{\infty} \left(p_{1,m}^{(r)} - B_{1,m}^{(r)}(t) \right) \mathrm{d}t, \ldots, \sum_{m=1}^{M} \int_0^{\infty} \left(p_{M,m}^{(r)} - B_{M,m}^{(r)}(t) \right) \mathrm{d}t \right)^T.$$

The average service time $b_1^{(r)}$ in the rth mode is defined as

$$b_1^{(r)} = \vec{\pi}^{(r)} \alpha^{(r)}.$$

We make the same assumptions about the matrix $B^{(r)}(t)$ as those in Lucantoni and Neuts [10].

The switching of operation mode $\#r$ to operation mode $\#r'$ is performed during the time which has a distribution function $G_r(t)$. Let $g_r(s)$ be its Laplace–Stieltjes transform and let the two first initial moments of distribution be finite. During the switching time, the service of customers is suspended and the directing process of service remains in its current state. The input to the system is continuing. For the sake of generality, we suppose that directing process ν_t of the $BMAP$ has matrix generating function $D^{(r,r')}(z)$, $r' \neq r$, during this switching time. The switching of modes does not cause any additional transitions of processes ν_t and m_t.

If we let $N = \infty$ and consider only one operation mode, the model described above coincides with the model of Lucantoni and Neuts [10].

Because we consider the model with a finite buffer and batch arrivals, it is necessary to have a convention concerning the case when the size of an arriving group exceeds the number of free waiting places in the buffer. We assume that a part of this group, which corresponds to the number of free places, is admitted to the system. The rest of the group is lost by the system.

We assume that the quality of the system operation is evaluated by the following cost criteria:

$$I = aL + c_1P_1 + c_2P_2 + dS + cR. \tag{1}$$

Here L is the mean queue length upon a service completion in the steady state; P_r is the probability of processing a customer in the rth mode, $r = 1, 2$; S is the probability of switching the mode at an arbitrary service completion epoch; R is the probability that the buffer is full at the service completion epoch; a, c_1, c_2, d, and c are the corresponding cost coefficients.

Note that it was preferable to include the loss probability of the system in cost criteria (1). But the problem of calculating such a probability for the systems with batch arrivals is not solved even for much simpler systems than ours. Only some approximate expressions are derived (see e.g., Gouweleeuw [6,7], Tijms [17]), and recently Kofman and Korezlioglu [8] derived this probability for the $BMAP/M/1$ queue by using techniques of marked point processes. So, we include into the cost criteria the value R, which characterizes the losses indirectly.

We make the following assumptions about the cost coefficients and traffic intensities:

$$a > 0, \quad d \geq 0, \quad c > 0, \quad c_1 < c_2, \quad \rho_1 > \rho_2,$$

where $\rho_r = \lambda^{(r)} b_1^{(r)}$ is the traffic intensity (load) in the rth mode, $r = 1, 2$.

So the first mode is cheaper but slower. The second one is faster but more expensive. Because the system is charged also for customers waiting in a queue and losses of customers, the problem of optimal switching the modes is not trivial.

Due to a given relation of the service costs and traffic intensities, it seems natural to exploit a so-called threshold strategy for the modes control. This strategy is determined as follows. An integer-valued threshold j is fixed, $j \geq 0$. If a queue length is i at a given service completion epoch and it does not exceed j, the first mode is selected for servicing the next customer. Otherwise, the second mode will be used.

Really, in case $d = 0$ and instantaneous switching of operation modes the optimality of threshold strategies in the class of homogeneous Markovian strategies was proved for several kinds of queueing models.

When $d > 0$ and (or) the system wastes time for the switchings, the threshold strategy turns out to be not optimal. The threshold strategy causes frequent switching of the modes. In this case, when switchings are charged, the strategy is expensive. So it seems natural to consider other strategies, which react by increasing the service rate if the queue length increases but do not cause frequent switchings. Such strategies are called hysteresis strategies. Hysteresis strategy is determined as follows. Two integer-valued thresholds j and k are fixed, $0 \leq j \leq k$. Let the queue length i at a given service completion epoch be less than or equal to j. Then the system will exploit the first mode of operation. If $i > k$, then the system exploits the second mode. And if $j < i \leq k$, the system retains a current mode of operation.

There are only a few papers where the optimality of hysteresis strategies is proved. And it was done only for very basic queueing models. But in situations, where only the information about the current value of the queue length and the current processing mode is available, the class of hysteresis strategies seems to be very appropriate for control of operation modes. Consequently, we exploit the class of hysteresis strategies in this paper. We use a so-called direct approach for determining the optimal hysteresis strategy. This approach is as follows. Fix the parameters (j, k) of hysteresis strategy. Investigate the Markov chain describing the system behavior. Calculate the stationary state probabilities of this chain. Using these probabilities, calculate the value of cost criteria under the fixed values of thresholds. So, the problem of determining the optimal hysteresis strategy is reduced to a problem of minimizing the known function of two integer-valued variables. This problem is solvable easily.

3 UNCONTROLLED VARIANT OF THE MODEL

To introduce the necessary notions and to illustrate the Principle of Disregarding, consider the model which exploits only one rth mode. We can omit index r, but we prefer to keep it when using the corresponding notations in the upcoming sections. However, this model is of independent value.

Let t_n be the nth service completion epoch in the system. Introduce the following three-dimensional stochastic process:

$$\{i_n, \nu_n, m_n\}, \quad n \geq 1,$$

where i_n is a queue length at the epoch $t_n + 0$, $i_n \geq 0$, ν_n is the state of process ν_t at the epoch t_n, $\nu_n = \overline{0, W}$, m_n is the state of process m_t at the epoch $t_n - 0$, $m_n = \overline{1, M}$.

This process is a Markov chain. We enumerate its states in lexicographic order. Denote by $P\{(i, \nu, m) \to (l, \nu', m')\}$ one-step transition probabilities:

$$P\{(i, \nu, m) \to (l, \nu', m')\}$$
$$= P\{i_{n+1} = l,\ \nu_{n+1} = \nu',\ m_{n+1} = m' \mid i_n = i,\ \nu_n = \nu, m_n = m\}, \quad n \geq 1.$$

These transition probabilities form matrix $W^{(r)}_{l-i+1}$, if $i > 0$, $l \geq i - 1$. The matrices $W^{(r)}_{l-i+1}$ can be determined from the following matrix expansion:

$$\sum_{n=0}^{\infty} W_n^{(r)} z^n = \int_0^{\infty} \mathrm{e}^{D^{(r)}(z)t} \otimes \mathrm{d}B^{(r)}(t) = \hat{\beta}_r(z), \tag{2}$$

where $\otimes$ stands for Kronecker product.

The algorithms for calculating these matrices are based on the approach of Lucantoni and Ramaswami [11].

Introduce also the matrices $\Psi_i^{(r)}$ with matrix generating function $\Psi_r(z)$. The entry of the matrix $\Psi_i^{(r)}$ has the following probabilistic sense. It is a probability of presenting exactly i customers in the corresponding system with unlimited waiting space upon beginning of a busy period and corresponding transition of processes ν_t, m_t during the idle state.

It is easy to see that

$$\Psi_i^{(r)} = ((-D_0^{(r)})^{-1} D_i^{(r)}) \otimes E_M, \quad i \geq 1,$$

where E_M stands for the identity matrix of size M.

The transition probabilities $P\{(0, \nu, m) \rightarrow (l, \nu', m')\}$ form the matrix

$$\sum_{i=1}^{l+1} \Psi_i^{(r)} W_{l-i+1}^{(r)}, \quad l \geq 0.$$

So, we defined the transition probabilities of Markov chain $\{i_n, \nu_n, m_n\}$ completely.

Introduce the following stationary state probabilities:

$$\pi(i, \nu, m) = \lim_{n \to \infty} P\{i_n = i, \nu_n = \nu, m_n = m\},$$
$$i \geq 0, \quad \nu = \overline{0, W}, \quad m = \overline{1, M}. \tag{3}$$

Due to finiteness of the queue and corresponding assumptions about the processes ν_t, m_t in [9,10], the limits (3) exist for any given set of input and service rates.

Denote also

$$\vec{\pi}(i, \nu) = (\pi(i, \nu, 1), \ldots, \pi(i, \nu, M)),$$
$$\vec{\pi}_i = (\vec{\pi}(i, 0), \ldots, \vec{\pi}(i, W)).$$

THEOREM 1 *The stationary state distribution vectors* $\vec{\pi}_l$, $l=\overline{0,N}$, *are defined as follows:*

$$\vec{\pi}_l = \vec{\pi}_0 A_l^{(r)}, \quad l=\overline{0,N}, \tag{4}$$

where the matrices $A_l^{(r)}$ *are calculated by the following recursive formulas:*

$$A_0^{(r)} = E,$$

$$A_{l+1}^{(r)} = \left(A_l^{(r)} - \sum_{i=1}^{l+1} \Psi_i^{(r)} W_{l-i+1}^{(r)} - \sum_{i=1}^{l} A_i^{(r)} W_{l-i+1}^{(r)}\right)(W_0^{(r)})^{-1}, \quad l \geq 0. \tag{5}$$

The entries of vector $\vec{\pi}_0$ *are calculated from the following system of linear algebraic equations:*

$$\vec{\pi}_0 \left\{ -A_N^{(r)} - \sum_{i=1}^{N} \Psi_i^{(r)} \sum_{m=0}^{N-i} W_m^{(r)} + \Psi_r(1)\hat{\beta}_r(1) + \sum_{i=1}^{N} A_i^{(r)} \left(\hat{\beta}_r(1) - \sum_{m=0}^{N-i} W_m\right)\right\} = \vec{0}_{(W+1)M}, \tag{6}$$

$$\vec{\pi}_0 \sum_{l=0}^{N} A_l^{(r)} \mathbf{1} = 1. \tag{7}$$

Here E is the identity matrix of size $(W+1)M$ and $\mathbf{1}$ is the unit column vector of this size.

Proof Having known one-step transition probabilities, it is very easy to derive the following system of equations for unknown vectors $\vec{\pi}_l$, $l=\overline{0,N}$:

$$\vec{\pi}_l = \vec{\pi}_0 \sum_{i=1}^{l+1} \Psi_i^{(r)} W_{l-i+1}^{(r)} + \sum_{i=1}^{l+1} \vec{\pi}_i W_{l-i+1}^{(r)}, \quad l=\overline{0,N-1}, \tag{8}$$

$$\vec{\pi}_N = \vec{\pi}_0 \sum_{i=1}^{N} \Psi_i^{(r)} \sum_{m=N+1-i}^{\infty} W_m + \vec{\pi}_0 \sum_{i=N+1}^{\infty} \Psi_i^{(r)} \hat{\beta}_r(1) + \sum_{i=1}^{N} \vec{\pi}_i \sum_{m=N+1-i}^{\infty} W_m. \tag{9}$$

Solve system (8), (9). The key point of our solution is the following. It is evident that Eq. (8) coincides with the analogous equations for the system with unlimited waiting space. So, because the probabilities $\vec{\pi}_l$, $l = 1, 2, \ldots$ are calculated step by step from (8) up to the value of unknown vector $\vec{\pi}_0$ in the form

$$\vec{\pi}_l = \vec{\pi}_0 A_l^{(r)}, \quad l = 1, 2, \ldots, \tag{10}$$

and only l first equations of system (8) are exploited for calculating the matrices $A_l^{(r)}$, then the matrices $A_l^{(r)}$ coincide with the corresponding matrices in representation (10) for the system with unlimited buffer.

It is known, that such matrices can be calculated from the following matrix expansion:

$$\sum_{l=0}^{\infty} A_l^{(r)} z^l = (E - \Psi^{(r)}(z))\hat{\beta}_r(z)(\hat{\beta}_r(z) - Ez)^{-1},$$

or from recurrent relations (5).

Recurrent relations like (5) are criticized for catastrophic cancellation (see Lucantoni [9]). The recursive scheme (49) from Lucantoni [9] can be exploited for calculating the matrices $A_l^{(r)}$ instead of (5). It is inessential for us. The essential thing is that these matrices coincide with corresponding matrices for the system with an infinite buffer and the problem of their calculation is already solved.

Having expression (4) for vectors $\vec{\pi}_l$, $l = \overline{0, N}$, we substitute them into (9) and derive Eq. (6) for the vector $\vec{\pi}_0$. Because (6) is just system (8)–(9) after implementing the elimination of unknowns, the rank of this system for a vector $\vec{\pi}_0$ is equal to W. So, we replace one equation of system (6) by normalization condition (7) and solve system (6)–(7). The system has an unique solution $\vec{\pi}_0$. By substituting this solution into (4), we finish the calculations.

Note that a critical point of our approach is an opportunity of presentation of vectors $\vec{\pi}_l$ in form (4). It is evident from (5), that such presentation is only possible if the matrix $W_0^{(r)}$ is nonsingular. The question of singularity of this matrix is common in papers concerning the systems with the $BMAP$-input. For our model, we proved the following sufficient condition for nonsingularity of matrix $W_0^{(r)}$.

LEMMA 1 *If the following conditions are fulfilled:*

(1) *the entries of a kernel* $B^{(r)}(t)$ *have form*

$$B^{(r)}_{m,m'}(t) = B^{(r)}_m(t)P^{(r)}_{m,m'}, \quad m, m' = \overline{1, M}, \tag{11}$$

i.e. the sojourn time of the process m_t *in the state* m *does not depend on the future state*;

(2) *matrix* $P^{(r)}$ *is nonsingular*;

(3) *distributions* $B^{(r)}_m(t)$ *belong to the class of infinitely divisible distributions*,

then matrix $W^{(r)}_0$ *is nonsingular.*

Note that condition (11) is rather usual in practical systems and that the class of infinitely divisible distributions is rather broad. For example, the Erlangian and degenerate distributions, which are widely used in the queueing theory, belong to this class. Note also that condition (3) can be omitted if all eigenvalues of matrix $D^{(r)}_0$ are real.

Although our idea of calculating the probabilities $\vec{\pi}_l$ in form (4) with consequent solving of system (6), (7) seems to be very transparent and trivial, in the classical paper Blondia [3] this idea is not noticed upon. So a system of $(W+1)\cdot M\cdot(N+1)$ equations should be solved instead of our system (6), (7) of $(W+1)M$ equations.

4 THRESHOLD CONTROL AND INSTANTANEOUS SWITCHING

Suppose now that two operation modes are available and the switching of modes is performed instantaneously. We find an optimal strategy in a class of threshold strategies which was defined above.

Fix a threshold $j, j \geq 0$.

THEOREM 2 *The stationary state distribution vectors* $\vec{\pi}_l$, $l = \overline{0, N}$, *are defined as follows:*

$$\vec{\pi}_l = \vec{\pi}_0 A^{(1)}_l, \quad l = \overline{0, j}, \tag{12}$$

$$\vec{\pi}_l = \vec{\pi}_0 \tilde{A}^{(2)}_l, \quad l = \overline{j+1, N}, \tag{13}$$

where the matrices $\tilde{A}_l^{(2)}$ are calculated by the following recurrent formulas:

$$\tilde{A}_{j+1}^{(2)} = A_{j+1}^{(1)} W_0^{(1)} (W_0^{(2)})^{-1},$$
$$\tilde{A}_l^{(2)} = \left(\tilde{A}_{l-1}^{(2)} - A_{l-1}^{(1)} + \sum_{m=j+1}^{l} A_m^{(1)} W_{l-m}^{(1)} \right. \tag{14}$$
$$\left. - \sum_{m=j+1}^{l-1} \tilde{A}_m^{(2)} W_{l-m}^{(2)} \right) (W_0^{(2)})^{-1}, \quad j+2 \leq l \leq N,$$

where the matrices $A_l^{(1)}$ are defined by Eq. (5).

The entries of vector $\vec{\pi}_0$ are calculated from the following system of linear algebraic equations:

$$\vec{\pi}_0 \left\{ - \tilde{A}_N^{(2)} + \Psi^{(1)}(1)\hat{\beta}_1(1) - \sum_{i=1}^{N} \Psi_i^{(1)} \sum_{m=0}^{N-i} W_m^{(1)} + \sum_{i=1}^{j} A_i^{(1)} \right.$$
$$\left. \times \left(\hat{\beta}_1(1) - \sum_{m=0}^{N-i} W_m^{(1)} \right) + \sum_{i=j+1}^{N} \tilde{A}_i^{(2)} \left(\hat{\beta}_2(1) - \sum_{m=0}^{N-i} W_m^{(2)} \right) \right\} = \vec{0}_{(W+1)M}, \tag{15}$$

$$\vec{\pi}_0 \left\{ \sum_{l=0}^{j} A_l^{(1)} + \sum_{l=j+1}^{N} \tilde{A}_l^{(2)} \right\} \mathbf{1} = 1. \tag{16}$$

Proof The system of equations for vectors $\vec{\pi}_l$, $l = \overline{0, N}$, has here the following form:

$$\vec{\pi}_l = \vec{\pi}_0 \sum_{i=1}^{l+1} \Psi_i^{(1)} W_{l-i+1}^{(1)} + \sum_{i=1}^{l+1} \vec{\pi}_i W_{l-i+1}^{(1)}, \quad l = \overline{0, j-1}, \tag{17}$$

$$\vec{\pi}_l = \vec{\pi}_0 \sum_{i=1}^{l+1} \Psi_i^{(1)} W_{l-i+1}^{(1)} + \sum_{i=1}^{j} \vec{\pi}_i W_{l-i+1}^{(1)} + \sum_{i=j+1}^{l+1} \vec{\pi}_i W_{l-i+1}^{(2)}, \quad l = \overline{j, N-1}, \tag{18}$$

$$\vec{\pi}_N = \vec{\pi}_0 \sum_{i=1}^{N} \Psi_i^{(1)} \sum_{m=N+1-i}^{\infty} W_m^{(1)} + \vec{\pi}_0 \sum_{i=N+1}^{\infty} \Psi_i^{(1)} \hat{\beta}_1(1)$$
$$+ \sum_{i=1}^{j} \vec{\pi}_i \sum_{m=N-i+1}^{\infty} W_m^{(1)} + \sum_{i=j+1}^{N} \vec{\pi}_i \sum_{m=N-i+1}^{\infty} W_m^{(2)}. \tag{19}$$

Solve system (17)–(19). Equation (17) coincides with (8), so we conclude automatically that probabilities $\vec{\pi}_l$, $l=\overline{0,j}$, have form (4) or, equivalently, form (12).

Consider now system (17), (18). Set temporarily $N=\infty$ in (18). Because finite system (17), (18) and a temporarily infinite system have the first N equations coinciding and all probabilities $\vec{\pi}_l$, $l=\overline{1,N}$ are defined by these N equations up to the value $\vec{\pi}_0$, this allows us to find the probabilities $\vec{\pi}_l$, $l=\overline{j+1,N}$. Such a way of neglecting and temporarily changing the tails of systems of equations was called the Principle of Disregarding; see Dudin [4] and Dudin and Nishimura [5].

We set $N=\infty$ and introduce partial generating functions

$$\vec{\Pi}_1(z)=\sum_{i=0}^{j}\vec{\pi}_i z^i, \quad \hat{\Pi}_2(z)=\sum_{i=j+1}^{\infty}\vec{\pi}_i z^i.$$

Multiplying Eqs. (17), (18) by the corresponding degrees of z and summing them up, we derive the equation

$$\hat{\Pi}_2(z)(Ez-\hat{\beta}_2(z))=\vec{\Pi}_1(z)(\hat{\beta}_1(z)-Ez)+\vec{\pi}_0(\Psi(z)-E)\hat{\beta}_1(z). \quad (20)$$

Because we introduced the generating function $\hat{\Pi}_2(z)$ only temporarily and we need only coefficients near z^l, $l=\overline{j,N}$ in its expansion, we remind of the matrix representation of $\sum_{i=0}^{\infty}A_i^{(1)}z^i$ given above and rewrite (20) into the form:

$$\sum_{i=j+1}^{\infty}\vec{\pi}_i z^i\left(\sum_{n=0}^{\infty}W_n^{(2)}z^n-Ez\right)=\vec{\pi}_0\sum_{i=j+1}^{\infty}A_i^{(1)}z^i\left(\sum_{n=0}^{\infty}W_n^{(1)}z^n-Ez\right). \quad (21)$$

Comparing the coefficients at equal degrees of z in (21), we conclude that probabilities $\vec{\pi}_l$, $l=\overline{j+1,N}$, are really defined by formula (13), where the matrices $\tilde{A}_l^{(2)}$ are calculated by means of recursion (14).

Because the values of all vectors $\vec{\pi}_l$, $l=\overline{0,N}$, are already derived up to the value of $\vec{\pi}_0$, we substitute (12) and (13) into (19) and derive (15). Equation (16) is just a consequence of normalization condition.

5 STATIONARY STATE PROBABILITIES FOR A GENERAL FORMULATION OF THE PROBLEM

Finally, consider the problem of optimal hysteresis control. Let the thresholds (j, k), $0<j\le k<N$, be fixed. When a hysteresis strategy is exploited, the process $\{i_n, \nu_n, m_n\}$, which was introduced in Section 3, is not Markovian. But the process $\{i_n, \nu_n, m_n, r_n\}$, where r_n is the number of the mode, which was exploited at the epoch t_n-0, is a Markov chain.

Denote by $P\{(i,\nu,m,r)\to(l,\nu',m',r')\}=P\{i_{n+1}=l,\quad \nu_{n+1}=\nu', m_{n+1}=m',\ r_{n+1}=r' \mid i_n=i,\ \nu_n=\nu,\ m_n=m,\ r_n=r\}$ one-step transition probabilities of this Markov chain.

LEMMA 2 *Transition probabilities $P\{(i,\nu,m,r)\to(l,\nu',m',r')\}$ are defined as follows:*

- *probabilities $P\{(i,\nu,m,1)\to(l,\nu',m',1)\}$ form matrix*

$$W^{(1)}_{l-i+1},\quad 0<i\le k,\ i-1\le l<N;$$

- *probabilities $P\{(i,\nu,m,1)\to(N,\nu',m',1)\}$ form matrix*

$$\sum_{n=N-i+1}^{\infty} W^{(1)}_n,\quad 0<i\le k;$$

- *probabilities $P\{(i,\nu,m,2)\to(l,\nu',m',2)\}$ form matrix*

$$W^{(2)}_{l-i+1},\quad j<i\le N,\ i-1\le l<N;$$

- *probabilities $P\{(i,\nu,m,2)\to(N,\nu',m',2)\}$ form matrix*

$$\sum_{n=N-i+1}^{\infty} W^{(2)}_n,\quad j<i\le N;$$

- *probabilities $P\{(0,\nu,m,1)\to(l,\nu',m',1)\}$ form matrix*

$$\sum_{n=1}^{l+1} \Psi^{(1)}_n W^{(1)}_{l-n+1},\quad 0\le l<N;$$

- *probabilities* $P\{(0,\nu,m,1)\rightarrow(N,\nu',m',1)\}$ *form matrix*

$$\sum_{i=1}^{N}\Psi_i^{(1)}\sum_{n=N-i+1}^{\infty}W_n^{(1)}+\sum_{i=N+1}^{\infty}\Psi_i^{(1)}\hat{\beta}_1(1);$$

- *probabilities* $P\{(i,\nu,m,1)\rightarrow(l,\nu',m',2)\}$ *form matrix*

$$\sum_{n=0}^{l-i+1}\Gamma_n^{(1)}W_{l-i+1-n}^{(2)},\quad N\geq i\geq k+1,\ i-1\leq l<N,$$

where the matrices $\Gamma_n^{(r)}$ *are defined from the following matrix expansion:*

$$\sum_{n=0}^{\infty}\Gamma_n^{(r)}z^n=\int_0^{\infty}\mathrm{e}^{(D^{(r,r')}(z))t}\,\mathrm{d}G_r(t)\otimes E_M;\tag{22}$$

- *probabilities* $P\{(i,\nu,m,1)\rightarrow(N,\nu',m',2)\}$ *form matrix*

$$\sum_{n=0}^{N-i}\Gamma_n^{(1)}\sum_{l=N-i-n+1}^{\infty}W_l^{(2)}+\sum_{n=N-i+1}^{\infty}\Gamma_n^{(1)}\hat{\beta}_2(1),\quad k<i\leq N;$$

- *probabilities* $P\{(j,\nu,m,2)\rightarrow(l,\nu,m,1)\}$ *form matrix*

$$\sum_{n=0}^{l-j+1}\Gamma_n^{(2)}W_{l+1-n-j}^{(1)},\quad \text{if } j>0,\ j-1\leq l<N,$$

or matrix

$$\Gamma_0^{(2)}\sum_{n=1}^{l+1}\Psi_n^{(1)}W_{l-n+1}^{(1)}+\sum_{n=1}^{l+1}\Gamma_n^{(2)}W_{l-n+1}^{(1)},\quad \text{if } j=0,\ 0\leq l<N;$$

- *probabilities* $P\{(j,\nu,m,2)\rightarrow(N,\nu,m,1)\}$ *form matrix*

$$\sum_{n=0}^{N-j}\Gamma_n^{(2)}\sum_{l=N-j-n+1}^{\infty}W_l^{(1)}+\sum_{n=N-j+1}^{\infty}\Gamma_n^{(2)}\hat{\beta}_1(1),\quad \text{if } j>0,$$

and

$$\Gamma_0^{(2)}\left(\sum_{i=1}^{N}\Psi_i^{(1)}\sum_{n=N-i+1}^{\infty}W_n^{(1)}+\sum_{i=N+1}^{\infty}\Psi_i^{(1)}\hat{\beta}_1(1)\right)$$
$$+\sum_{n=1}^{N}\Gamma_n^{(2)}\sum_{l=N-n+1}^{\infty}W_l^{(1)}+\sum_{n=N+1}^{\infty}\Gamma_n^{(2)}\hat{\beta}_1(1),\quad if\, j=0.$$

Lemma 2 is proved by analyzing the corresponding transitions of the Markov chain.

Denote by

$$\pi(i,\nu,m)=\lim_{n\to\infty}P\{i_n=i,\ \nu_n=\nu,\ m_n=m,\ r_n=1\},$$
$$\chi(i,\nu,m)=\lim_{n\to\infty}P\{i_n=i,\ \nu_n=\nu,\ m_n=m,\ r_n=2\},$$

the stationary distribution of the Markov chain $\{i_n,\nu_n,m_n,r_n\}$, $n\geq 1$.

Introduce vectors

$$\vec{\pi}(i,\nu)=(\pi(i,\nu,1),\ldots,\pi(i,\nu,M)),$$
$$\vec{\chi}(i,\nu)=(\chi(i,\nu,1),\ldots,\chi(i,\nu,M)),$$
$$\vec{\pi}_i=(\vec{\pi}(i,0),\ldots,\vec{\pi}(i,W)),\quad \vec{\chi}_i=(\vec{\chi}(i,0),\ldots,\vec{\chi}(i,W)).$$

THEOREM 3 *Vectors $\vec{\pi}_i,\vec{\chi}_i$, $i=\overline{0,N}$, of the stationary state distribution are defined as follows:*

$$\vec{\pi}_l=\vec{\pi}_0A_l^{(1)},\quad l=\overline{0,j-1},\tag{23}$$
$$\vec{\pi}_l=\vec{\pi}_0\Theta_l,\quad l=\overline{j,k},\tag{24}$$
$$\vec{\pi}_l=\vec{\pi}_0\left(\Theta_l-\sum_{n=k+1}^{l+1}\Theta_nW_{l-n+1}^{(1)}\right),\quad l=\overline{k+1,N-1},\tag{25}$$
$$\vec{\pi}_N=\vec{\pi}_0\tilde{\Theta}_N,\tag{26}$$
$$\vec{\chi}_l=\vec{\pi}_0\nu(k,j)\tilde{\Omega}_{l-j}^{(2)},\quad l=\overline{j,k},\tag{27}$$
$$\vec{\chi}_l=\vec{\pi}_0\left[\nu(k,j)\tilde{\Omega}_{l-j}^{(2)}-\sum_{i=k+1}^{l}\left(\Theta_i-\sum_{n=k+1}^{i+1}\Theta_nW_{i+1-n}^{(1)}\right)\Omega_{l-i}^{(2)}\right],$$
$$l=\overline{k+1,N-1},\tag{28}$$

$$\vec{\chi}_N = \vec{\pi}_0 \left[v(k,j)\tilde{\Omega}^{(2)}_{N-j} - \sum_{i=k+1}^{N-1} \left(\Theta_i - \sum_{n=k+1}^{i+1} \Theta_n W^{(1)}_{i-n+1} \right) \Omega^{(2)}_{N-i} - \tilde{\Theta}_N \Omega^{(2)}_0 \right], \tag{29}$$

where the matrices $A^{(1)}_l$ *are defined by formula* (5), *matrices* Θ_l *are defined by*

$$\Theta_l = A^{(1)}_l - v(k,j)\Omega^{(1)}_{l-j}, \quad l = \overline{j,N}, \tag{30}$$

$$v(k,j) = A^{(1)}_{k+1}(\Omega_{k-j+1})^{-1}, \tag{31}$$

the matrices $\Omega^{(r)}_l$ *are defined by*

$$\Omega^{(r)}_l = \sum_{i=0}^{l} \Gamma^{(r')}_i \tilde{\Omega}^{(r)}_{l-i}, \quad l \geq 0, \ r' \neq r, \ r, r' = \overline{1,2}, \tag{32}$$

matrices $\tilde{\Omega}^{(r)}_l$ *are calculated by recurrent formulas*

$$\begin{aligned} \tilde{\Omega}^{(r)}_0 &= E, \\ \tilde{\Omega}^{(r)}_l &= A^{(r)}_l + \sum_{i=0}^{l-1} \Psi^{(1)}_{l-i} \tilde{\Omega}^{(r)}_i, \quad l \geq 1, \end{aligned} \tag{33}$$

the matrices $A^{(r)}_l$ *are defined by formula* (5), *and the matrix* $\tilde{\Theta}_N$ *is defined by*

$$\begin{aligned} \tilde{\Theta}_N = {} & \Psi(1)\hat{\beta}_1(1) - \sum_{i=1}^{N} \Psi_i \sum_{n=0}^{N-i} W^{(1)}_n \\ & + \sum_{i=1}^{j-1} A^{(1)}_i \left(\hat{\beta}_1(1) - \sum_{n=0}^{N-i} W^{(1)}_n \right) + \sum_{i=j}^{k} \Theta_i \left(\hat{\beta}_1(1) - \sum_{n=0}^{N-i} W^{(1)}_n \right) \\ & + v(k,j) \left(\hat{g}_2(1)\hat{\beta}_1(1) - \sum_{n=0}^{N-j} \Gamma^{(2)}_n \sum_{l=0}^{N-j-n} W^{(1)}_l \right). \end{aligned} \tag{34}$$

The entries of vector $\vec{\pi}_0$ satisfy the following system:

$$\vec{\pi}_0\left\{v(k,j)\sum_{l=j+1}^{N}\tilde{\Omega}_{l-j}^{(2)}\left(\hat{\beta}_2(1)-\sum_{n=0}^{N-l}W_n^{(2)}\right)-v(k,j)\tilde{\Omega}_{N-j}^{(2)}\right.$$
$$+\sum_{i=k+1}^{N-1}\left(\Theta_i-\sum_{n=k+1}^{i+1}\Theta_n W_{i-n+1}^{(1)}\right)\left((\hat{g}_1(1)-\Gamma_0^{(1)})\hat{\beta}_2(1)-E\right.$$
$$\left.+\sum_{l=i+1}^{N}\Omega_{l-i}^{(2)}\left(\hat{\beta}_2(1)-\sum_{n=0}^{N-l}W_n^{(2)}\right)-\sum_{n=0}^{N-i}\Gamma_n^{(1)}\sum_{l=0}^{N-i-n}W_l^{(2)}\right)$$
$$\left.+\tilde{\Theta}_N((\hat{g}_1(1)-\Gamma_0^{(1)})\hat{\beta}_2(1)+\Gamma_0^{(1)})\right\}=\vec{0}_{(W+1)M}, \tag{35}$$

$$\vec{\pi}_0\left\{\sum_{l=0}^{j-1}A_l^{(1)}+\sum_{l=j}^{k}\Theta_l+\sum_{l=k+1}^{N-1}\left(\Theta_l-\sum_{n=k+1}^{l+1}\Theta_n W_{l-n+1}^{(1)}\right)\right.$$
$$+\tilde{\Theta}_N+v(k,j)\sum_{l=j}^{N}\tilde{\Omega}_{l-j}^{(2)}-\sum_{l=k+1}^{N-1}\sum_{i=k+1}^{l}\left(\Theta_i-\sum_{n=k+1}^{i+1}\Theta_n W_{i+1-n}^{(1)}\right)\Omega_{l-i}^{(2)}$$
$$\left.-\sum_{i=k+1}^{N-1}\left(\Theta_i-\sum_{n=k+1}^{i+1}\Theta_n W_{i-n+1}^{(1)}\right)\Omega_{N-i}^{(2)}-\tilde{\Theta}_N\Omega_0^{(2)}\right\}\mathbf{1}=1. \tag{36}$$

Proof Taking into account Lemma 2, we have the following system of equations in vectors $\vec{\pi}_l$, $\vec{\chi}_l$, $l=\overline{0,N}$:

$$\vec{\pi}_l=\vec{\pi}_0\sum_{i=1}^{l+1}\Psi_i^{(1)}W_{l-i+1}^{(1)}+\sum_{i=1}^{l+1}\vec{\pi}_i W_{l-i+1}^{(1)},\quad l=\overline{0,j-2}, \tag{37}$$

$$\vec{\pi}_l=\vec{\pi}_0\sum_{i=1}^{l+1}\Psi_i^{(1)}W_{l-i+1}^{(1)}+\sum_{i=1}^{l+1}\vec{\pi}_i W_{l-i+1}^{(1)}+\vec{\chi}_j\sum_{n=0}^{l+1-j}\Gamma_n^{(2)}W_{l-n-j+1}^{(1)},$$
$$l=\overline{j-1,k-1}, \tag{38}$$

$$\vec{\pi}_l=\vec{\pi}_0\sum_{i=1}^{l+1}\Psi_i^{(1)}W_{l-i+1}^{(1)}+\sum_{i=1}^{k}\vec{\pi}_i W_{l-i+1}^{(1)}+\vec{\chi}_j\sum_{n=0}^{l+1-j}\Gamma_n^{(2)}W_{l-n-j+1}^{(1)},$$
$$l=\overline{k,N-1}, \tag{39}$$

$$\vec{\pi}_N = \vec{\pi}_0\left(\sum_{i=1}^{N}\Psi_i\sum_{n=N-i+1}^{\infty}W_n^{(1)} + \sum_{i=N+1}^{\infty}\Psi_i\hat{\beta}_1(1)\right) + \sum_{i=1}^{k}\vec{\pi}_i\sum_{n=N-i+1}^{\infty}W_n^{(1)}$$
$$+\vec{\chi}_j\left(\sum_{n=0}^{N-j}\Gamma_n^{(2)}\sum_{u=N-j}^{\infty}W_u^{(1)} + \sum_{n=N-j+1}^{\infty}\Gamma_n^{(2)}\hat{\beta}_1(1)\right), \tag{40}$$

$$\vec{\chi}_l = \sum_{i=j+1}^{l+1}\vec{\chi}_i W_{l-i+1}^{(2)}, \quad l=\overline{j,k-1}, \tag{41}$$

$$\vec{\chi}_l = \sum_{i=j+1}^{l+1}\vec{\chi}_i W_{l-i+1}^{(2)} + \sum_{i=k+1}^{l+1}\vec{\pi}_i\sum_{n=0}^{l+1-i}\Gamma_n^{(1)}W_{l-n-i+1}^{(2)},$$
$$l=\overline{k,N-1}, \tag{42}$$

$$\vec{\chi}_N = \sum_{i=j+1}^{N}\vec{\chi}_i\sum_{n=N-i+1}^{\infty}W_n^{(2)} + \sum_{i=k+1}^{N}\vec{\pi}_i$$
$$\times\left[\sum_{n=0}^{N-i}\Gamma_n^{(1)}\sum_{u=N-i-n+1}^{\infty}W_u^{(2)} + \sum_{n=N-i+1}^{\infty}\Gamma_n^{(1)}\hat{\beta}_2(1)\right]. \tag{43}$$

We solve system (37)–(43) by repeated use of the Principle of Disregarding. Equation (23) follows from Eq. (37). Consider system (37), (38), set temporarily $k=\infty$ and introduce temporary generating function $\hat{\Pi}(z) = \sum_{i=0}^{\infty}\vec{\pi}_i z^i$. Multiplying Eqs. (37) and (38) by the corresponding degrees of z and summing them up, we derive:

$$\hat{\Pi}_2(z) = \vec{\pi}_0(\Psi_1(z) - E)\hat{\beta}_1(z)(\hat{\beta}_1(z) - zE)^{-1}$$
$$-\vec{\chi}_j z^j \hat{g}_2(z)\hat{\beta}_1(z)(\hat{\beta}_1(z) - Ez)^{-1}. \tag{44}$$

Expanding (44) in series, we have

$$\vec{\pi}_l = \vec{\pi}_0 A_l^{(1)} - \vec{\chi}_j\Omega_{l-j}^{(1)}, \quad l\geq j. \tag{45}$$

where matrices $\Omega_{l-j}^{(1)}$ are defined by formulas (32), (33). Because the real tail of system (37), (38) is not infinite, we derived relation (45) only for $l=\overline{j,k}$.

Substituting (23) and (45) into Eq. (39) for $l=k$, after some tedious algebra we derive the relation:

$$\vec{\chi}_j = \vec{\pi}_0 v(k,j), \tag{46}$$

where matrix $v(k,j)$ is defined by formula (31).

Substituting (46) into (45) we prove (24).

Now consider system (37)–(39). Also, temporarily set $N=\infty$ and denote $\vec{\Pi}_1(z) = \sum_{i=0}^{k} \vec{\pi}_i z^i$ and $\hat{\Pi}_2(z) = \sum_{i=k+1}^{\infty} \vec{\pi}_i z^i$. By the standard way we get from (37) to (39) the following relation:

$$\begin{aligned}\hat{\Pi}_2(z) = \vec{\Pi}_1(z)(\hat{\beta}_1(z) - Ez)z^{-1} + \vec{\pi}_0(\Psi_1(z)\\ - E + z^i v(k,j)\hat{g}_2(z))\hat{\beta}_1(z)z^{-1}.\end{aligned} \tag{47}$$

As we already know,

$$\vec{\Pi}_1(z) = \vec{\pi}_0\left(\sum_{i=0}^{j} A_i^{(1)} z^i + \sum_{i=j+1}^{k} \Theta_i z^i\right). \tag{48}$$

Expanding (47) in series, after some calculations, we get formula (25). Formula (26) is derived by substituting (23)–(25) and (46) into (40).

Consider system (41). Set temporarily $k=\infty$ and introduce temporary generating function $\hat{K}(z)$. From (41) we derive that

$$\hat{K}(z) = \vec{\chi}_j z^j \hat{\beta}_2(z)(\hat{\beta}_2(z) - Ez)^{-1}. \tag{49}$$

Expanding (49) in series we prove (27).

Consider now system (41), (42). Set temporarily $N=\infty$ and introduce temporary generating function $\hat{K}(z)$. It follows from (41), (42), that

$$\begin{aligned}\hat{K}(z) = \vec{\pi}_0 v(k,j) z^j \hat{\beta}_2(z)(\hat{\beta}_2(z) - zE)^{-1}\\ - \sum_{i=k+1}^{\infty} \vec{\pi}_i z^i \hat{g}_1(z)\hat{\beta}_2(z)(\hat{\beta}_2(z) - zE)^{-1}.\end{aligned} \tag{50}$$

Expanding (50) in series, we derive (28), (29). Finally, we derive (35) just by substituting (23)–(29) into (43). Equation (36) follows from the normalization condition.

So, we can calculate the stationary state probabilities for Markov chain $\{i_n, \nu_n, m_n, r_n\}$ under any fixed value of (j, k), $0 < j \le k < N$. It can be verified that Theorem 3 is valid for $j = 0$ also, but two modifications should be implemented. Matrix function $\hat{g}_2(z)$ should be replaced by $\hat{g}_2(z) + \Gamma_0^{(2)}(\Psi_1(z) - E)$ and matrices $\Omega_l^{(1)}$ should be replaced by $\Omega_l^{(1)} - \Gamma_0^{(2)} A_l^{(1)}$. The case $k = N$ is trivial.

6 DEPENDENCE OF COST CRITERIA ON THRESHOLDS

Having formulas (23)–(29), (35)–(36), we can calculate a value of the cost criteria for any given set (j, k) of thresholds.

Consequently,

$$L = \left(\sum_{i=0}^{N} i\vec{\pi}_i + \sum_{i=j}^{N} i\vec{\chi}_i\right)\mathbf{1}, \tag{51}$$

$$P_1 = \left(\sum_{i=0}^{N} \vec{\pi}_i\right)\mathbf{1}, \tag{52}$$

$$P_2 = \left(\sum_{i=j}^{N} \vec{\chi}_i\right)\mathbf{1}, \tag{53}$$

$$S = 2\vec{\pi}_0 v(k,j)\mathbf{1}, \tag{54}$$

$$R = (\vec{\pi}_N + \vec{\chi}_N)\mathbf{1}. \tag{55}$$

Substituting (51)–(55) into (1) we have a required dependence. The problem of determining the optimal set (j^*, k^*) of thresholds turned out to be rather trivial.

7 CONCLUDING REMARKS

In this paper, the following results are obtained. The algorithm for calculating the stationary state probabilities of embedded queue length for the system *BMAP/SM/1/N* is obtained. It is simpler than the known algorithm of Blondia [3] for the special case: the system *MAP/G/1/N*. The algorithms for calculating the stationary state probabilities of embedded queue length for the system $BMAP/SM/1/N$ and controlled modes of operation are elaborated for two strategies of control. The first strategy is the threshold combined with the case of instantaneous switching. The second strategy is a more general hysteresis strategy with switching times involved. For the latter model, formulas for calculating the dependence of cost criteria on thresholds are presented. Two important contributions in comparison to known results are made in this paper. At first, a more general service process in comparison to other papers devoted to operation mode control is considered. Real-life systems can be described more adequately by it. And the second accomplishment is that a controlled system with finite buffer is investigated. In some real-life systems, the buffer is rather small and the models with infinite buffer produce not accurate results.

The elaborated algorithms are simpler than those for corresponding systems with an infinite buffer. The problem of calculating the roots of an equation in the unit disc is avoided.

We use notation $\Psi_i^{(r)}$ instead of its explicit form intentionally. By corresponding definitions of these matrices, all our results are easily extendable to systems with different kinds of vacations and breakdowns.

References

[1] A. Baiocchi, Analysis of the loss probability of the $MAP/G/1/K$ queue. Part I: Asymptotic Theory, *Communications in Statistics–Stochastic Models* **10** (1994), 867–893.

[2] A. Baiocchi and N. Blefari-Melezzi, Analysis of the loss probability of the $MAP/G/1/K$ queue. Part 2: Approximations and numerical results, *Communications in Statistics–Stochastic Models* **10** (1994), 895–925.

[3] C. Blondia, The $N/G/1$ finite capacity queue, *Communications in Statistics–Stochastic Models* **5** (1989), 273–294.

[4] A.N. Dudin, Optimal control for an $M^X/G/1$ queue with two operation modes, *Probability in the Engineering and Informational Sciences* **11** (1997), 255–265.

[5] A.N. Dudin and S. Nishimura, Optimal control for a $BMAP/G/1$ queue with two service modes, *Mathematical Problems in Engineering* (to appear).

[6] F.N. Gouweleeuw, Calculating the loss probability in a $BMAP/G/1/N+1$ queue, *Communications in Statistics–Stochastic Models* **12** (1996), 473–492.

[7] F.N. Gouweleeuw, The loss probability in finite-buffer queues with batch arrivals and complete rejection, *Probability in the Engineering and Informational Sciences* **8** (1994), 221–227.

[8] D. Kofman and H. Korezlioglu, Some EATA properties for marked point processes, *Journal of Applied Probability* **32** (1995), 922–929.

[9] D.M. Lucantoni, New results on the single server queue with a batch Markovian arrival process, *Communications in Statistics–Stochastic Models* **7** (1991), 1–46.

[10] D.M. Lucantoni and M.F. Neuts, Some steady-state distributions for the $MAP/SM/1$ queue, *Communications in Statistics–Stochastic Models* **10** (1994), 575–578.

[11] D.M. Lucantoni and V. Ramaswami, Efficient algorithms for solving the non-linear matrix equation arising in phase type queues, *Communications in Statistics–Stochastic Models* **1** (1985), 29–51.

[12] F.A. Machihara, *A G/SM/1* queue with vacations depending on service times, *Communications in Statistics–Stochastic Models* **11** (1995), 671–690.

[13] S. Nishimura and Y. Jiang, An $M/G/1$ vacation model with two service modes, *Probability in the Engineering and Informational Sciences* **9** (1995), 355–374.

[14] R.D. Nobel, A regenerative approach for an $M^X/G/1$ queue with two service modes, *Automatic Control and Computer Sciences* **32** (1998), 3–14.

[15] R.D. Nobel and H. Tijms, Optimal control for a $M^X/G/1$ queue with two service modes, *European Journal of the Operations Research* (submitted).

[16] T. Suzuki and M. Ebe, Decision rules for the queueing system $M/G/1$ with service depending on queue-length, *Memoirs of the Defense Academy of Japan* **7** (1967), 1263–1273.

[17] H. Tijms, Heuristics for finite-buffer queues, *Probability in the Engineering and Informational Sciences* **6** (1992), 277–285.