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Analysis of the Self-similar Solutions of a Generalized Burgers Equation with Nonlinear Damping

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The nonlinear ordinary differential equation resulting from the self-similar reduction of a generalized Burgers equation with nonlinear damping is studied in some detail. Assuming initial conditions at the origin we observe a wide variety of solutions – (positive) single hump, unbounded or those with a finite zero. The existence and non-existence of positive bounded solutions with different types of decay (exponential or algebraic) to zero at infinity for specific parameter ranges are proved.

Keywords: Burger's equations; Initial value problem; Generalized Burger's equation; Self-similar solutions

1. INTRODUCTION

In the present paper we study the self-similar solutions of the generalized Burgers equation, namely

$$u_t + u^\beta u_x + \lambda u^\alpha = \frac{\delta}{2} u_{xx}, \quad -\infty < x < \infty, \ t > 0 \tag{1.1}$$

where $\alpha > 0$, $\beta > 0$, $\lambda \in \mathbb{R}$ and $\delta > 0$ (small) are constants. Here λu^α is the nonlinear damping term. Equation (1.1) reduces to the standard Burgers equation when $\beta = 1$ and $\lambda = 0$.

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Lardner and Arya [5] studied a special case of (1.1), namely

$$u_t = uu_x - \lambda u + \frac{\delta}{2} u_{xx}, \tag{1.2}$$

where $\lambda > 0$ and $\delta > 0$ (small) are constants. The transformation $x \to -x$ changes (1.2) to (1.1) with $\beta = 1$ and $\alpha = 1$. Equation (1.2) with $\lambda = 0$ arises when considering the plane motions of a continuous medium for which the constitutive relation for the stress contains a large linear term proportional to the strain, a small term which is quadratic in the strain and a small dissipative term proportional to the strain rate. The λ-term in (1.2) would arise in such a system if the equation of motion includes a small viscous damping term proportional to the velocity. Lardner and Arya [5] used the method of matched asymptotic expansions to connect the solution inside the shock layer with that outside.

Sachdev, Nair and Tikekar [9] reduced (1.1) to the ODE

$$f'' + 2\eta f' + \frac{4}{\alpha - 1} f - 2^{3/2} \delta^{-1/2} f^{(\alpha-1)/2} f' - 4\lambda f^{\alpha} = 0, \tag{1.3}$$

by a similarity transformation

$$u = t^{1/(1-\alpha)} f(\eta), \quad \eta = \frac{x}{(2\delta t)^{1/2}}, \tag{1.4}$$

provided that $\beta = (\alpha - 1)/2$. By a simple scaling, (1.3) can be changed to

$$g'' + 2\eta g' + \frac{4}{\alpha - 1} g - 2^{3/2} g^{(\alpha-1)/2} g' - 4\lambda g^{\alpha} = 0. \tag{1.5}$$

Note that λ's in (1.3) and (1.5) are different. Equation (1.5) is an important special case which after a simple transformation belongs to a class of nonlinear ordinary differential equations, called Euler–Painlevé equations first introduced by Sachdev and his collaborators in a series of papers [9, 10] and [11]. These equations are much more general than the equation studied by Euler and Painlevé (see Kamke [4], p. 574), which is exactly linearizable; hence the solutions of the former class are referred to as Euler–Painlevé transcendents. It was also brought out by Sachdev [7] that this class

covers a large number of equations enumerated by Kamke [4] (see also the more recent work of Sachdev [8]). One of the purposes of studying this class of ODEs is to characterize the generalized Burgers equations by Euler–Painlevé transcendents in the manner that K-dV type of equations are characterized by Painlevé transcendents (see Ablowitz *et al.* [1,2]). In the present case however we do not have solitary type of solutions. It becomes possible to replace these special solutions by the so called single hump solutions or positive solutions over the whole real line. The study of (1.5) with initial conditions at zero brings out a rich structure of Euler–Painlevé transcendents, as evidenced by Theorems 1–5 and Theorems A, B and C.

Here we study the following initial value problem both analytically and numerically:

$$g'' + 2\eta g' + \frac{4}{\alpha - 1} g - 2^{3/2} g^{(\alpha-1)/2} g' - 4\lambda g^{\alpha} = 0, \quad -\infty < \eta < \infty \tag{1.6}$$

$$g(0) = \nu, \quad g'(0) = 0, \tag{1.7}$$

where $\nu > 0$, is a constant. Even with these simple initial conditions with $g'(0) = 0$ a wide variety of solutions with complicated structure are observed (see Section 2) (see Peletier and Serafini [6] for related problems). For the special case $\alpha \geq 3$ and $\lambda = 0$ in (1.6), we have the following results from Srinivasa Rao, Sachdev and Mythily Ramaswamy [13]:

THEOREM A *Assume that* $\alpha > 3$ *and* $\lambda = 0$. *Then there exists a unique* ν^* *such that*:

(i) *If* $\nu > \nu^*$, *the solution of* (1.6)–(1.7) *decays algebraically to zero as* $\eta \to -\infty$ *and is positive on* $(-\infty, 0)$; *g has a zero at a finite point* $\eta = \eta_1 > 0$.

(ii) *If* $\nu = \nu^*$, *the solution g of* (1.6)–(1.7) *decays exponentially to zero as* $\eta \to \infty$ *and algebraically to zero as* $\eta \to -\infty$; *g is positive on* $(-\infty, \infty)$.

(iii) *If* $\nu < \nu^*$, *the solution g of* (1.6)–(1.7) *decays algebraically to zero as* $\eta \to \pm\infty$.

THEOREM B *Assume that* $\alpha = 3$ *and* $\lambda = 0$. *Then the solution* g *of* (1.6)–(1.7) *decays algebraically to zero as* $\eta \to -\infty$ *and is positive on* $(-\infty, 0)$; g *has a finite zero on* $(0, \infty)$.

Combining the results of Srinivasa Rao, Sachdev and Mythily Ramaswamy [13] and Peletier and Serafini [6], we have the following theorem when $1 < \alpha < 3$ and $\lambda = 0$.

THEOREM C *Assume that* $1 < \alpha < 3$ *and* $\lambda = 0$. *Then there exists a* $\nu^* > 0$ *such that*:

(i) *If* $\nu > \nu^*$, *the solution* g *of* (1.6)–(1.7) *decays algebraically to zero as* $\eta \to -\infty$ *and vanishes at a finite point* $\eta = \eta_1 > 0$; g *is positive on* $(-\infty, \eta_1)$.

(ii) *If* $\nu = \nu^*$, *the solution* g *of* (1.6)–(1.7) *decays exponentially to zero as* $\eta \to -\infty$ *and vanishes at a finite point* $\eta = \eta_1 > 0$; g *is positive on* $(-\infty, \eta_1)$.

(iii) *If* $\nu < \nu^*$, *the solution of* (1.6)–(1.7) *has a zero at a finite point on* $(0, \infty)$ *and* $(-\infty, 0)$ *each.*

In the manner of Brezis, Peletier and Terman [3], we give the asymptotic behaviour of the positive solution g of (1.6) that decays to 0 as $\eta \to \infty$ exponentially or algebraically, depending on whether (g'/g) tends to $-\infty$ or 0 as η tends to ∞ in the following two theorems.

THEOREM 1.1 *If* $g(\eta) > 0$ *for all* η *and* $\lim_{\eta \to \infty} g'/g = -\infty$, *then*

$$g(\eta) = Ae^{-\eta^2}\eta^{(3-\alpha)/(\alpha-1)} \left(1 - \frac{(\alpha-2)(\alpha-3)}{(\alpha-1)^2}\frac{1}{2\eta^2} + o\left(\frac{1}{\eta^2}\right)\right) \quad \text{as} \quad \eta \to \infty, \tag{1.8}$$

where $A > 0$ *is a constant.*

THEOREM 1.2 *If* $g(\eta) > 0$ *for all* η *and* $\lim_{\eta \to \infty} g'/g = 0$, *for some* $0 < K < 2$, *then*

$$g(\eta) = A\eta^{-2/(\alpha-1)}\left(1 + o(\eta^{-K})\right) \quad \text{as} \quad \eta \to \infty, \tag{1.9}$$

where $A > 0$ *is a constant.*

Similar results can also be given for the solutions of (1.6) as $\eta \to -\infty$.

Srinivasa Rao, Sachdev and Mythily Ramaswamy [12] analysed the positive solutions of (1.6) decaying exponentially to zero as $\eta \to \infty$, using shooting arguments.

In the present paper we prove the existence of following types of solutions of the IVP (1.6)–(1.7), which depend crucially on the parameters α, λ and ν:

(i) Positive solutions decaying exponentially to zero as $\eta \to \infty$ and algebraically to zero as $\eta \to -\infty$.
(ii) Positive solutions decaying algebraically to zero as $\eta \to \pm\infty$.
(iii) Solutions decaying algebraically (exponentially) to zero at one end and having a finite zero on the other side.
(iv) Unbounded solutions.

More precisely, we prove the following theorems for $\lambda \neq 0$.

THEOREM 1 *Assume that* $\alpha > 3$ *and* $\lambda > 0$. *Then there exist* ε *and* ν_1 *such that, if* $\nu \in (0, \nu_1) \cup (\nu_0 - \varepsilon, \nu_0)$, *the solution* g *of* (1.6)–(1.7) *decays algebraically to zero as* $\eta \to \pm\infty$ *and is positive on* $(-\infty, \infty)$. *Here* $\nu_0 \equiv (\lambda(\alpha-1))^{-1/(\alpha-1)}$, *the constant solution of* (1.6).

THEOREM 2 *Assume that* $\alpha > 3$ *and* $\lambda > 0$. *For sufficiently small* λ, *there exists a* $\nu > 0$ *such that the solution* g *of* (1.6)–(1.7) *is positive, decays exponentially to zero as* $\eta \to \infty$ *and algebraically to zero as* $\eta \to -\infty$.

THEOREM 3 *Assume that* $\alpha > 3$ *and* $\lambda > 0$. *If* λ *is sufficiently large, the solution* g *of* (1.6)–(1.7) *is positive on* $(-\infty, \infty)$ *and decays algebraically to zero as* $\eta \to \pm\infty$ *for all* ν.

THEOREM 4 *Assume that* $\alpha > 3$ *and* $\lambda < 0$. *Then there exists a* ν^* *such that for* $\nu < \nu^*$, *the solution* g *of* (1.6)–(1.7) *is positive and decays algebraically to zero as* $\eta \to \pm\infty$.

THEOREM 5 *Assume that* $1 < \alpha \leq 3$ *and* $\lambda \leq 0$. *If* $g(\eta, \nu)$ *is a solution of the IVP* (1.6)–(1.7), g *cannot remain positive on* $(0, \infty)$.

The scheme of the present paper is as follows. In Section 2 we present the numerical study of (1.6)–(1.7). Section 3 details the analysis. Section 4 contains the conclusions of the present paper.

2. NUMERICAL STUDY OF THE NONLINEAR EQUATION (1.6)

We present here qualitative results following from the numerical study of the IVP

$$g'' + 2\eta g' + \frac{4}{\alpha - 1} g - 2^{3/2} g^{(\alpha-1)/2} g' - 4\lambda g^{\alpha} = 0, \quad \eta > 0, \tag{2.1}$$

$$g(0) = \nu, \quad g'(0) = 0, \tag{2.2}$$

where $\alpha > 1$, $\nu > 0$ and λ is real. Let $g(\eta, \nu)$ be the solution of (2.1)–(2.2) on the maximum possible interval of existence. Recall that

$$\nu_0 \equiv (\lambda(\alpha - 1))^{-1/(\alpha-1)} \tag{2.3}$$

is an exact solution of (2.1). The following distinct cases are observed from the numerical integration of (2.1)–(2.2).

I. $\alpha > 3$

(a) $\lambda = 0$ There exists a positive value of $\nu = \nu^*$ such that

(i) If $\nu < \nu^*$, $g(\eta, \nu)$ decays algebraically to zero as $\eta \to \pm\infty$; $g(\eta, \nu)$ is positive on $(-\infty, \infty)$.

(ii) If $\nu = \nu^*$, $g(\eta, \nu^*)$ decays exponentially to zero as $\eta \to \infty$ and decays algebraically to zero as $\eta \to -\infty$; $g(\eta, \nu^*)$ is positive on $(-\infty, \infty)$.

(iii) If $\nu > \nu^*$, $g(\eta, \nu)$ has a finite zero on $(0, \infty)$ and decays algebraically to zero as $\eta \to -\infty$; $g(\eta, \nu)$ is positive on $(-\infty, 0)$.

For the case $\alpha = 5$, $\lambda = 0$, $\nu^* \approx 0.97995$ (see Fig. 1).

It is observed that all the solutions are bounded on $(-\infty, 0)$. Also, the solutions of IVP (2.1)–(2.2) are bounded on $(0, \infty)$ for $\alpha = 7, 11$. There are solutions which tend to $-\infty$ at a finite point $\eta = \eta_0 > 0$ for $\alpha = 5, 9$. In fact, our numerical experiments suggest that when $g^{(\alpha-1)/2}$ is replaced by $|g|^{(\alpha-1)/2}$ in (2.1), there are solutions which tend to $-\infty$ at a finite point for any $\alpha > 3$. If $g^{(\alpha-1)/2}$ is replaced by $|g|^{(\alpha-1)/2}$ sign(g), the solutions are bounded on $(0, \infty)$; when the singularity (that is, blowing up) is observed on $(0, \infty)$, the interval (ν^*, ∞) contains ν's for which $g(\eta, \nu)$ is bounded or tends to $-\infty$ at a finite point. We conjecture that there exists a $\bar{\nu} \in (\nu^*, \infty)$ such that $g(\eta, \bar{\nu})$ exists for all

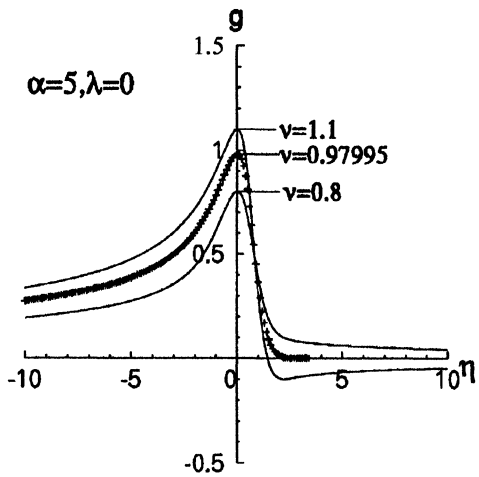

FIGURE 1 Numerical solution of IVP (2.1)–(2.2) for $\alpha=5$, $\lambda=0$; $\alpha=9$, $\lambda=0.01$; $\alpha=9$, $\lambda=0.01$; $\alpha=9$, $\lambda=1$.

η, has a finite zero and $g(\eta,\bar{\nu}) \to -\infty$ as $\eta\to\infty$. If $\nu>\bar{\nu}$, $g(\eta,\nu)$ tends to $-\infty$ at a finite point.

(b) $\lambda>0$ sufficiently small On $(-\infty,0)$, all the solutions are positive, bounded and decay algebraically to zero as $\eta\to-\infty$.

On $(0,\infty)$, there exists ν_1, ν_2 with $\nu_1<\nu_2$ such that

(i) If $\nu<\nu_1$, g is positive on $(0,\infty)$ and decays algebraically to zero as $\eta\to\infty$

(ii) If $\nu=\nu_1$, $g(\eta,\nu_1)$ is positive on $(0,\infty)$ and decays exponentially to zero as $\eta\to\infty$.

(iii) If $\nu\in(\nu_1,\nu_2)$, $g(\eta,\nu)$ has a finite zero.

(iv) If $\nu=\nu_2$, $g(\eta,\nu_2)$ decays exponentially to zero as $\eta\to\infty$; g is positive on $(0,\infty)$.

(v) If $\nu_2<\nu<\nu_0$, where ν_0 is as in (2.3), $g(\eta,\nu)$ decays algebraically to zero as $\eta\to\infty$; $g(\eta,\nu)$ is positive on $(0,\infty)$.

For $\alpha=9$, $\lambda=0.01$, we found that $\nu_1\approx 1.27858$, $\nu_2\approx 1.366448$ (see Figs. 2 and 3).

It is interesting to observe that for λ positive and small, solutions with exponential decay to zero are not unique, which is in contrast to the case $\lambda=0$, $\alpha>3$, for which solutions of (2.1)–(2.2) are unique.

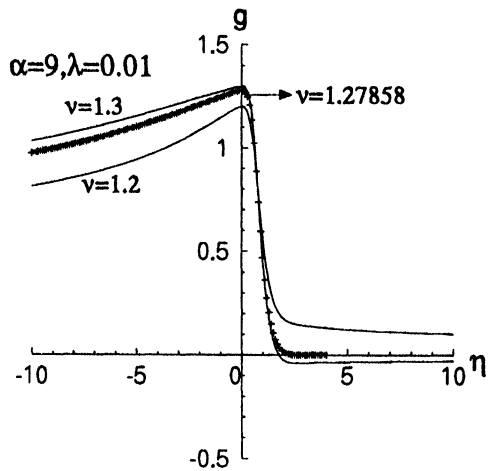

FIGURE 2 Numerical solution of IVP (2.1)–(2.2) for $\alpha=5$, $\lambda=0$; $\alpha=9$, $\lambda=0.01$; $\alpha=9$, $\lambda=0.01$; $\alpha=9$, $\lambda=1$.

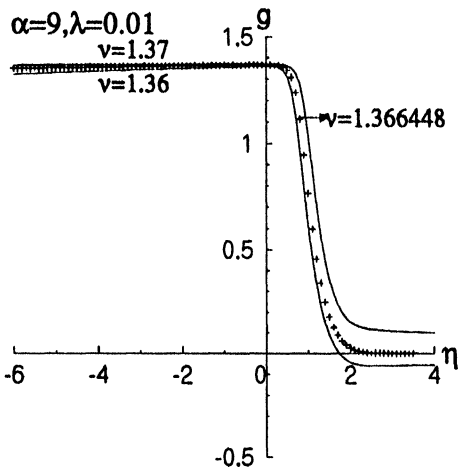

FIGURE 3 Numerical solution of IVP (2.1)–(2.2) for $\alpha=5$, $\lambda=0$; $\alpha=9$, $\lambda=0.01$; $\alpha=9$, $\lambda=0.01$; $\alpha=9$, $\lambda=1$.

When λ is large, all the solutions decay algebraically to zero as $\eta\to\infty$ and also ordered with respect to ν (see Fig. 4). As for the case $\lambda=0$, there are solutions which tend to $-\infty$ at some $\eta_0>0$ for $\alpha=5, 9$. For $\alpha=7, 11$, all the solutions are bounded on $(-\infty,\infty)$. In the light of

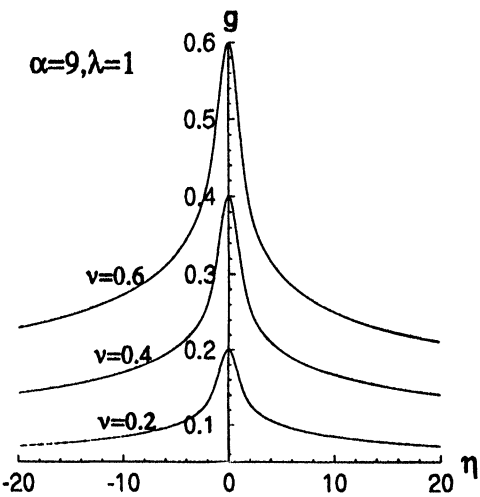

FIGURE 4 Numerical solution of IVP (2.1)–(2.2) for $\alpha=5$, $\lambda=0$; $\alpha=9$, $\lambda=0.01$; $\alpha=9$, $\lambda=0.01$; $\alpha=9$, $\lambda=1$.

our numerical study, we conjecture that there exists a $\lambda_0 > 0$ for which all the solutions decay algebraically to zero as $\eta \to \infty$ except for one ν, for which $g(\eta, \nu)$ is with exponential decay to zero as $\eta \to \infty$.

(c) $\lambda < 0$ When λ is sufficiently small, there exists $\nu^* > 0$ such that

(i) If $\nu < \nu^*$, g is positive on $(-\infty, \infty)$ and decays algebraically to zero as $\eta \to \pm\infty$.
(ii) If $\nu = \nu^*$, $g(\eta, \nu^*)$ decays exponentially to zero as $\eta \to \infty$ and algebraically to zero as $\eta \to -\infty$; g is positive on $(-\infty, \infty)$.
(iii) If $\nu > \nu^*$, $g(\eta, \nu)$ has a finite zero on $(0, \infty)$; $g(\eta, \nu)$ is positive on $(-\infty, 0)$ and decays algebraically to zero as $\eta \to -\infty$.

For the case $\alpha=7$, $\lambda=-0.001$, $\nu^* \approx 1.18471$ (see Fig. 5). As ν was increased, no second zero was observed. This situation is similar to that for the case $\lambda=0$.

When $\lambda < 0$, $|\lambda|$ is not small, there exist ν_1 and ν_2, $\nu_1 < \nu_2$, such that

(i) If $\nu < \nu_1$, $g(\eta, \nu)$ is positive on $(-\infty, \infty)$ and decays algebraically to zero as $\eta \to \pm\infty$.
(ii) If $\nu = \nu_1$, $g(\eta, \nu_1)$ decays exponentially to zero as $\eta \to \infty$ and algebraically to zero as $\eta \to -\infty$; $g(\eta, \nu_1)$ is positive on $(-\infty, \infty)$.

(iii) If $\nu \in (\nu_1, \nu_2)$, $g(\eta, \nu)$ has a finite zero on $(0, \infty)$; $g(\eta, \nu)$ decays algebraically to zero as $\eta \to -\infty$ and is positive on $(-\infty, 0]$.

(iv) If $\nu = \nu_2$, $g(\eta, \nu_2)$ has a finite zero on $(0, \infty)$; $g(\eta, \nu_2)$ decays exponentially to zero as $\eta \to -\infty$ and is positive on $(-\infty, 0)$.

(v) If $\nu > \nu_2$, g has a finite zero on $(0, \infty)$ and $(-\infty, 0)$ each.

For $\alpha = 7$, $\lambda = -0.4$, we found that $\nu_1 \approx 0.97081$, $\nu_2 = 1.71276$ (see Figs. 6 and 7).

As for the cases $\lambda \geq 0$, there are values of ν for which $g(\eta, \nu)$ tends to $-\infty$ at a finite point for $\alpha = 5$, 9 at some $\eta_0 > 0$ which is in contrast to the case $\alpha = 7$, 11 for which all the solutions are bounded on $(-\infty, \infty)$. It should be noted that all the solutions are bounded on $(-\infty, 0)$ for all α's. When $|\lambda|$ is large, the number of zeros increases with increase in ν. We conjecture that there exists $\bar{\nu} > \nu_1$ such that $g(\eta, \bar{\nu})$ exists for all η, g has a finite zero on $(0, \infty)$ and $g \downarrow -\infty$ as $\eta \to \infty$ for $\alpha = 5$.

II. $\alpha = 3$

(a) $\lambda = 0$ All the solutions are positive on $(-\infty, 0]$ and decay algebraically to zero as $\eta \to -\infty$; they have a finite zero on $(0, \infty)$ (see Fig. 8).

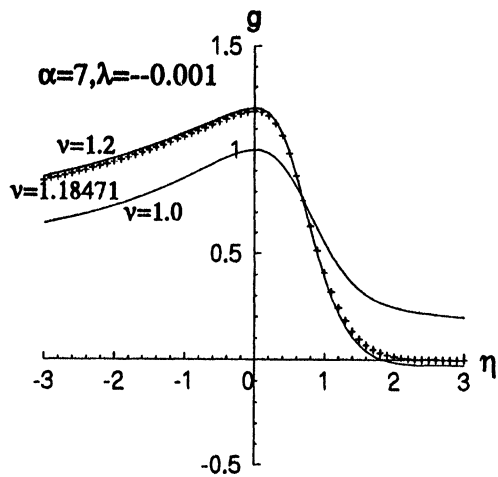

FIGURE 5 Numerical solution of IVP (2.1)–(2.2) for $\alpha = 7$, $\lambda = -0.001$; $\alpha = 7$, $\lambda = -0.4$; $\alpha = 7$, $\lambda = -0.4$; $\alpha = 3$, $\lambda = 0$.

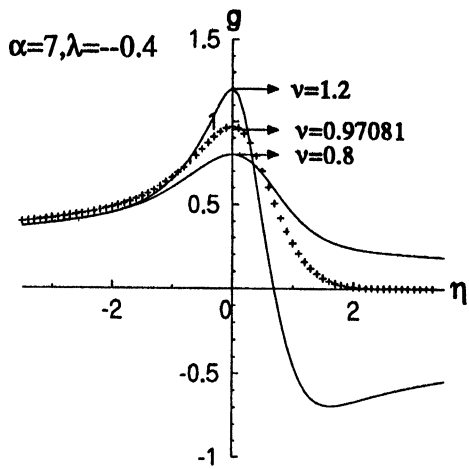

FIGURE 6 Numerical solution of IVP (2.1)–(2.2) for $\alpha=7$, $\lambda=-0.001$; $\alpha=7$, $\lambda=-0.4$; $\alpha=7$, $\lambda=-0.4$; $\alpha=3$, $\lambda=0$.

(b) $\lambda>0$ There exists $\nu^*>0$ such that

(i) If $\nu<\nu^*$, $g(\eta,\nu)$ has a finite zero on $(0,\infty)$; $g(\eta,\nu)$ is positive on $(-\infty,0]$ and decays algebraically to zero as $\eta\to-\infty$.
(ii) If $\nu=\nu^*$, $g(\eta,\nu)$ decays exponentially to zero as $\eta\to\infty$ and algebraically to zero as $\eta\to-\infty$; $g(\eta,\nu)$ is positive on $(-\infty,\infty)$.
(iii) If $\nu>\nu^*$, $g(\eta,\nu)$ is positive on $(-\infty,\infty)$ and decays algebraically to zero as $\eta\to\pm\infty$.

For $\alpha=3$, $\lambda=3$, it is found that $\nu^*\approx 0.2109$ (see Fig. 9).

(c) $\lambda<0$ When $|\lambda|$ is sufficiently small, all the solutions have a finite zero on $(0,\infty)$; they are positive on $(-\infty,0)$ and decay algebraically to zero $\eta\to-\infty$.

When $|\lambda|$ is large, there exists ν^* such that

(i) If $\nu<\nu^*$, $g(\eta,\nu)$ has a finite zero on $(0,\infty)$; $g(\eta,\nu)$ is positive on $(-\infty,0]$ and decays algebraically to zero as $\eta\to-\infty$.
(ii) If $\nu=\nu^*$, $g(\eta,\nu)$ has a finite zero on $(0,\infty)$; $g(\eta,\nu)$ is positive on $(-\infty,0]$ and decays exponentially to zero as $\eta\to-\infty$.
(iii) If $\nu>\nu^*$, $g(\eta,\nu)$ has one zero on $(-\infty,0)$ and $(0,\infty)$ each.

For $\alpha=3$, $\lambda=-3$, $\nu^*\approx 0.2542$ (see Fig. 10).
Note that in this case, all the solutions are bounded.

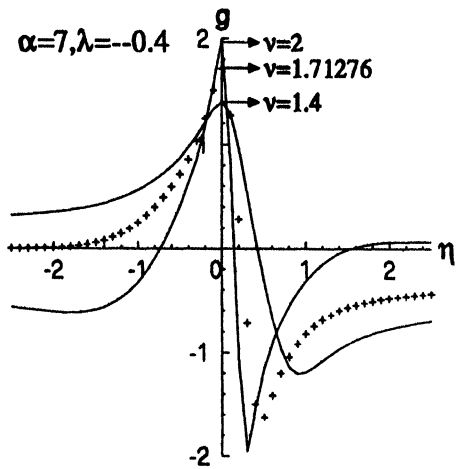

FIGURE 7 Numerical solution of IVP (2.1)–(2.2) for $\alpha=7$, $\lambda=-0.001$; $\alpha=7$, $\lambda=-0.4$; $\alpha=7$, $\lambda=-0.4$; $\alpha=3$, $\lambda=0$.

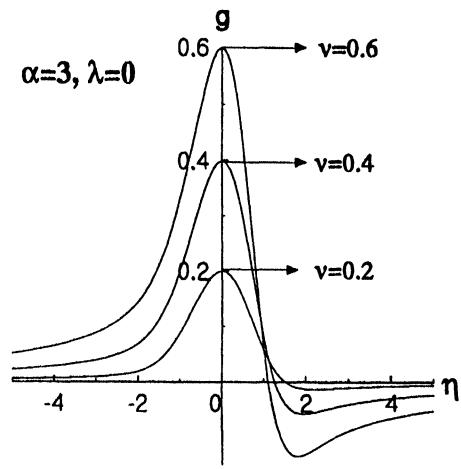

FIGURE 8 Numerical solution of IVP (2.1)–(2.2) for $\alpha=7$, $\lambda=-0.001$; $\alpha=7$, $\lambda=-0.4$; $\alpha=7$, $\lambda=-0.4$; $\alpha=3$, $\lambda=0$.

III. $1<\alpha<3$ For this case, we have taken $\alpha=2$. Also we replaced $g^{(\alpha-1)/2}$ in (2.1) with $|g|^{(\alpha-1)/2}$ sign(g) for the Eq. (2.1) to be defined when g becomes negative. The following was observed from our numerical study of (2.1)–(2.2).

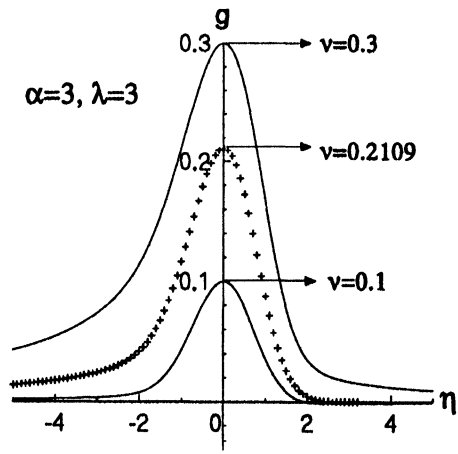

FIGURE 9 Numerical solution of IVP (2.1)–(2.2) for $\alpha=3$, $\lambda=3$; $\alpha=3$, $\lambda=-3$; $\alpha=2$, $\lambda=0$; $\alpha=2$, $\lambda=1$.

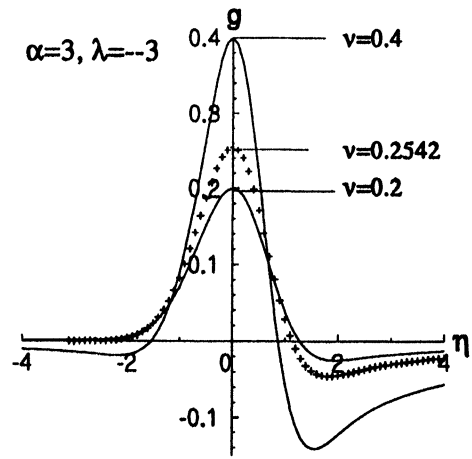

FIGURE 10 Numerical solution of IVP (2.1)–(2.2) for $\alpha=3$, $\lambda=3$; $\alpha=3$, $\lambda=-3$; $\alpha=2$, $\lambda=0$; $\alpha=2$, $\lambda=1$.

(a) $\lambda=0$ There exists ν^* such that

(i) If $\nu<\nu^*$, $g(\eta,\nu)$ has one finite zero on $(0,\infty)$ and $(-\infty,0)$ each.
(ii) If $\nu=\nu^*$, $g(\eta,\nu^*)$ is positive on $(-\infty,0]$ and decays exponentially to zero as $\eta\to-\infty$; $g(\eta,\nu)$ has a finite zero on $(0,\infty)$.
(iii) If $\nu>\nu^*$, $g(\eta,\nu)$ is positive on $(-\infty,0]$ and decays algebraically to zero as $\eta\to-\infty$ $g(\eta,\nu)$ has a finite zero on $(0,\infty)$.

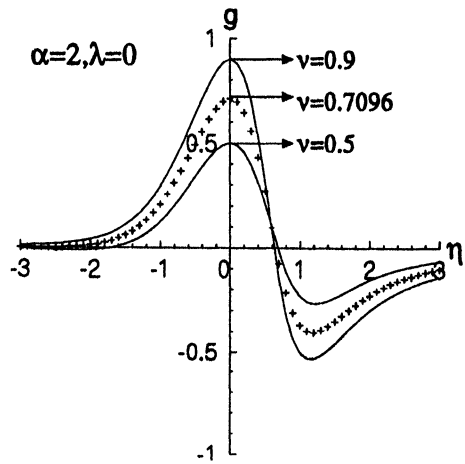

FIGURE 11 Numerical solution of IVP (2.1)–(2.2) for $\alpha=3$, $\lambda=3$; $\alpha=3$, $\lambda=-3$; $\alpha=2$, $\lambda=0$; $\alpha=2$, $\lambda=1$.

For $\alpha=2$, $\lambda=0$, $\nu^*\approx 0.7096$ (see Fig. 11).

(b) $\lambda>0$ There exist ν_1, ν_2 with $\nu_1<\nu_2$ such that

(i) If $\nu<\nu_1$, $g(\eta,\nu)$ has a finite zero on $(0,\infty)$ and $(-\infty,0)$ each.

(ii) If $\nu=\nu_1$, $g(\eta,\nu_1)$ is positive on $(-\infty,0]$ and decays exponentially to zero as $\eta\to-\infty$; $g(\eta,\nu_1)$ has a finite zero on $(0,\infty)$.

(iii) If $\nu\in(\nu_1,\nu_2)$, $g(\eta,\nu)$ is positive on $(-\infty,0]$ and decays algebraically to zero as $\eta\to-\infty$; $g(\eta,\nu)$ has a finite zero on $(0,\infty)$.

(iv) If $\nu=\nu_2$, then $g(\eta,\nu_2)$ decays exponentially to zero as $\eta\to\infty$ and algebraically to zero as $\eta\to-\infty$; $g(\eta,\nu_2)$ is positive on $(-\infty,\infty)$.

(v) If $\nu_2<\nu<\nu_0$, then $g(\eta,\nu)$ is positive on $(-\infty,\infty)$ and decays algebraically to zero as $\eta\to\pm\infty$. Here ν_0 is the value of ν for which an exact solution exists (see (2.3)).

For the case $\alpha=2$, $\lambda=1$, we found that $\nu_1\approx 0.2973$, $\nu_2\approx 0.97506$ (see Figs. 12 and 13).

(c) $\lambda<0$ When $|\lambda|$ is sufficiently small, then there exists ν^* such that

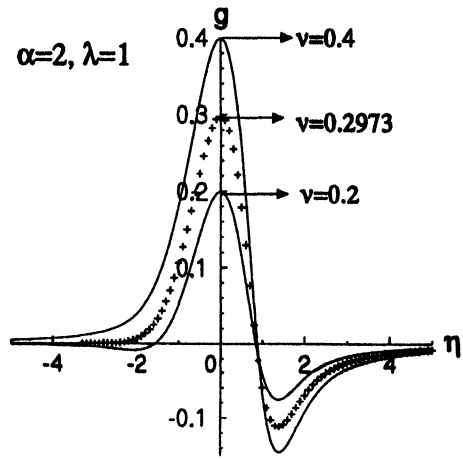

FIGURE 12 Numerical solution of IVP (2.1)–(2.2) for $\alpha=3$, $\lambda=3$; $\alpha=3$, $\lambda=-3$; $\alpha=2$, $\lambda=0$; $\alpha=2$, $\lambda=1$.

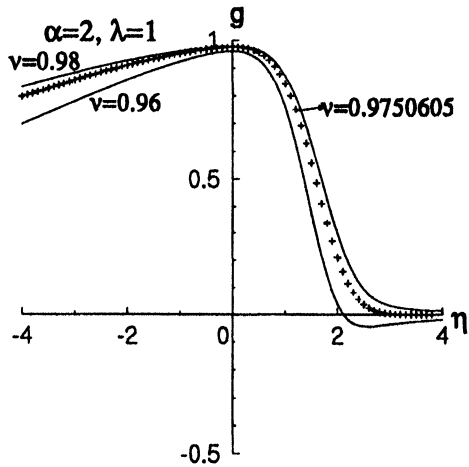

FIGURE 13 Numerical solution of IVP (2.1)–(2.2) for $\alpha=2$, $\lambda=1$; $\alpha=2$, $\lambda=-0.001$; $\alpha=2$, $\lambda=-1$.

(i) If $\nu<\nu^*$, $g(\eta,\nu^*)$ has a finite zero on $(0,\infty)$ and $(-\infty,0)$ each.
(ii) If $\nu=\nu^*$, $g(\eta,\nu)$ has finite zero on $(0,\infty)$; it is positive on $(-\infty,0]$ and decays exponentially to zero as $\eta\to-\infty$.
(iii) If $\nu>\nu^*$, $g(\eta,\nu)$ has finite zero on $(0,\infty)$; $g(\eta,\nu)$ is positive on $(-\infty,0)$ and decays algebraically to zero as $\eta\to-\infty$.

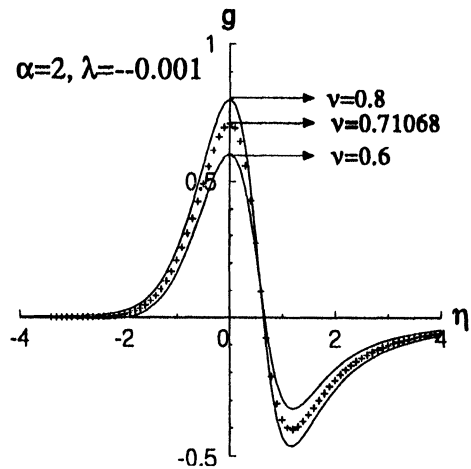

FIGURE 14 Numerical solution of IVP (2.1)–(2.2) for $\alpha=2$, $\lambda=1$; $\alpha=2$, $\lambda=-0.001$; $\alpha=2$, $\lambda=-1$.

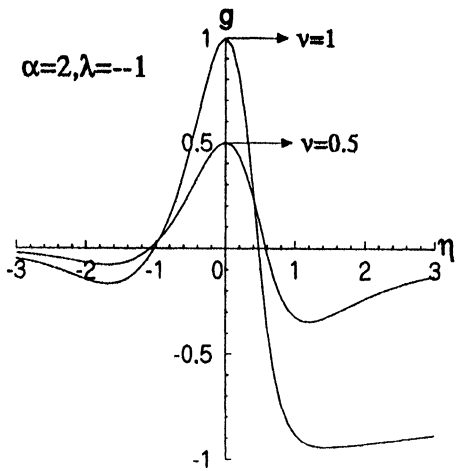

FIGURE 15 Numerical solution of IVP (2.1)–(2.2) for $\alpha=2$, $\lambda=1$; $\alpha=2$, $\lambda=-0.001$; $\alpha=2$, $\lambda=-1$.

For $\alpha=2$, $\lambda=-0.001$, we found that $\nu^* \approx 0.71068$ (see Fig. 14).

When $|\lambda|$ is large, all the solutions have a finite zero on $(-\infty, 0)$ and $(0, \infty)$ each (see Fig. 15).

3. ANALYSIS OF THE SOLUTIONS OF IVP (1.6)–(1.7)

In this section, we study the behaviour of the solutions of the initial value problem

$$g''(\eta) + 2\eta g'(\eta) + \frac{4}{\alpha - 1} g(\eta) - 2^{3/2} g^{(\alpha-1)/2}(\eta) g'(\eta) - 4\lambda g^{\alpha}(\eta) = 0, \tag{3.1}$$

$$g(0) = \nu, \quad g'(0) = 0, \tag{3.2}$$

on $(-\infty, \infty)$. For this purpose, we study the IVP (3.1)–(3.2) on $(0, \infty)$ and $(-\infty, 0)$, separately.

3.1. Analysis of IVP (3.1)–(3.2) on [0, ∞)

Local existence and uniqueness of the solution of the initial value problem (3.1)–(3.2) are guaranteed by the standard existence theorems. Let the unique solution defined on the largest possible interval be $g(\eta, \nu)$. For simplicity we shall write $g(\eta, \nu)$ as $g(\eta)$ when there is no ambiguity.

LEMMA 3.1 *Let $\alpha > 1$, $\lambda \leq 0$. Then no solution of* (3.1) *can have a positive local minimum and hence $g' < 0$ for all $\eta > 0$.*

Proof Suppose η_0 is a point such that $g(\eta_0) > 0$ and $g'(\eta_0) = 0$. Then

$$g''(\eta_0) = -\frac{4}{\alpha - 1} g(\eta_0) + 4\lambda g^{\alpha}(\eta_0) < 0.$$

Thus g can only have local maximum at $\eta = \eta_0$. ■

LEMMA 3.2 *Let $\alpha > 1$ and $\lambda > 0$. Then*

(i) *If $\nu > (\lambda(\alpha - 1))^{-1/(\alpha-1)}$, $g(\eta) > \nu$ for all $\eta > 0$*
(ii) *If $\nu = (\lambda(\alpha - 1))^{-1/(\alpha-1)} \equiv \nu_0$, $g \equiv \nu_0$.*
(iii) *If $\nu < (\lambda(\alpha - 1))^{-1/(\alpha-1)}$, $g(\eta) < \nu$ for all $\eta > 0$ and $g' < 0$ as long as g is positive.*

Proof Suppose that at $\eta = \eta_0$, $g'(\eta_0) = 0$. This by (3.1) implies that

$$\begin{aligned} g''(\eta_0) &= -\frac{4}{\alpha - 1} g(\eta_0) + 4\lambda g^{\alpha}(\eta_0) \\ &= 4\lambda g(\eta_0) \left(-\frac{1}{\lambda(\alpha - 1)} + g^{\alpha-1}(\eta_0) \right). \end{aligned} \tag{3.3}$$

Thus

$$g''(\eta_0) < 0 \quad \text{if } g(\eta_0) < \nu_0,$$
$$g''(\eta_0) = 0 \quad \text{if } g(\eta_0) = \nu_0,$$
$$g''(\eta_0) > 0 \quad \text{if } g(\eta_0) > \nu_0.$$

If $\nu > \nu_0$, we have that $g''(0) > 0$. Thus g has a local minimum at $\eta = 0$. It also follows from (3.3) that g cannot have a positive local maximum. Thus the solution g of (3.1)–(3.2) is always greater than ν_0.

If $\nu = \nu_0$ and $g'(0) = 0$, by the uniqueness of the solutions, we have $g \equiv \nu_0$. This solution was noted earlier by Sachdev, Nair and Tikekar [9].

If $\nu < \nu_0$, g has a local maximum at $\eta = 0$ and, in addition, by (3.3) g cannot have a positive local minimum.

Therefore, $g' < 0$ if $g > 0$. ■

Since we are interested in the solutions of IVP (3.1)–(3.2) that vanish as $\eta \to -\infty$, we consider in detail the case $g(0) < \nu_0$ for $\alpha > 1$, $\lambda > 0$. These two lemmas show that both the cases $\alpha > 1$, $\lambda > 0$, $\nu < \nu_0$ (see (2.3)) and $\alpha > 1$, $\lambda \leq 0$, the solutions are monotonic decreasing provided g is positive. They may either become zero at a finite point or remain positive and decreasing for all $\eta > 0$.

LEMMA 3.3 *Let* $\alpha > 1$ *and* $\lambda \in \mathbb{R}$. *If* $g(\eta, \nu) > 0$ *for all* $\eta > 0$, g, $g' \to 0$ *as* $\eta \to \infty$.

Proof Integrating (3.1) from η_0 to η, we get

$$g'(\eta) = g'(\eta_0) + 2\eta_0 g(\eta_0) - \frac{2^{5/2}}{\alpha+1} g^{(\alpha+1)/2}(\eta_0) + \frac{2^{5/2}}{(\alpha+1)} g^{(\alpha+1)/2}(\eta)$$
$$+ \left(-2\eta g + \frac{2(\alpha-3)}{(\alpha-1)} \int_{\eta_0}^{\eta} g\,ds + 4\lambda \int_{\eta_0}^{\eta} g^{\alpha}\,ds \right). \tag{3.4}$$

Since $g' < 0$ for all $\eta > 0$ and g is bounded below, $g \to \bar{g} \geq 0$ as $\eta \to \infty$. We will show that $\bar{g} = 0$. Suppose on the contrary that $\bar{g} \neq 0$. This implies from (3.4) that

$$g'(\eta) \sim C_1 + \left(-2\eta\bar{g} + \frac{2(\alpha-3)}{(\alpha-1)} \eta\bar{g} + 4\lambda\eta\bar{g}^{\alpha} \right) \quad \text{as} \quad \eta \to \infty$$

where C_1 is a constant consisting of the terms evaluated at η_0 and $(2^{5/2}\bar{g}^{(\alpha+1)/2})/(\alpha+1)$. Therefore,

$$g'(\eta) \sim C_1 + \left(-\frac{4}{\alpha-1} + 4\lambda\bar{g}^{\alpha-1} \right)\eta\bar{g} \quad \text{as} \quad \eta \to \infty.$$

If $\alpha > 1$, $\lambda \leq 0$ and $\bar{g} \neq 0$ by assumption, $g'(\eta) \to -\infty$ as $\eta \to \infty$. This is also true for $\alpha > 1$, $\lambda > 0$ (since $\bar{g} < \nu_0$). This implies that g cannot be positive for all $\eta \geq 0$, leading to a contradiction. Hence $g \to 0$ as $\eta \to \infty$. We will now show that $g' \to 0$ as $\eta \to \infty$. We note that there exists a point η_1 such that $g''(\eta_1) = 0$, otherwise $g'' < 0$ for all η and hence g cannot always remain positive, a contradiction to our assumption. From the differential Eq. (3.1), we have

$$g'''(\eta_1) = -\frac{2(\alpha+1)}{(\alpha-1)}g'(\eta_1) + 2^{1/2}(\alpha-1)g^{(\alpha-3)/2}(\eta_1)g'^2(\eta_1) + 4\lambda\alpha g^{\alpha-1}(\eta_1)g'(\eta_1). \tag{3.5}$$

When $\alpha > 1$, $\lambda \leq 0$, we use the fact that $g'(\eta_1) < 0$. We thus have $g'''(\eta_1) \geq 0$, implying that g'' has exactly one zero. It follows that g'' is ultimately positive.

When $\alpha > 1$, $\lambda > 0$,

$$g'''(\eta_1) = g'(\eta_1)\left(-\frac{2(\alpha+1)}{(\alpha-1)} + 4\lambda\alpha g^{\alpha-1}(\eta_1) \right) + 2^{1/2}(\alpha-1)g^{(\alpha-3)/2}(\eta_1)g'^2(\eta_1). \tag{3.6}$$

Since $g(\eta_1) \to 0$ as $\eta_1 \to \infty$, we have $-2(\alpha+1)/(\alpha-1) + 4\lambda\alpha g^{\alpha-1}(\eta_1) < 0$ for large η_1 and hence $g'''(\eta_1) \geq 0$ at η_1 where $g''(\eta_1) = 0$, η_1 large. Hence for η sufficiently large g'' cannot change sign. We claim that g'' is ultimately positive. Otherwise $g' < 0$, $g'' < 0$ for $\eta > \eta_2$ implying that g cannot remain positive, a contradiction. Therefore, for $\alpha > 1$, λ arbitrary, $g'' > 0$ for η sufficiently large. Since g' is ultimately increasing and bounded above by 0, g' should converge to a constant g_1. We shall show that $g_1 = 0$. Suppose on the contrary that $g_1 \neq 0$; let $g_1 < 0$, since $g' < 0$ for all $\eta > 0$. This implies $g' < -K$, for some $K > 0$, η sufficiently large. An integration of $g' < -K$ from η_0 to η with respect to η gives

$$g(\eta) - g(\eta_0) < -K(\eta - \eta_0), \quad \text{for} \quad \eta > \eta_0$$

which implies that $g(\eta)$ cannot remain positive. This is a contradiction to our assumption that g is positive on $(0, \infty)$. Therefore g' should tend to zero as $\eta \to \infty$. Hence the lemma. ■

To understand how the solution decays as $\eta \to \infty$, we analyse as a first step, the asymptotic behaviour of g and g' as $\eta \to \infty$, in the case when $g(\eta) > 0$ for all $\eta > 0$. We show that $\lim_{\eta \to \infty} g'/g$ can be either 0 or $-\infty$ and correspondingly the positive solution decays algebraically or exponentially to zero as $\eta \to \infty$, respectively. Following the work of Brezis, Peletier and Terman [3], we have the Lemmas 3.4–3.6.

LEMMA 3.4 (Trapping Region for $\alpha > 1$, $\lambda \leq 0$) *For given $\mu > 0$ and $\alpha > 1$, $\lambda \leq 0$, define*

$$D_\mu^1 = \{(g_1, g_2) \in \mathbb{R}^2 : g_1 > 0,\ g_2 < 0 \text{ and } g_2 + \mu g_1 > 0\}.$$

Let g be a solution of the IVP (3.1)–(3.2). *Further assume that there exists a number*

$$\tau > \frac{1}{2\mu}\left(\frac{4}{\alpha - 1} + 2^{3/2}\mu\nu^{(\alpha-1)/2} - 4\lambda\nu^{\alpha-1} + \mu^2\right) \tag{3.7}$$

such that $(g(\tau),\ g'(\tau)) \in D_\mu^1$, then $(g(\eta),\ g'(\eta)) \in D_\mu^1$ for all $\eta \geq \tau$.

LEMMA 3.5 (Trapping Region for $\alpha > 1$, $\lambda > 0$) *Let $\mu > 0$, and $\alpha > 1$, $\lambda > 0$. Define*

$$D_\mu^2 = \{(g_1, g_2) \in \mathbb{R}^2 : 0 < g_1 < \nu_0,\ g_2 < 0 \text{ and } g_2 + \mu g_1 > 0\}.$$

If g is a solution of the IVP (3.1)–(3.2) *and if there exists*

$$\tau > \frac{1}{2\mu}\left(\frac{4}{\alpha - 1} + 2^{3/2}\mu\nu^{(\alpha-1)/2} + \mu^2\right) \tag{3.8}$$

such that $(g(\tau),\ g'(\tau)) \in D_\mu^2$, then $(g(\eta),\ g'(\eta)) \in D_\mu^2$ for all $\eta \geq \tau$.

LEMMA 3.6 *Suppose that $\alpha > 1$ and $\lambda \in \mathbb{R}$ and $g(\eta, \nu) > 0$ for all $\eta \geq 0$. Then $\lim_{\eta \to \infty} g'/g$ exists and is either $-\infty$ or 0.*

Remark 1 (Asymptotic Behaviour of the Solution of (3.1)–(3.2) as $\eta \to \infty$) From Lemma 3.6, we have that if $g(\eta, \nu) > 0$, $g'/g \to 0$ or

$-\infty$ as $\eta \to \infty$, when $\alpha > 1$ and $\lambda \in \mathbb{R}$. From Theorems 1.1 and 1.2 of Section 1, we know that

(i) If $\lim_{\eta\to\infty} g'/g = 0$, g decays algebraically to zero as $\eta \to \infty$.
(ii) If $\lim_{\eta\to\infty} g'/g = -\infty$, g decays exponentially to zero as $\eta \to \infty$.

Remark 2 For $\alpha > 3$, we have the following asymptotic characterisation.

(i) g decays algebraically to zero as $\eta \to \infty$ if and only if $\exp(\eta^2)\, g$ is an increasing function for η sufficiently large.
(ii) g decays exponentially to zero as $\eta \to \infty$ if and only if $\exp(\eta^2)\, g$ is a decreasing function for η sufficiently large.

We now look for positive solutions with algebraic decay to zero as $\eta \to \infty$ in the next four lemmas.

LEMMA 3.7 *Assume that* $\alpha > 3$ *and* $\lambda > 0$. *If* g *is a solution of the IVP* (3.1)–(3.2) *with* $g(0) < min(((\alpha-3)^3/8(\alpha-1)^2)^{1/(\alpha-1)}, \nu_0)$, *then* g *is positive on* $[0, \infty)$ *and decays algebraically to zero as* $\eta \to \infty$. *Here,* ν_0 *is as in* (2.3).

Proof We shall show that $g' + 2\eta g > 0$ for all $\eta > 0$, *i.e.* $(g \exp(\eta^2))' > 0$ for all $\eta > 0$; this, in view of Remark 2(i), would imply that g decays algebraically to zero as $\eta \to \infty$.

An integration of (3.1) with initial conditions (3.2) gives

$$g' + 2\eta g = \int_0^\eta \left(\frac{2(\alpha-3)}{(\alpha-1)} g + 2^{3/2} g^{(\alpha-1)/2} g' \right) ds + 4\lambda \int_0^\eta g^\alpha ds. \quad (3.9)$$

Define

$$h(\eta) = \frac{2(\alpha-3)}{(\alpha-1)} g + 2^{3/2} g^{(\alpha-1)/2} g'. \quad (3.10)$$

This implies that

$$h(0) = \frac{2(\alpha-3)}{(\alpha-1)} \nu > 0,$$

since $\alpha > 3$. We claim that $h(\eta) > 0$ for all $\eta \geq 0$. If this is not the case, there exists η_1 such that $h(\eta_1) = 0$, $h'(\eta_1) \leq 0$. We shall now show that this is not possible under the assumptions of the lemma.

By the definition of h,

$$h'(\eta) = \frac{2(\alpha-3)}{(\alpha-1)} g' + 2^{1/2}(\alpha-1) g^{(\alpha-3)/2} g'^2 + 2^{3/2} g^{(\alpha-1)/2} g''. \quad (3.11)$$

Using (3.1), (3.11) becomes

$$\begin{aligned} h'(\eta) &= \frac{2(\alpha-3)}{(\alpha-1)} g' + 2^{1/2}(\alpha-1) g^{(\alpha-3)/2} g'^2 - 2^{5/2} \eta g^{(\alpha-1)/2} g' \\ &\quad - \frac{2^{7/2}}{\alpha-1} g^{(\alpha+1)/2} + 8 g^{\alpha-1} g' + 2^{7/2} \lambda g^{(3\alpha-1)/2} \\ &> \frac{2(\alpha-3)}{(\alpha-1)} g' + 2^{1/2}(\alpha-1) g^{(\alpha-3)/2} g'^2 \\ &\quad - \frac{2^{7/2}}{\alpha-1} g^{(\alpha+1)/2} + 8 g^{\alpha-1} g' + 2^{7/2} \lambda g^{(3\alpha-1)/2}, \end{aligned} \quad (3.12)$$

since $g' < 0$ as long as $g > 0$. Now, $h(\eta_1) = 0$ implies that

$$g'(\eta_1) = -\frac{(\alpha-3)}{2^{1/2}(\alpha-1)} g^{-(\alpha-3)/2}(\eta_1). \quad (3.13)$$

On using (3.13) in (3.12), we get

$$\begin{aligned} h'(\eta_1) &> g^{-(\alpha-3)/2}(\eta_1) \left(\frac{(\alpha-3)^3}{2^{1/2}(\alpha-1)^2} \right) \\ &\quad - 2^{5/2} g^{(\alpha+1)/2}(\eta_1) + 2^{7/2} \lambda g^{(3\alpha-1)/2}(\eta_1). \end{aligned} \quad (3.14)$$

Thus, if $g(0) < ((\alpha-3)^3/8(\alpha-1)^2)^{1/(\alpha-1)}$, then from (3.14) $h'(\eta_1) > 0$, contradicting that $h'(\eta_1) \leq 0$. Therefore $h(\eta)$ cannot have a zero and remains positive throughout.

From (3.9), since $\lambda \geq 0$ and the integrals are greater than 0, $g' + 2\eta g > 0$ for all $\eta > 0$. Hence the lemma. ■

The following lemma shows that for $\alpha > 3$ and λ sufficiently large, every solution of the IVP (3.1)–(3.2) decays algebraically to zero as $\eta \to \infty$.

LEMMA 3.8 *Assume that* $\alpha > 3$ *and* $\lambda > 0$. *If* $\lambda > 8(\alpha-1)/(\alpha-3)^3$, *all the solutions with* $\nu < \nu_0$ (*see* (2.3)) *are positive on* $(0, \infty)$ *and decay algebraically to zero as* $\eta \to \infty$.

Proof A simple manipulation shows that

$$\nu_0 \equiv (\lambda(\alpha-1))^{-1/(\alpha-1)} < \left(\frac{(\alpha-3)^3}{8(\alpha-1)^2}\right)^{1/(\alpha-1)},$$

provided that $\lambda > 8(\alpha-1)/(\alpha-3)^3$. This implies by Lemma 3.7 that all solutions decay algebraically to zero as $\eta \to \infty$. Hence the lemma. ■

In the following lemma, we prove that, when $\alpha > 1$, $\lambda > 0$, $\nu < \nu_0$ and ν is sufficiently close to ν_0, every solution of the IVP (3.1)–(3.2) decays algebraically to zero as $\eta \to \infty$. To prove this we follow the work of Brezis, Peletier and Terman [3] (see Lemmas 3 and 4 there).

LEMMA 3.9 *If $\alpha > 1$ and $\lambda > 0$, there exists $\varepsilon > 0$ such that if $\nu \in (\nu_0 - \varepsilon, \nu_0)$, the solution of the IVP (3.1)–(3.2) is positive for all $\eta > 0$ and decays algebraically to zero as $\eta \to \infty$.*

Proof For $\delta > 0$, define

$$A_\delta = \{(g_1, g_2) : g_1 > 0,\ g_2 < 0,\ \|(g_1, g_2) - (\nu_0, 0)\| < \delta\},$$

where $\|\ \|$ is the Euclidean norm. Choose δ so small that $A_\delta \subset D_\mu^2$. By using the continuous dependence of the solutions on initial conditions, it is possible to find ε_μ such that if $\nu \in (\nu_0 - \varepsilon\mu, \nu_0)$, $(g, g') \in A_\delta \subset D_\mu^2$ for $0 \le \eta \le \tau$. Thus by Lemma 3.5, for $\nu \in (\nu_0 - \varepsilon\mu, \nu_0)$ the solution does not leave D_μ^2. This implies that $g'/g \ge -\mu$ for $\eta \ge 0$. Therefore $g > 0$ on $(0, \infty)$ and g decays algebraically to zero as $\eta \to \infty$. ■

For $1 < \alpha \le 3$, $\lambda \le 0$, there are no solutions, which are positive for all $\eta > 0$. We show in Lemma 3.11 later on that in this case every solution of IVP (3.1)–(3.2) has a finite zero.

Remark 3 We observe that g cannot have a zero before

$$h_1(s) \equiv \frac{2(\alpha-3)}{(\alpha-1)} g(s) + 2^{3/2} g^{(\alpha-1)/2}(s) g'(s) + 4\lambda g^\alpha(s) \tag{3.15}$$

has a zero. Suppose this is not the case. Then there exists η_1 such that $g(\eta_1) = 0$ but $h_1(\eta_1) \ge 0$ on $[0, \eta_1]$. By (3.9),

$$g'(\eta_1) = \int_0^{\eta_1} h_1(s)\,ds > 0.$$

This contradicts that $g'(\eta_1) \le 0$ at $\eta = \eta_1$, the first zero of g.

LEMMA 3.10 *Assume that* $\alpha > 3$ *and* $\lambda < 0$. *For* $g(0)$ *sufficiently small, solutions of the initial value problem* (3.1)–(3.2) *are positive on* $(0, \infty)$ *and decay algebraically to zero as* $\eta \to \infty$.

Proof An integration of (3.1) with initial conditions (3.2) implies that

$$g' + 2\eta g = \int_0^\eta \left(\frac{2(\alpha - 3)}{\alpha - 1} g + 2^{3/2} g^{(\alpha-1)/2} g' + 4\lambda g^\alpha \right) ds$$
$$= \int_0^\eta 2g^{(\alpha-1)/2}(s) h(s) ds = \int_0^\eta h_1(s) ds \tag{3.16}$$

where h_1 is as in (3.15) and

$$h(s) \equiv \frac{(\alpha - 3)}{(\alpha - 1)} g^{(3-\alpha)/2}(s) + 2^{1/2} g'(s) + 2\lambda g^{(\alpha+1)/2}(s). \tag{3.17}$$

It suffices to show that for ν sufficiently small the integrand on the right hand side of (3.16) is positive for all $\eta > 0$, which in turn would imply that $g' + 2\eta g > 0$ for all $\eta > 0$ and the lemma would follow.

Note that $h(0) = \nu^{(3-\alpha)/2}((\alpha - 3)/(\alpha - 1) + 2\lambda \nu^{\alpha - 1})$. Thus $h(0) > 0$ if

$$\nu < \left(\frac{\alpha - 3}{(\alpha - 1)(-2\lambda)} \right)^{1/(\alpha-1)} \tag{3.18}$$

Hence there exists a neighbourhood $(0, \varepsilon)$ such that h is positive on $(0, \varepsilon)$. We know from Remark 3 that g cannot have a zero before the integrand of the right hand side of (3.16) becomes zero. This implies that as long as the integrand in (3.16) is non-negative, $g > 0$ and hence $h(s)$ in (3.17) is well defined.

We will show that for ν sufficiently small, $h(s)$ cannot become zero on $[0, \infty)$. By (3.17) and (3.1),

$$h'(\eta) = -\frac{(\alpha - 3)^2}{2(\alpha - 1)} g^{(1-\alpha)/2} g' - 2^{3/2} \eta g' - \frac{2^{5/2}}{\alpha - 1} g$$
$$+ (4 + \lambda(\alpha + 1)) g^{(\alpha-1)/2} g' + 2^{5/2} \lambda g^\alpha. \tag{3.19}$$

Let η_1 be the first zero of h, if possible. Then necessarily $h'(\eta_1) \leq 0$. Further

$$g'(\eta_1) = -\frac{1}{2^{1/2}} \frac{(\alpha - 3)}{(\alpha - 1)} g^{(3-\alpha)/2}(\eta_1) - 2^{1/2} \lambda g^{(\alpha+1)/2}(\eta_1). \tag{3.20}$$

Using (3.20) in (3.19) and dropping the second term in (3.19), we get

$$h'(\eta_1) > \frac{1}{2^{3/2}}\frac{(\alpha-3)^3}{(\alpha-1)^2}g^{(2-\alpha)}(\eta_1) - 2^{1/2}\lambda^2(\alpha+1)g^{\alpha}(\eta_1)$$
$$-\frac{1}{\alpha-1}(2^{3/2}(\lambda+1)(\alpha-3)+2^{5/2})g(\eta_1)$$
$$> g^{2-\alpha}(\eta_1)\left(\frac{(\alpha-3)^3}{2^{3/2}(\alpha-1)^2} - 2^{1/2}\lambda^2(\alpha+1)g^{2(\alpha-1)}\right.$$
$$\left.-\frac{1}{(\alpha-1)}(2^{3/2}(\alpha-3)+2^{5/2})g^{\alpha-1}(\eta_1)\right)$$

Thus, since $g(\eta_1) < \nu$,

$$h'(\eta_1) > g^{2-\alpha}(\eta_1)\left(\frac{(\alpha-3)^3}{2^{3/2}(\alpha-1)^2} - 2^{1/2}\lambda^2(\alpha+1)\nu^{2(\alpha-1)}\right.$$
$$\left.-\frac{1}{(\alpha-1)}(2^{3/2}(\alpha-3)+2^{5/2})\nu^{\alpha-1}\right).$$

Choose ν sufficiently small so that $h'(\eta_1) > 0$ and the condition (3.18) is satisfied. Thus h cannot have a zero for ν sufficiently small. Hence the lemma. ■

Now we look for situations where the solutions of the IVP (3.1)–(3.2) become zero at a finite η.

LEMMA 3.11 *Assume that* $1 < \alpha \leq 3$ *and* $\lambda \leq 0$. *Then all the solutions of the IVP* (3.1)–(3.2) *have a finite zero.*

Proof By the differential Eq. (3.1),

$$g'(\eta) = -2\eta g + \frac{2(\alpha-3)}{(\alpha-1)}\int_0^{\eta} g\,ds + \frac{2^{5/2}}{\alpha+1}g^{(\alpha+1)/2}$$
$$-\frac{2^{5/2}}{\alpha+1}\nu^{(\alpha+1)/2} + 4\lambda\int_0^{\eta} g^{\alpha}ds. \tag{3.21}$$

Suppose $g > 0$ for all η for contradiction. Then

$$g' < -K, \quad \text{for} \quad \eta > \eta_1,$$

where K is a positive number. This implies that g cannot remain positive, a contradiction. Thus g must have a finite zero. ■

LEMMA 3.12 *Assume that* $\alpha > 3$. *For* $|\lambda|$ *sufficiently small, there exists a* ν *such that the solution of* (3.1)–(3.2) *has a finite zero.*

Proof By Theorem A in Section 1, there exists a solution $g_0(\eta, \nu_0)$ of (3.1)–(3.2) such that $g_0(\eta_1, \nu_0) < 0$ for $\alpha > 3$ and $\lambda = 0$. By using the continuous dependence of the solutions on λ, there exists a sufficiently small λ such that $g(\eta_1, \nu_0) < 0$. Hence the lemma. ■

In the next three lemmas, we show that there exists some ν for which the corresponding solution of the IVP (3.1)–(3.2) has exponential decay at $\eta = +\infty$ when $\alpha > 3$ and λ sufficiently small.

Suppose $\nu \in \mathbb{R}^+$. Define

$S_1 = \{\nu > 0 : g(\eta, \nu)$ is positive on $[0, \eta_1]$ and $g(\eta_1, \nu) = 0$ for some $\eta_1 > 0\}$,

$S_2 = \{\nu > 0 : g(\eta, \nu)$ is positive on $(0, \infty)$ and decays algebraically to zero as $\eta \to \infty\}$, and

$S_3 = \{\nu > 0 : g(\eta, \nu)$ is positive on $(0, \infty)$ and decays exponentially to zero as $\eta \to \infty\}$

Clearly S_1, S_2, and S_3 are disjoint. Also, $S_1 \cup S_2 \cup S_3 = (0, \infty)$.

LEMMA 3.13 S_1 *is open.*

Proof Suppose $g(\eta, \nu_1)$ has a finite zero at $\eta = \eta_1$. Then by the uniqueness of the solutions, we have $g'(\eta_1, \nu_1) \neq 0$ and hence $g'(\eta_1, \nu_1) < 0$. This implies that there exists $\eta_2 > \eta_1$ such that $g(\eta_2, \nu_1) < 0$. By the continuous dependence on the initial data, there exists a neighbourhood U around ν_1 such that for all ν in U, $g(\eta_2, \nu) < 0$. Hence the proof is complete. ■

LEMMA 3.14 S_2 *is open.*

Proof Suppose $\nu_1 \in S_2$. By the definition of S_2, for any $\mu > 0$, there exists $\eta_\mu > \tau_\mu$ (see Eqs. (3.7) and (3.8)) such that $(g(\eta_\mu, \nu_1), g'(\eta_\mu, \nu_1)) \in D_\mu^1$ or D_μ^2. By the continuous dependence on the initial data, there exists a neighbourhood U around ν_1 such that $(g(\eta_\mu, \nu), g'(\eta_\mu, \nu)) \in D_\mu^1$ or D_μ^2 for all $\nu \in U$. It follows from Lemmas 3.4 and

3.5 that $(g, g') \in D_\mu^1$ or D_μ^2 for all $\eta \geq \eta_\mu$. This implies that g decays algebraically to zero as $\eta \to \infty$. ■

LEMMA 3.15 *Assume that either* $\alpha > 3$, $\lambda \in \mathbb{R}$ *and* $|\lambda|$ *is sufficiently small. Then there exists an exponentially decaying solution as* $\eta \to \infty$.

Proof Lemmas 3.7, 3.8, 3.9, 3.12, 3.13, 3.14 imply that S_1, S_2 are non-empty and open. Since $(0, \infty)$ is connected, there exists $\nu^* \notin S_1 \cup S_2$. Thus $g(\eta, \nu^*)$ decays exponentially to zero as $\eta \to \infty$. ■

3.2. Analysis of IVP (3.1)–(3.2) on $(-\infty, 0)$

Here, we consider IVP (3.1)–(3.2) for $\eta \leq 0$. Let $\eta = -s$. Then the differential Eq. (3.1) becomes

$$g_{ss} + 2sg_s + \frac{4}{\alpha - 1} g + 2^{3/2} g^{(\alpha-1)/2} g_s - 4\lambda g^\alpha = 0. \tag{3.22}$$

The initial conditions are

$$g(0) = \nu, \quad g'(0) = 0. \tag{3.23}$$

Notice that (3.22) and (3.1) differ only in the sign of the term $2^{3/2} g^{(\alpha-1)/2} g_s$. Therefore the analysis regarding the local extrema is the same as before. In particular a lemma analogous to Lemma 3.2 is true here also. Further the above term on integration yields $cg^{(\alpha+1)/2}$ where c is a constant. In the analysis of a solution $g > 0$ for all $s > 0$ and $g \to \bar{g} \geq 0$ as $s \to \infty$, this term remains bounded. Hence following the analysis of Lemma 3.3, we can show that $g \to 0$ as $s \to \infty$. It is easy to see that $g \to 0$ as $s \to \infty$ implies that $g' \to 0$ as $s \to \infty$. Thus we have the following lemma.

LEMMA 3.16 *Assume that* $\alpha > 1$ *and* $\lambda \in \mathbb{R}$. *If g is a solution of* (3.22)–(3.23),

(i) $g' < 0$ *as long as* $g > 0$ *when* $\lambda \leq 0$ *or* $\lambda > 0$ *and* $g(0) < \nu_0$ (*see Eq.* (2.3)).

Using the trapping region argument for $\lambda \leq 0$ and $\lambda > 0$ as in Lemmas 3.4 and 3.5, respectively, we can prove the asymptotic behaviour as in Lemma 3.6, when $s \to \infty$. Thus we have

LEMMA 3.17 (Asymptotic behaviour as $s \to \infty$) *Assume that* $\alpha > 1$ *and* $\lambda \in R$. *Suppose* $g(s, \nu) \geq 0$ *for all* $s \geq 0$. *Then*

(i) *If* $\lim_{s\to\infty} g'/g = 0$, g *decays algebraically to zero as* $s \to \infty$ *and* (1.9) *gives the asymptotic behaviour for* g *with* η *replaced by* s.
(ii) *If* $\lim_{s\to\infty} g'/g = -\infty$, g *decays exponentially to zero as* $s \to \infty$ *and the asymptotic expansion is given by* (1.8) *with* η *replaced by* s.

For the case $\alpha > 3$ and $\lambda > 0$, we can prove that every solution of the IVP (3.22)–(3.23) decays algebraically to zero as $s \to \infty$ for any $\nu > 0$.

LEMMA 3.18 *Assume that* $\alpha > 3$ *and* $\lambda > 0$. *If* g *is a solution of the IVP* (3.22)–(3.23), g *is positive and decays algebraically to zero as* $s \to \infty$, *for* $\nu > 0$.

Proof An integration of (3.22) with initial conditions (3.23) gives

$$g' + 2sg = \int_0^s \left(\frac{2(\alpha - 3)}{(\alpha - 1)} g - 2^{3/2} g^{(\alpha-1)/2} g' + 4\lambda g^{\alpha} \right) dt. \tag{3.24}$$

We claim that g is positive for all $s \geq 0$ and decays algebraically to zero as $s \to \infty$.

Suppose, on the contrary, that $g > 0$ on $(0, s_1)$ and $g(s_1) = 0$. Then by Eq. (3.24), we have

$$g'(s_1) = \int_0^{s_1} \left(\frac{2(\alpha - 3)}{(\alpha - 1)} g - 2^{3/2} g^{(\alpha-1)/2} g' + 4\lambda g^{\alpha} \right) dt > 0,$$

since $g' < 0$, $g \geq 0$ on $(0, s_1)$ and $\alpha > 3$, $\lambda > 0$. On the otherhand $g'(s_1) \leq 0$, because s_1 is the first zero of g. We arrive at a contradiction. Hence g is positive on $(0, \infty)$.

Since $g' < 0$ when $g > 0$ by Lemma 3.16, it follows from (3.24) that $g' + 2sg > 0$ for all $s \geq 0$. This in turn implies that $g \exp(s^2)$ is an increasing function on $(0, \infty)$. Hence by Lemma 3.17, and the arguments as in Remark 2, it follows that g decays algebraically to zero as $s \to \infty$. Hence the lemma. ■

In the following lemma we give the existence of a positive solution with algebraic decay as $s \to \infty$, for the case $\alpha > 3$ and $\lambda < 0$.

LEMMA 3.19 *Assume that* $\alpha > 3$ *and* $\lambda < 0$. *For* ν *sufficiently small,* $g(s, \nu)$ *decays algebraically to zero as* $s \to \infty$.

Proof From (3.24), if $\nu < ((\alpha-3)/(-2\lambda(\alpha-1)))^{1/(\alpha-1)}$, we have $g'+2sg>0$ provided that $g>0$. This implies that g is positive and decays algebraically to zero as $s\to\infty$. ■

Theorem 1 is a consequence of Lemmas 3.7, 3.9 and 3.18. Theorem 2 follows from Lemmas 3.15 and 3.18. Theorem 3 results from Lemmas 3.8 and 3.18, while Theorem 4 follows from Lemmas 3.10 and 3.19. Theorem 5 is a simple consequence of Lemma 3.11.

4. CONCLUSIONS

In this paper, we have studied the self-similar form of solutions of GBE (1.1) governed by (1.6), subject to the initial conditions at $x=0$. We have shown the existence of bounded, positive solutions with different types of decay (exponential or algebraic) to zero as $x\to+\infty$ or $x\to-\infty$ for different sets of parameters α and λ. The analysis shows a rich structure of solutions.

References

[1] Ablowitz, M. J., Ramani, A. and Segur, H. (1978). Nonlinear evolution equations and ordinary differential equations of Painlevé type, *Lett. Nuovo Cimento*, **23**, 333–338.

[2] Ablowitz, M. J., Ramani, A. and Segur, H. (1980). A connection between nonlinear evolution equations and ordinary differential equations of P-type. I., *J. Math. Phys.*, **21**, 715–721.

[3] Brezis, H., Peletier, L. A. and Terman, D. (1986). A very singular solution of the heat equation with absorption, *Arch. Rat. Mech. Anal.*, **95**, 185–209.

[4] Kamke, E. (1943). Differential Gleichungen: Lösungsmethoden und Lösungen. Akademische Verlagsgesellschaft, Leipzig.

[5] Lardner, R. W. and Arya, J. C. (1980). Two generalisations of Burgers' equation, *Acta Mechanica*, **37**, 179–190.

[6] Peletier, L. A. and Serafini, H. C. (1995). A very singular solution and other self-similar solutions of the heat equation with convection, *Nonlin. Anal. Th. Meth. Appl.*, **24**, 29–49.

[7] Sachdev, P. L. (1991). Nonlinear ordinary differential equations and their applications, Marcel Dekker, Inc., New York.

[8] Sachdev, P. L. (1997). A compendium on nonlinear ordinary differential equations. John Wiley & Sons, Inc., New York.

[9] Sachdev, P. L., Nair, K. R. C. and Tikekar, V. G. (1986). Generalized Burgers equations and Euler–Painlevé transcendents I, *J. Math. Phys.*, **27**, 1506–1522.

[10] Sachdev, P. L. and Nair, K. R. C. (1987). Generalized Burgers equations and Euler–Painlevé transcendents II, *J. Math. Phys.*, **28**, 997–1004.

[11] Sachdev, P. L., Nair, K. R. C. and Tikekar, V. G. (1988). Generalized Burgers equations and Euler – Painlevé transcendents III, *J. Math. Phys.*, **29**, 2397–2404.

[12] Srinivasa Rao, Ch., Sachdev, P. L. and Mythily Ramaswamy (1999). Self-similar solutions of a generalized Burgers equation with nonlinear damping, communicated.
[13] Srinivasa Rao, Ch., Sachdev, P. L. and Mythily Ramaswamy (1999). Analysis of the self-similar solutions of the nonplanar Burgers equation, communicated.