GAPS IN DENSE SIDON SETS

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Abstract

We prove that if $A \subset [1, N]$ is a Sidon set with $N^{1/2} - L$ elements, then any interval $I \subset [1, N]$ of length cN contains $c|A| + E_I$ elements of A, with $|E_I| \le 52N^{1/4}(1 + c^{1/2}N^{1/8})(1 + L_+^{1/2}N^{-1/8})$, $L_+ = \max\{0, L\}$. In particular, if $|A| = N^{1/2} + O(N^{1/4})$, and $g(A)$ is the maximum gap in A, we deduce that $g(A) \ll N^{3/4}$. Also we prove that, under this condition, the exponent 3/4 is sharp.

1. Introduction

We say that A is a Sidon set if all the sums $a + a'$, $a \le a'$, are different. Erdős and Turan [5] proved that if $A \subset [1, N]$ is a Sidon set then $|A| \le N^{1/2} + O(N^{1/4})$. On the other hand, Bose and Chowla [1] proved that if $N = p^2 + p + 1$, then there exists a Sidon set $A \subset [1, N]$ with p elements; i.e, the upper bound (1.1) is sharp except for the error term.

Sidon sets of large size have notable properties of regularity. In [7], M. Koluntzakis proved that the elements of a Sidon set of large size, $|A| \sim N^{1/2}$, are well distribuited in the classes of residues of small modulo. See [5] for an elementary proof of this result.

Erdős and Freud [4] proved that if $|A| \sim N^{1/2}$ then the elements of A are well distributed in the interval $[1, N]$.

Theorem A (Erdős-Freud). *Let $c > 0$ and $A \subset [1, N]$ a Sidon set with $|A| \sim N^{1/2}$ elements. Then, any interval of length cN contains $\sim cN^{1/2}$ elements.*

S.W. Graham [6] has proved a more precise result.

Theorem B (S. Graham). *Let $A \subset [1, N]$ be a Sidon set with $N^{1/2} + O(N^{1/4})$ elements. Then, any interval of length cN contains $cN^{1/2} + O(N^{3/8})$ elements.*

If we denote by $g(A) = \max_{a_{k-1}, a_k \in A}\{a_k - a_{k-1}\}$ the maximum gap in A, from the Theorem B it is easy to deduce that if A is a Sidon set $A \subset [1, N]$ with $N^{1/2} + O(N^{1/4})$, then $g(A) \ll N^{7/8}$.

In this paper we shall use an identity (Lemma 2.1), which was introduced in [2] and [3], to obtain a better result.

Theorem 1.1. *Let $A \subset [1, N]$ a Sidon set with $N^{1/2} - L$ elements. Then, any interval of length cN contains $c|A| + E_I$ elements of A, with*

$$|E_I| \leq 52N^{1/4}(1 + c^{1/2}N^{1/8})(1 + L_+^{1/2}N^{-1/8}), \quad L_+ = \max\{0, L\}.$$

In particular we deduce from this theorem the following corollary for gaps.

Corollary 1.1. *If $A \subset [1, N]$ is a Sidon set and $|A| = N^{1/2} + O(N^{1/4})$, then $g(A) \ll N^{3/4}$.*

It is easy to see that the exponent 3/4 is the best possible if $A \subset [1, N]$ is a Sidon set with $|A| = N^{1/2} + O(N^{1/4})$. Consider $N = p^2 + p + 1$, and a Sidon set A, $A \subset [1, N]$ with $p \geq \sqrt{N} - 1$ elements. If we split the interval $[1, N]$ in intervals of length $[N^{3/4}]$, then, one of them contains less than $2N^{1/4}$ elements. If we remove these elements from A we have a Sidon set A' with $|A'| = N^{1/2} + O(N^{1/4})$ elements and $g(A') \gg N^{3/4}$.

We don't know how to derive a better estimate for $g(A)$ when the error term is less than $N^{1/4}$. It is related with the difficulty of improving the error term in the upper bound for finite Sidon sets. It would be interesting to know a good upper bound for $g(A)$ when A is a Sidon set of maximal size. Maybe, it is possible an upper bound like $g(A) \ll N^{1/2+\epsilon}$.

It should be noted that the classical construction of Erdős and Turan [5] of Sidon sets, $A_p = \{2kp + (k^2)_p : k = 0, 1, \ldots, p - 1\}$, gives $g(A) \ll N^{1/2}$ for these sets. It seems not to be the case for the Ruzsa's construction [8] of finite Sidon sets. Numerical and heuristic arguments suggest that $g(A)/N^{1/2} \to \infty$ in this case. In particular, it would imply that the Erdős's Conjecture, $F(N) \leq N^{1/2} + O(1)$, is not true.

2. Proofs

The proof of the following lemma can be found in [2] or [3].

Lemma 2.1. *Let $A \subset [1, N]$ be a sequence of integers. Then, for any integer $H \geq 1$ we have*

$$2 \sum_{1 \leq h \leq H} d(h)(H - h) = \frac{H^2|A|^2}{N + H - 1} - H|A| + D_H,$$

where

$$D_H = \sum_{1\le n\le N+H-1} \left(A(n) - A(n-H) - \frac{H|A|}{N+H-1}\right)^2 ,$$

$A(n)$ *is the counting function of* A *and* $d(h) = \#\{h = a - a'; \quad a, a' \in A\}$. □

$A(n) - A(n-H)$ is the number of elements of A lying in the interval $(n-H, n]$ and the quantity $\frac{H|A|}{N+H-1}$ is the expected value of $A(n) - A(n-H)$. Then, D_H is a measure of the distribution of the elements of A in the interval $[1, N+H-1]$.

The argument of the proof of the Theorem 1.1 is the following: If $|A|$ is close to $N^{1/2}$, (L small), then D_H is "small" and consequently, the number of elements of A lying in intervals of length H is "close", at least in average, to the expected number. From that we can deduce a "good" distribution in any interval $I = (\alpha N, \beta N]$. Upper and lower bounds for the error $E_I = |A \cap I| - (\beta - \alpha)|A|$ are obtained in two different steps (Lemma 2.3 and Lemma 2.4).

Lemma 2.2. *If* $A \subset [1, N]$ *is a Sidon set with* $|A| = N^{1/2} - L$ *then, for any integer* H *we have*

$$D_H \le \frac{3H^2 L_+}{N^{1/2}} + \frac{H^3}{N} + 2HN^{1/2}$$

where $L_+ = \max\{0, L\}$.

Proof. We apply Lemma 2.1 to the sequence A. Since A is a Sidon set, hence $d(h) \le 1$ for any integer $h \ge 1$ and $2\sum_{1\le h\le H-1} d(h)(H-h) \le H^2$. Also we use the trivial estimate for the size of a Sidon set, $|A| \le 2N^{1/2}$.

$$D_H \le H^2 - \frac{H^2|A|^2}{N+H-1} + H|A| = \frac{H^2N + H^3 - H^2 - H^2|A|^2}{N+H-1} + H|A|$$

$$\le \tfrac{H^2(N-|A|^2)}{N} + \tfrac{H^3}{N} + 2HN^{1/2}.$$

If $L \le 0$, then $D_H \le \frac{H^3}{N} + 2HN^{1/2}$.

If $L > 0$, then $D_H \le \frac{H^2}{N}(N^{1/2} + |A|)L_+ + \frac{H^3}{N} + 2HN^{1/2} \le \frac{3H^2L_+}{N^{1/2}} + \frac{H^3}{N} + 2HN^{1/2}$. □

Let $I = (\alpha N, \beta N]$, $c = \beta - \alpha$ and we write $|A \cap I| = c|A| + E_I$. We will choose $H = [N^{3/4}]$ in all the proofs.

Lemma 2.3. $E_I \leq 10N^{1/4}(c^{1/2}N^{1/8}+1)(L_+^{1/2}N^{-1/8}+1)$.

Proof. We write $I_H = (\alpha N, \beta N + H]$, then $cN + H - 1 \leq |I_H| \leq cN + H + 1$. We have

$$\sum_{n\in I_H} A(n) - A(n-H) \geq H|A\cap I|,$$

since each $a \in A \cap I$ is counted H times in the sum. Then,

$$\sum_{n\in I_H}\left(A(n) - A(n-H) - \frac{H|A|}{N+H-1}\right) \geq H|A\cap I| - \frac{|I_H|H|A|}{N+H-1}$$

$$= E_I H + H|A|\left(c - \frac{|I_H|}{N+H-1}\right) \geq E_I H - H|A|\frac{(1-c)(H+1)}{N+H-1} \geq E_I H - \frac{H^2|A|}{N}.$$

Then

$$E_I \leq H^{-1}\sum_{n\in I_H}\left(A(n) - A(n-H) - \frac{H|A|}{N+H-1}\right) + \frac{H|A|}{N}.$$

Now we apply Cauchy's inequality, Lemma 2.1 and the trivial estimates $|A| \leq 2N^{1/2}$, $N^{3/4}/2 \leq H \leq N^{3/4}$ to get

$$E_I \leq H^{-1}|I_H|^{1/2}D_H^{1/2} + \frac{H|A|}{N}$$

$$\leq H^{-1}\left((cN)^{1/2} + (H+1)^{1/2}\right)\left(\frac{\sqrt{3}HL_+^{1/2}}{N^{1/4}} + \frac{H^{3/2}}{N^{1/2}} + \sqrt{2}H^{1/2}N^{1/4}\right) + \frac{H|A|}{N}$$

$$\leq 2N^{-3/4}\left(c^{1/2}N^{1/2} + \sqrt{2}N^{3/8}\right)\left(\sqrt{3}N^{1/2}L_+^{1/2} + N^{5/8} + \sqrt{2}N^{5/8}\right) + 2N^{1/4}$$

$$\leq 10N^{1/4}\left(c^{1/2}N^{1/8}+1\right)\left(L_+^{1/2}N^{-1/8}+1\right). \quad \square$$

Lemma 2.4. $-E_I \leq 52N^{1/4}(c^{1/2}N^{1/8}+1)(L_+^{1/2}N^{-1/8}+1)$.

Proof.

$$\sum_{n\in I_H} A(n) - A(n-H) \leq H\left(|A\cap I| + |A\cap(\alpha N - H, \alpha N]| + |A\cap(\beta N, \beta N + H]|\right).$$

We apply Lemma 2.3 to the intervals $(\alpha N - H, \alpha N]$ and $(\beta N, \beta N + H]$ to obtain an upper bound for the last two terms.

$$|A\cap(\alpha N - H, \alpha N]| + |A\cap(\beta N, \beta N + H]| \leq 2\frac{H}{N}|A| + 20N^{1/4}\left(\frac{H^{1/2}N^{1/8}}{N^{1/2}}+1\right)\left(L_+^{1/2}N^{-1/8}+1\right)$$

$$\leq 4N^{1/4} + 40N^{1/4}(L_+^{1/2}N^{-1/8}+1) \leq 44N^{1/4} + 40N^{1/8}L_+^{1/2}.$$

Then,

$$\sum_{n\in I_H}\left(A(n)-A(n-H)-\frac{H|A|}{N+H-1}\right)\le H|A\cap I|-\frac{|I_H|H|A|}{N+H-1}+H\left(44N^{1/4}+40N^{1/8}L_+^{1/2}\right)$$

$$=E_IH+H|A|\left(c-\frac{|I_H|}{N+H-1}\right)+H\left(44N^{1/4}+40N^{1/8}L_+^{1/2}\right)$$

$$\le E_IH+H(44N^{1/4}+40N^{1/8}L_+^{1/2}),$$

because $|I_H|\ge cN+H-1$.

Finally we apply Cauchy inequality and Lemma 2.2 to obtain

$$-E_I\le 44N^{1/4}+40N^{1/8}L_+^{1/2}+H^{-1}\sum_{n\in I_H}\left|A(n)-A(n-H)-\frac{H|A|}{N+H-1}\right|$$

$$\le 44N^{1/4}+40N^{1/8}L_+^{1/2}+2N^{-3/4}|I_H|^{1/2}D_H^{1/2}$$

$$\le 44N^{\frac14}+40N^{\frac18}L_+^{\frac12}+2N^{\frac{-3}{4}}\left((cN)^{1/2}+(H+1)^{1/2}\right)\left(\frac{\sqrt{3}HL_+^{1/2}}{N^{1/4}}+\frac{H^{3/2}}{N^{1/2}}+\sqrt{2}H^{1/2}N^{1/4}\right)$$

$$\le 44N^{1/4}+40N^{1/8}L_+^{1/2}+2N^{-3/4}\left(c^{1/2}N^{1/2}+\sqrt{2}N^{3/8}\right)\left(\sqrt{3}N^{1/2}L_+^{1/2}+N^{5/8}+\sqrt{2}N^{5/8}\right)$$

$$\le 52N^{1/4}(1+c^{1/2}N^{1/8})(1+L_+^{1/2}N^{-1/8}).\quad\square$$

Lemma 2.3 and Lemma 2.4 imply Theorem 1.1. To prove Corollary 1.1, suppose that $A=N^{1/2}-L$, with $L_+\le kN^{1/4}$, and let I be any interval of length $k'N^{3/4}$. If we apply Lemma 2.4 we have

$$|A\cap I|>\frac{k'}{N^{1/4}}|A|-52N^{1/4}(1+k'^{1/2})(1+k^{1/2})>k'N^{1/4}-kk'-52N^{1/4}(1+k'^{1/2})(1+k^{1/2}).$$

If we take k' large enough, $k'>10000k$, then $|A\cap I|>0$ for any interval of length greater than $k'N^{3/4}$.

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