A UNIFIED APPROACH TO THE STUDY OF GENERAL AND PALINDROMIC COMPOSITIONS [1]

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Abstract

We study many properties of compositions of integers by using the symbolic method, multivariate generating functions and Riordan arrays. In particular, we study palindromic compositions with respect to various parameters and present a bijection between walks, compositions and bit strings. The results obtained for compositions can thus be exported to the corresponding objects. We also present a conjecture about a combinatorial interpretation of an algebraic relation involving the number of compositions and the square of the number of palindromic compositions.

Keywords: Compositions, bit strings, palindromes, generating functions, Riordan arrays.

1. Introduction

In the area of combinatorial number theory, particular attention has been given to partitions and compositions of integers. A *composition* of a positive integer n is an ordered sequence of positive integers whose sum is n. Compositions for which the sequence from

[1] or, *Everything you always wanted to know about compositions.*

partitions	compositions
4	**4**
3+1	3+1 1+3
2+2	**2+2**
2+1+1	2+1+1 **1+2+1** 1+1+2
1+1+1+1	**1+1+1+1**

Table 1.1: Partitions and compositions of 4: palindromic compositions are marked in bold

left to right is the same as from right to left are called *palindromic*. Sequences having sum n and considered without order, give the *partitions* of n. As an example, in Table 1.1, partitions and compositions of number 4 are reported. A first well-known result is about the number of compositions: as shown for example in [12], the number of compositions of n is 2^{n-1} and those with m summands are $\binom{n-1}{m-1}$. In the last years, many studies investigated compositions of n having specific properties: in [8], Grimaldi counts the compositions of n which do not contain the summand 1; in [6] Chinn and Heubach study compositions without 2 having j summands and palindromic compositions. In [3] the three previous authors together show various ways of generating all palindromic compositions and count the number of times each integer appears as a summand among the palindromic composition of n. In another paper, Chinn and Heubach [5], explore the problem of how many compositions of n exist with no occurrences of summand i. Recently, Heubach and Mansour [9] derived properties for compositions and palindromic compositions with parts from a general set. Most of these results are proved from a combinatorial point of view.

We were puzzled by the specificity of the results given in the literature and we wondered if there exists a more general way to study some properties of compositions. We did not want just a way to count compositions without 1's, another way to count compositions without 2's, etc., but a way to count compositions with or without $k_1, k_2, \ldots, k_s$ or having j summands, or being palindromic, or even having a particular summand in a specific position. As we will show in Section 2, we are able to give an answer to our problems using the symbolic method (see [7, 13], where the method is explained and applied in the context of bit strings and compositions, respectively). In particular, multivariate generating functions allow us to represent properties of compositions and to enumerate them according to the desired characteristics: the generating functions shown in Theorems 2.1 and 2.6 allow us to find any kind of information about general and palindromic compositions by differentiation and evaluation in specific values of the variables. Our approach is algebraic and, as we will show in Section 4, we are often able to give a combinatorial interpretation of the involved algebraic formulas. As particular cases, we find again the results of [3, 4, 5, 6, 8].

Another aspect we wish to point out is that the enumeration of compositions with respect to many parameters is related to the concept of Riordan matrices (see, e.g. [10]). We believe this relation is interesting on its own, but can also be used to easily extract some coefficients of generating functions and to compute combinatorial sums by a simple transformation of generating functions (see formula 2.3). The relation between compositions and Riordan arrays is treated in Section 2 while, in Section 4, we illustrate some examples in which the Riordan array approach is used.

In the literature, attention has been given to combinatorial structures which can be associated to compositions: the simplest graphical representation of compositions are *bargraphs* (see, e.g., [2]). A bargraph represents a composition by a sequence of columns composed by cells, where column j has k cells if and only if the j-th summand in the composition is equal to k: this object is actually a column convex polyomino whose lower edge lies on the horizontal axis. Figure 1.1 illustrates the bargraphs corresponding to compositions of number 4.

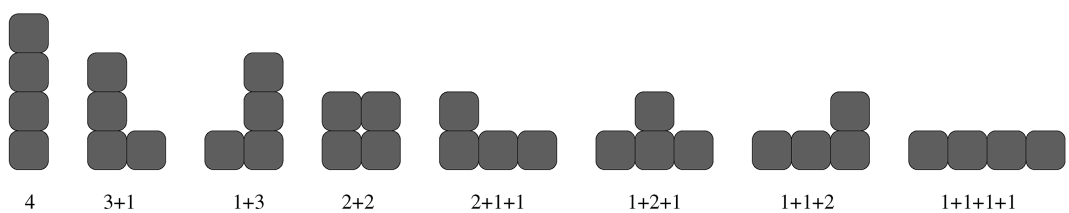

Figure 1.1: Bargraphs associated to compositions of number 4

Bargraphs are not the only combinatorial structure which can be associated to compositions and, in Section 3, we present a bijection between compositions (and therefore bargraphs) and bit strings, or, equivalently, with walks on the square lattice made up of two kinds of steps. Thanks to this correspondence, we can extend the results obtained for compositions (bargraphs) to bit strings and walks. For example, we can count walks with or without k consecutive equal steps and bit strings with or without runs of length k (a *k length run* is a maximal sequence of k consecutive 0 or k consecutive 1); we can count the number of the sequences of k equal steps contained into the walks of length n and many other properties of walks and bit strings. Moreover we present a conjecture which gives a combinatorial interpretation of an algebraic formula relating the number of compositions to the square of the number of palindromic compositions (see Theorem 2.7). Finally, in Section 4, we manipulate the multivariate generating functions examined in Section 2 thus finding many properties for compositions and palindromic compositions. In this way, we find and unify certain results on compositions, palindromic compositions and bit strings, scattered in the literature and present new results which extend and generalize the previous ones.

2. Compositions, palindromic compositions and Riordan arrays

A composition λ of a positive integer n is an ordered collection of integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ such that $\sum_{i=1}^{m} \lambda_i = n$; m is said to be the length of λ (we also say that λ has m terms or summands). Let $\mathcal{C}_m$ be the class of compositions of length m: $\mathcal{C}_m$ can be seen as the Cartesian product of the set $\mathcal{N} = \{1, 2, 3, \ldots\}$, m times:

$$\mathcal{C}_m = \underbrace{\mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N}}_{m}.$$

The symbolic method (see, e.g., [7, 13]), or equivalently the Schützenberger methodology, allows us to derive functional relationships on generating functions directly from the constructive definitions of the objects of the combinatorial class. Let $f(t) = t + t^2 + t^3 + \cdots = t/(1-t)$ be the generating function associated with $\mathcal{N}$, and $f_m(t)$ the generating function for the compositions of length m, then we have:

$$f_m(t) = \sum_{n \geq 0} C_{n,m} t^n = f(t)^m = \left(\frac{t}{1-t} \right)^m,$$

where we define $C_{0,0} = 1$ and $C_{0,m} = 0$ for $m > 0$. Hence the number of compositions of n in m terms is

$$C_{n,m} = \binom{n-1}{m-1}.$$

In order to find the generating function for compositions with no restrictions on the number of parts we have to sum over m and we get:

$$F(t) = \sum_{m \geq 0} f_m(t) = \sum_{n \geq 0} C_n t^n = \frac{1-t}{1-2t},$$

where $C_0 = 1$ and $C_n = 2^{n-1}$ for $n > 0$.

We can also define the bivariate generating function $F(t, z)$ for the number of composition of n with m terms:

$$F(t, z) = \sum_{m \geq 0} f(t)^m z^m = \frac{1}{1 - zf(t)} = \frac{1}{1 - \frac{zt}{1-t}}.$$

The previous results are well known and can be found for example in Riordan [12].

The numbers $C_{n,m}$ constitute a matrix (see Table 2.2) which in the literature is known as a *Riordan array*.

The concept of Riordan array was introduced in 1991 by Shapiro, Getu, Woan and Woodson [14] (they chose this name in honour of John Riordan) with the aim of defining a class of infinite lower triangular arrays having properties analogous to those of the Pascal triangle. This concept has been successively studied in [10, 15].

n/m	0	1	2	3	4	5
0	1					
1	0	1				
2	0	1	1			
3	0	1	2	1		
4	0	1	3	3	1	
5	0	1	4	6	4	1

Table 2.2: Numbers $C_{n,m}$

A Riordan array is an infinite lower triangular array $\{d_{n,m}\}_{n,m\ \in \mathbf{N}}$, defined by a pair of formal power series $D = (d(t), h(t))$, such that the generic element $d_{n,m}$ is the n-th coefficient in the series $d(t)(th(t))^m$:

$$d_{n,m} = [t^n]d(t)(th(t))^m, \qquad n, m \geq 0. \tag{2.1}$$

From this definition we have $d_{n,m} = 0$ for $m > n$. The bivariate generating function of a Riordan array is given by:

$$d(t, z) = \sum_{n,m\geq 0} d_{n,m}t^n z^m = \frac{d(t)}{1 - zth(t)}. \tag{2.2}$$

In the sequel we always assume that $d(0) \neq 0$; if we also have $h(0) \neq 0$ then the Riordan array is said to be *proper*. These matrices have interesting properties, for example, if $D = (d(t), h(t))$ is a Riordan array and $f(t)$ is the generating function for the sequence $\{f_m\}_{m\in\mathbf{N}}$, then every combinatorial sum involving these numbers can be computed as follows:

$$\sum_{m=0}^{n} d_{n,m}f_m = [t^n]d(t)f(th(t)). \tag{2.3}$$

In this context, the number of compositions of n with m terms $\{C_{n,m}\}_{n,m\in\mathbf{N}}$ corresponds to the Riordan array $C = (1, 1/(1 - t))$, as can be easily checked by the previous considerations. As we shall see, the enumeration of compositions from different points of view gives rise to many Riordan matrices.

Coming back to our initial reasoning, when we perform the power $f(t)^m$ we loose information about the kind of terms constituting the composition. To keep this information, we introduce an infinite number of indeterminates denoting the particular terms of the composition (an analogous approach has been used in [7, Chapter III, Example 2]). More precisely, let $w = (w_1, w_2, w_3, \ldots)$ and

$$\hat{f}(t, w) = \sum_{k=1}^{\infty} w_k t^k;$$

this generating function enumerates the whole class $\mathcal{N}$ and also keeps track of each integer k with the variable w_k. Similarly, we can define $\hat{f}_m(t,w) = \hat{f}(t,w)^m$ and

$$\hat{F}(t,z,w) = \sum_{m\geq 0} \hat{f}(t,w)^m z^m = \frac{1}{1 - z\sum_{k=1}^{\infty} w_k t^k} =$$

$$= 1 + w_1 zt + \left(w_2 z + {w_1}^2 z^2\right) t^2 + \left(w_3 z + 2\, w_1 w_2 z^2 + {w_1}^3 z^3\right) t^3 + O\left(t^4\right).$$

For example, when $n = 3$ we have the compositions

$$3, \quad 2+1, \quad 1+2, \quad 1+1+1$$

and this result is enclosed in the coefficient of t^3 in $\hat{F}(t,z,w)$, where the exponent of z marks the length of the composition and the summands are marked by the various w_k without taking care of their order. The Riordan array nature of the coefficients of $\hat{F}(t,z,w)$ is clear: they, in fact, correspond to the array $C = (1, \sum_{k=0}^{\infty} w_{k+1} t^k)$, shown in Table 2.3.

n/m	0	1	2	3	4	5
0	1					
1	0	w_1				
2	0	w_2	w_1^2			
3	0	w_3	$2w_1w_2$	w_1^3		
4	0	w_4	$2w_1w_3 + w_2^2$	$3w_1^2w_2$	w_1^4	
5	0	w_5	$2w_1w_4 + 2w_2w_3$	$3w_1^2w_3 + 3w_1w_2^2$	$4w_1^3w_2$	w_1^5

Table 2.3: Numbers $C_{n,m}$ in terms of w_k's

We can summarize the previous results in the following theorem:

Theorem 2.1 *Let $\hat{F}(t,z,w)$ be the multivariate generating functions for the compositions of n with m terms where each w_k denotes the presence of term k in the composition. Then we have:*

$$\hat{F}(t,z,w) = \frac{1}{1 - z\sum_{k=1}^{\infty} w_k t^k},$$

that is, the coefficients $[t^n z^m]\hat{F}(t,z,w)$ correspond to the elements of the Riordan array $(1, \sum_{k=0}^{\infty} w_{k+1} t^k)$.

These simple arguments allow us to find many properties of compositions by specializing some of the variables as will be shown in Section 4.

We use the previous formulas to find some enumerative results on *palindromic compositions*. A palindromic composition is a composition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ for which the sequence from left to right is the same as from right to left, i.e.:

$$\lambda_i = \lambda_{m-i+1}, \quad \forall i = 1, \ldots, m.$$

The number of palindromic compositions of n with m terms satisfies the following theorems.

Theorem 2.2 *Let $P_{n,m}$ be the number of palindromic compositions of n with m terms. Then we have:*

$$P_{2n,2m} = C_{n,m}, \quad n \geq 0,$$

$$P_{2n+1,2m} = 0, \quad n \geq 0,$$

$$P_{2n,2m+1} = \sum_{k=1}^{n} C_{n-k,m}, \quad n \geq 1,$$

$$P_{2n+1,2m+1} = \sum_{k=0}^{n} C_{n-k,m}, \quad n \geq 0.$$

Proof. The first two formulas in the statement of the theorem are trivial since a palindromic composition with an even number of summands corresponds to two copies of a same composition. Instead, a palindromic composition with an odd number of summands can be seen as two copies of a same composition with a summand in the middle which can be of any value. More precisely, let $p_1 \ldots p_m y p_m \ldots p_1$ be a palindromic composition of number s, that is, $2\sum_{i=1}^{m} p_i + y = s$ with $y \geq 1$ and $p_i \geq 1$, $1 \leq i \leq m$. The summand y can be an even or an odd number:

- if $y = 2k$, with $k \geq 1$, then number $s = 2(\sum_{i=1}^{m} p_i + k)$ is even, say $s = 2n$, and $\sum_{i=1}^{m} p_i = n - k$. Hence, the number $P_{2n,2m+1}$ of palindromic compositions of $2n$ with $2m+1$ summands can be found by summing over $k \geq 1$ the number of compositions of $n-k$ with m terms.
- if $y = 2k+1$, with $k \geq 0$, then number $s = 2(\sum_{i=1}^{m} p_i + k) + 1$ is odd, say $s = 2n+1$, and $\sum_{i=1}^{m} p_i = n - k$. Hence, the number $P_{2n+1,2m+1}$ of palindromic compositions of $2n+1$ with $2m+1$ summands can be found by summing over $k \geq 0$ the number of compositions of $n-k$ with m terms. □

Theorem 2.3 *Let $P_m(t)$ be the generating functions for the palindromic compositions of length m. Then we have:*

$$P_{2m}(t) = f_m(t^2) = \left(\frac{t^2}{1-t^2}\right)^m,$$

$$P_{2m+1}(t) = \frac{t}{1-t} f_m(t^2) = \frac{t}{1-t}\left(\frac{t^2}{1-t^2}\right)^m,$$

that is, functions $P_{2m}(t)$ and $P_{2m+1}(t)$ correspond to the column generating functions of the Riordan arrays

$$\left(1, \frac{t}{1-t^2}\right), \quad \left(\frac{t}{1-t}, \frac{t}{1-t^2}\right),$$

respectively.

Proof. For the palindromic compositions of length $2m$, by using Theorem 2.2, we simply have:

$$P_{2m}(t) = \sum_{n\geq 0} P_{2n,2m}t^{2n} = \sum_{n\geq 0} C_{n,m}t^{2n} = f_m(t^2) = \left(\frac{t^2}{1-t^2}\right)^m.$$

The computation of the generating function $P_{2m+1}(t)$ for the palindromic compositions of length $2m+1$ is a bit more complicated. In fact we have:

$$\begin{aligned} P_{2m+1}(t) &= \sum_{n\geq 0} P_{n,2m+1}t^n = \sum_{n\geq 1} P_{2n,2m+1}t^{2n} + \sum_{n\geq 0} P_{2n+1,2m+1}t^{2n+1} = \\ &= \sum_{n\geq 1}(\sum_{k=1}^{n} C_{n-k,m})t^{2n} + \sum_{n\geq 0}(\sum_{k=0}^{n} C_{n-k,m})t^{2n+1} = \\ &= \sum_{n\geq 0}\left((\sum_{k=0}^{n} C_{n-k,m})t^{2n} - C_{n,m}t^{2n}\right) + \sum_{n\geq 0}(\sum_{k=0}^{n} C_{n-k,m})t^{2n+1} = \\ &= \sum_{n\geq 0}(\sum_{k=0}^{n} C_{k,m})t^{2n} + \sum_{n\geq 0}(\sum_{k=0}^{n} C_{k,m})t^{2n+1} - \sum_{n\geq 0} C_{n,m}t^{2n} = \\ &= \frac{1}{1-t^2}f_m(t^2) + \frac{t}{1-t^2}f_m(t^2) - f_m(t^2) = \frac{t}{1-t}f_m(t^2) = \frac{t}{1-t}\left(\frac{t^2}{1-t^2}\right)^m. \end{aligned}$$

The Riordan array nature of $P_{2m}(t)$ and $P_{2m+1}(t)$ can be deduced directly from (2.1). □

In order to find the total number of palindromic compositions of n with m terms we have to extract the $[t^n]$ coefficient from $P_m(t)$. Alternatively, one could proceed by directly computing a closed form for the sums in the statement of Theorem 2.2. First, we consider palindromic compositions of even numbers with an odd number of summands:

$$\begin{aligned} P_{2n,2m+1} &= [t^{2n}]\frac{t}{1-t}\left(\frac{t^2}{1-t^2}\right)^m = [t^{2n-2m}](t+t^2)(1-t^2)^{-(m+1)} = \\ &= [t^{2(n-m)-1}](1-t^2)^{-(m+1)} + [t^{2(n-m-1)}](1-t^2)^{-(m+1)} = \\ &= 0 + [y^{(n-m-1)}](1-y)^{-m-1} = \\ &= \binom{-m-1}{n-m-1}(-1)^{n-m-1} = \binom{n-1}{m} \end{aligned}$$

Then, we compute the number of palindromic compositions of odd numbers with an odd number of summands.

$$\begin{aligned} P_{2n+1,2m+1} &= [t^{2n+1}]\frac{t}{1-t}\left(\frac{t^2}{1-t^2}\right)^m = [t^{2n+1-2m}](t+t^2)(1-t^2)^{-(m+1)} = \\ &= [t^{2(n-m)}](1-t^2)^{-(m+1)} + [t^{2(n-m)-1}](1-t^2)^{-(m+1)} = \\ &= [y^{(n-m)}](1-y)^{-m-1} + 0 = \binom{-m-1}{n-m}(-1)^{n-m} = \binom{n}{m}. \end{aligned}$$

Summarizing, we have the proof of the following theorem:

Theorem 2.4 *The number $P_{n,m}$ of palindromic compositions of n with m summands satisfies:*

$$P_{n,m} = \begin{cases} \binom{k-1}{s-1} & \text{if } n = 2k, m = 2s \\ \binom{k-1}{s} & \text{if } n = 2k, m = 2s+1 \\ \binom{k}{s} & \text{if } n = 2k+1, m = 2s+1 \\ 0 & \text{if } n = 2k+1, m = 2s \end{cases}$$

As done in Theorem 2.1, we can keep track of each integer k with the variable w_k also in the case of palindromic compositions.

Theorem 2.5 *Let $w = (w_1, w_2, w_3 \ldots)$, $w^2 = (w_1^2, w_2^2, w_3^2 \ldots)$, and let $\hat{P}_m(t, w)$ be the generating function for the palindromic compositions of length m, where each w_k denotes the presence of term k in the composition. Then we have:*

$$\hat{P}_{2m}(t, w) = \hat{f}_m(t^2, w^2) = \left(\sum_{k=1}^{\infty} w_k^2 t^{2k}\right)^m,$$

$$\hat{P}_{2m+1}(t, w) = \hat{f}(t, w)\hat{f}_m(t^2, w^2) = \left(\sum_{k=1}^{\infty} w_k t^k\right)\left(\sum_{k=1}^{\infty} w_k^2 t^{2k}\right)^m,$$

that is, functions $\hat{P}_{2m}(t)$ and $\hat{P}_{2m+1}(t)$ correspond to the column generating functions of the Riordan arrays:

$$\left(1, \sum_{k=1}^{\infty} w_k^2 t^{2k-1}\right), \quad \left(\sum_{k=1}^{\infty} w_k t^k, \sum_{k=1}^{\infty} w_k^2 t^{2k-1}\right),$$

respectively.

Proof. If we want to keep track of each integer k with the variable w_k we can proceed as follows. A palindromic composition with an even number of summands corresponds to two copies of a same composition, hence, each summand is repeated twice and this can be marked with a w_k^2. In Theorem 2.3 we found $P_{2m}(t) = f_m(t^2)$ hence we have $\hat{P}_{2m}(t, w) = \hat{f}_m(t^2, w^2)$. When the palindromic composition has an odd number of summands things are more complicated. We can follow the proof of Theorem 2.2 by taking into consideration a palindromic composition $p_1 \ldots p_m y p_m \ldots p_1$ with $2m+1$ terms and by distinguishing the cases $y = 2k$ and $y = 2k+1$: these two summands can be marked with w_{2k} and w_{2k+1}, respectively. We begin by keeping track of the summand in the middle of the palindromic composition and proceed as in Theorem 2.3, assuming $w_0 = 1$:

$$\sum_{n\geq 1}(\sum_{k=1}^{n} w_{2k} C_{n-k,m}) t^{2n} + \sum_{n\geq 0}(\sum_{k=0}^{n} w_{2k+1} C_{n-k,m}) t^{2n+1} =$$

$$= \sum_{n \geq 1} \left((\sum_{k=0}^{n} w_{2k} C_{n-k,m}) t^{2n} - w_0 C_{n,m} t^{2n} \right) + \sum_{n \geq 0} (\sum_{k=0}^{n} w_{2k+1} C_{n-k,m}) t^{2n+1} =$$

$$= \sum_{n \geq 0} \left((\sum_{k=0}^{n} w_{2k} C_{n-k,m}) t^{2n} - C_{n,m} t^{2n} \right) + \sum_{n \geq 0} (\sum_{k=0}^{n} w_{2k+1} C_{n-k,m}) t^{2n+1} =$$

$$= \sum_{n \geq 0} (\sum_{k=0}^{n} w_{2k} C_{n-k,m}) t^{2n} + \sum_{n \geq 0} (\sum_{k=0}^{n} w_{2k+1} C_{n-k,m}) t^{2n+1} - \sum_{n \geq 0} C_{n,m} t^{2n} =$$

$$= (1 + w_2 t^2 + w_4 t^4 + \cdots) f_m(t^2) + t(w_1 + w_3 t^2 + w_5 t^4 \cdots) f_m(t^2) - f_m(t^2) =$$

$$= \left(\sum_{k=1}^{\infty} w_k t^k \right) f_m(t^2) = \hat{f}(t,w) f_m(t^2).$$

In order to mark all the summands we need to substitute $f_m(t^2)$ with $\hat{f}_m(t^2, w^2)$ in the previous formula and we obtain the desired expression for $\hat{P}_{2m+1}(t,w)$. The relation with Riordan arrays comes from (2.1). □

Theorem 2.6 *Let $\hat{P}(t,z,w)$ be the multivariate generating functions for the palindromic compositions of n with m terms where each w_k denotes the presence of term k in the composition. Then we have:*

$$\hat{P}(t,z,w) = \frac{1 + z \sum_{k=1}^{\infty} w_k t^k}{1 - z^2 \sum_{k=1}^{\infty} w_k^2 t^{2k}}.$$

Proof. The proof follows from Theorem 2.5:

$$\hat{P}(t,z,w) = \sum_{m \geq 0} \hat{P}_{2m}(t,w) z^{2m} + \sum_{m \geq 0} \hat{P}_{2m+1}(t,w) z^{2m+1} =$$

$$= \sum_{m \geq 0} \left(\sum_{k=1}^{\infty} w_k^2 t^{2k} \right)^m z^{2m} + \sum_{m \geq 0} \left(\sum_{k=1}^{\infty} w_k t^k \right) \left(\sum_{k=1}^{\infty} w_k^2 t^{2k} \right)^m z^{2m+1} =$$

$$= \sum_{m \geq 0} \left(z^2 \sum_{k=1}^{\infty} w_k^2 t^{2k} \right)^m + z \left(\sum_{k=1}^{\infty} w_k t^k \right) \sum_{m \geq 0} \left(z^2 \sum_{k=1}^{\infty} w_k^2 t^{2k} \right)^m =$$

$$= \left(1 + z \sum_{k=1}^{\infty} w_k t^k \right) \frac{1}{1 - z^2 \sum_{k=1}^{\infty} w_k^2 t^{2k}}.$$

□

By developing $\hat{P}(t,z,w)$ into series we find:

$$\hat{P}(t,z,w) = 1 + w_1 z t + \left({w_1}^2 z^2 + w_2 z \right) t^2 + \left({w_1}^3 z^3 + w_3 z \right) t^3 +$$

$$+ \left({w_1}^4 z^4 + {w_1}^2 w_2 z^3 + {w_2}^2 z^2 + w_4 z \right) t^4 + O(t^5).$$

and, for example, the coefficient of t^4 can be easily checked with Figure (1.1) and Table (1.1).

The total number P_n of palindromic compositions of n can be found by extracting the $[t^n]$ coefficient from $\hat{P}(t,1,1)$. We have:

$$P_n = [t^n]\hat{P}(t,1,1) = [t^n]\frac{1+\frac{t}{1-t}}{1-\frac{t^2}{1-t^2}} = [t^n]\frac{(1-t+t)(1-t^2)}{(1-t)(1-t^2-t^2)} = [t^n]\frac{1+t}{1-2t^2}.$$

In order to extract this coefficient it is convenient to distinguish between even and odd values of n :

$$P_{2k} = [t^{2k}]\frac{1}{1-2t^2} + [t^{2k-1}]\frac{1}{1-2t^2} = [t^{2k}]\frac{1}{1-2t^2} + 0 = [t^k]\frac{1}{1-2t} = 2^k,$$

$$P_{2k+1} = [t^{2k+1}]\frac{1}{1-2t^2} + [t^{2k}]\frac{1}{1-2t^2} = 0 + [t^k]\frac{1}{1-2t} = 2^k.$$

These results can also be found in [3, Theorem 4]. Finally, if we remember that the number of compositions of number n is $C_n = 2^{n-1}$, we can conclude this section with the following:

Theorem 2.7 *The total number P_n of palindromic compositions of n satisfies the following formula:*

$$P_n = 2^{\lfloor n/2 \rfloor},$$

hence, the following relation between compositions and palindromic compositions holds true:

$$C_n = \begin{cases} \frac{1}{2}P_n^2 & \text{if } n = 2k \\ P_n^2 & \text{if } n = 2k+1 \end{cases}. \tag{2.4}$$

Obviously, the ratio P_n/C_n gives the probability to find a palindromic composition among the compositions of number n.

Theorem 2.7 gives a nice formula involving the number of compositions and the number of palindromic compositions: we believe this is not casual, in the sense that we expect a combinatorial interpretation of this relation can be found. In the next section, we will illustrate a conjecture about a possible combinatorial interpretation of formula (2.4).

3. Compositions, bargraphs, walks and bit strings

As we said in the Introduction, compositions of integers can be easily seen as bargraphs by associating to each summand λ_i in the composition a column of λ_i cells in the bargraph. In addition, a bargraph can be easily seen as a pair of walks on the integer lattice $\mathbf{N} \times \mathbf{Z}$ starting at the origin and made up of north-east (N) and south-east (S) steps. If we look at a bargraph, we can obtain the corresponding pair of walks by associating a step to each cell in the following way: by starting from the left, we associate λ_i steps of the same type (N or S) to the λ_i cells of the i-th column; when we change column, we change the type of step. Depending on the starting step (N or S), we obtain two walks, symmetric with respect to the x-axis, having as length the area, i.e. the total number of cells, of the corresponding bargraph.

Since a walk can be simply viewed as a bit string (by associating 1 to a N step and 0 to a S step), we can extend the previous correspondence to bit strings too, obtaining a correlation between walks, compositions and bit strings. Figure 3.2 illustrates a composition of number 15, the associated bargraph and the corresponding pairs of walks and bit strings. As pointed out in the figure with the mark of summand 3, the bijection preserves a correspondence between components of the objects. So, for example, when considering bargraphs, walks and bit strings associated to a composition of n, we can state that:

- the area of each bargraph (i.e. the number of cells), the length of the walks and the number of bits in the bit strings are n;
- the number of columns in a bargraph corresponds to i) the number of summands in the composition, ii) the number of rises and drops in the walks and iii) the number of runs in the bit strings;
- the number of k length runs in the bit strings is twice i) the number of terms equal to k in the composition, ii) the number of columns of length k in the bargraph, iii) the number of rises or drops having length k in the walks.

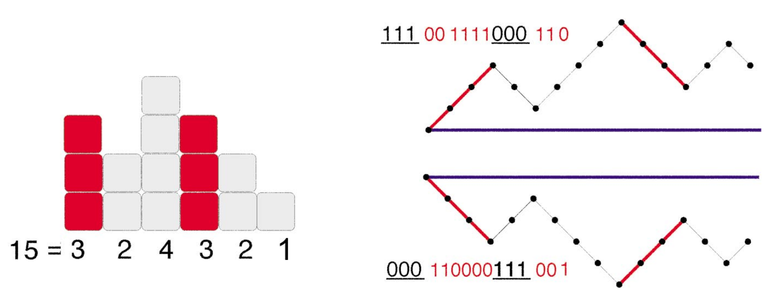

Figure 3.2: A composition of 15 and the corresponding bargraph, walks and bit strings

When a palindromic composition has odd length, the corresponding bit strings are also palindromic while, in the case of even length, the bit strings becomes palindromic if one swaps 0 and 1 in the second half of the bit strings. Figure 3.3 illustrates an example of two palindromic correspondences between compositions, bargraphs, walks and bit strings.

Thanks to this correspondence, formula (2.4) can be read as a relation between area of bargraphs and the number of palindromic bargraphs. In fact, since the area of a bargraph

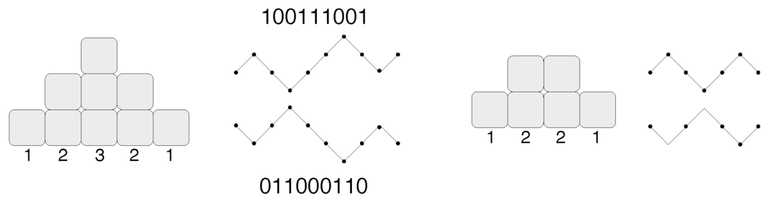

Figure 3.3: Two palindromic compositions and the corresponding objects

associated to a composition of n is obviously n, then the total area A_n of the bargraphs corresponding to the C_n compositions of n is nC_n. Using Theorem 2.7 we obtain

$$A_{2k+1} = (2k+1)P_{2k+1}^2$$

$$2A_{2k} = (2k)P_{2k}^2.$$

These relations, proved from an algebraic point of view, imply a possible combinatorial interpretation. In fact, they suggest that a rectangle having dimensions $P_n \times nP_n$ is equivalent to the total area of bargraphs corresponding to the C_n compositions of n if n is odd, and to the double of the same total area if n is even. These considerations led us to suppose that the rectangle $P_n \times nP_n$ could be exactly tiled with the C_n bargraphs if n is odd and with the C_n bargraphs used twice, if n is even. In fact, for small values of n we verified the following conjecture:

Conjecture 3.8 *For each integer n there exists at least one tiling of the rectangle having dimensions P_n and nP_n using:*

- *the C_n bargraphs corresponding to the compositions of n, if n is odd;*
- *the C_n bargraphs corresponding to the compositions of n used twice, if n is even.*

Each bargraph can be rotated.

Figures 3.4 and 3.5 illustrate a possible tiling satisfying the conjecture in the cases $n = 4$ and $n = 5$ respectively.

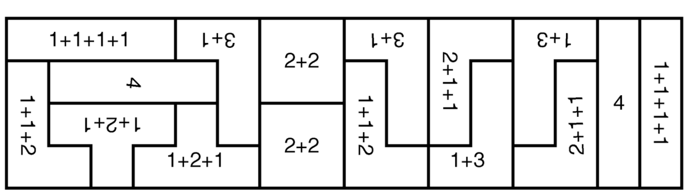

Figure 3.4: A tiling of the $P_4 \times 4P_4$ rectangle using twice the $C_4 = 8$ bargraphs

This conjecture is similar to the one illustrated in the paper of Woan et al. [16]; as far as we know, the conjecture is still unsolved.

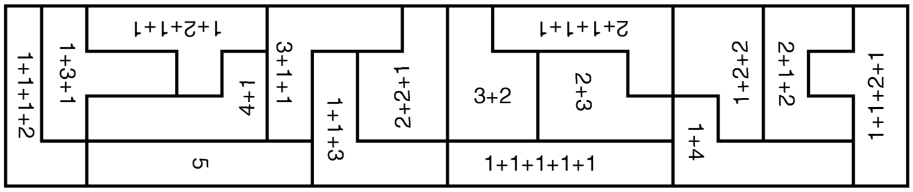

Figure 3.5: A tiling of the $P_5 \times 5P_5$ rectangle using the $C_5 = 16$ bargraphs

4. Some results on compositions, bit strings and palindromic compositions

In Section 2 we have found the generating functions $\hat{F}(t,z,w)$ and $\hat{P}(t,z,w)$ whose $[t^n z^m]$ coefficient counts the number of general and palindromic compositions of n with exactly m terms. These coefficients are expressed in terms of the indeterminate w_i^j which represents summand i repeated j times. By differentiating and/or evaluating in specific values of these indeterminate it is possible to find many properties about compositions. Moreover, thanks to the bijection illustrated in Section 3, the counting results on compositions can be easily exported to walks and bit strings. The following subsections collect some examples which point out the generality of our approach.

4.1 Compositions

By using the results of Section 2, we are able to find many properties of compositions, starting from the trivial result $[t^n]\hat{F}(t,1,1) = 2^{n-1}$, $n \geq 1$, where, from now on, $w = 1$ stands for $w = (w_1, w_2, w_3, \ldots) = (1,1,1,\ldots)$.

We can go on and count the total number of summands i in all the compositions of number n. To this purpose, we have to differentiate with respect to w_i and then put $w = 1$ in $\hat{F}(t,z,w)$; we get the following generating function:

$$S^{[i]}(t) = \left[\frac{\partial}{\partial w_i}\hat{F}(t,1,w) \;\Big|\; w = 1\right] = \frac{t^i(1-t)^2}{(1-2t)^2}. \tag{4.5}$$

The $[t^n]$ coefficient can be extracted by using the following relation:

$$[t^n]\frac{1}{(1-2t)^2} = \binom{-2}{n}(-2)^n = \binom{2+n-1}{n}2^n = \frac{(n+1)!}{n!}2^n = (n+1)2^n \tag{4.6}$$

obtaining:

$$S_n^{[i]} = [t^n]\frac{t^i(1-t)^2}{(1-2t)^2} =$$

$$= (n-i+1)2^{n-i} - 2(n-i)2^{n-i-1} + (n-i-1)2^{n-i-2} = (n-i+3)2^{n-i-2},$$

with initial conditions $S_n^{[i]} = 0$ if $n < i$, $S_i^{[i]} = 1$ and $S_{i+1}^{[i]} = 2$.

Equivalently, if we want to count the total number of summands $i_1, i_2, \ldots, i_k$ in the compositions of n, we simply extend the previous formula:

$$S^{[i_1,i_2,\ldots,i_k]}(t) = \sum_{j=1}^{k}\left[\frac{\partial}{\partial w_{i_j}}\hat{F}(t,1,w)\ \Big|\ w=1\right] = \frac{(t^{i_1}+t^{i_2}+\ldots+t^{i_k})(1-t)^2}{(1-2t)^2}. \qquad (4.7)$$

In order to find the number of m length compositions without term i, we have to compute $\hat{F}(t,z,w)$ by substituting w_i with 0, w_j with 1 for $j \neq i$, and extracting the coefficient $[t^n z^m]$ from the function:

$$C^{[\bar{i}]}(t,z) = \left[\hat{F}(t,z,w)\Big|\ w_i = 0, w_j = 1,\ j \neq i\right] = \frac{1}{1 - z\frac{t-t^i+t^{i+1}}{1-t}}.$$

When $i = 1$, by using properties (2.1) and (2.2) of Riordan arrays, we can extract the $[t^n z^m]$ coefficient from the previous function, obtaining the number of m length compositions of number n without summand 1:

$$\begin{aligned} C^{[\bar{1}]}_{n,m} = [t^n z^m]C^{[\bar{1}]}(t,z) &= [t^n]\left(\frac{t^2}{1-t}\right)^m = [t^{n-2m}](1-t)^{-m} = \binom{-m}{n-2m}(-1)^{n-2m} = \\ &= \binom{m+n-2m-1}{n-2m} = \binom{n-m-1}{m-1}. \end{aligned}$$

If $i > 1$ the computations are a bit more complicated, but the extraction of the coefficient $[t^n z^m]C^{[\bar{i}]}(t,z)$ can be found analogously as follows:

$$[t^n z^m]C^{[\bar{i}]}(t,z) = [t^n]\left(\frac{t-t^i+t^{i+1}}{1-t}\right)^m = [t^n]\left(\frac{t}{1-t} - t^i\right)^m =$$

$$= [t^n]\sum_{l=0}^{m}\binom{m}{l}\left(\frac{t}{1-t}\right)^l(-t^i)^{m-l} = \sum_{l=0}^{m}\binom{m}{l}(-1)^{m-l}[t^{n-l-i(m-l)}](1-t)^{-l} =$$

$$= \sum_{l=\lceil\frac{mi-n}{i-1}\rceil}^{m}\binom{m}{l}(-1)^{m-l}\binom{-l}{n-l-i(m-l)}(-1)^{n-l-i(m-l)},$$

so we have the following non closed expression:

$$C^{[\bar{i}]}_{n,m} = \sum_{l=\lceil\frac{mi-n}{i-1}\rceil}^{m}\binom{m}{l}(-1)^{m-l}\binom{n-1-i(m-l)}{l-1}. \qquad (4.8)$$

If we put $z = 1$ in $C^{[\bar{i}]}(t,z)$, we obtain the total number of compositions of n without summand i, thus we can generalize the results described in [6, 8]; in particular, the generating function:

$$C^{[\bar{1}]}(t) = C^{[\bar{1}]}(t,1) = \frac{1-t}{1-t-t^2} \qquad (4.9)$$

represents the sequence studied in paper [8]. In this case we obtain a link between the compositions without summand 1 and Fibonacci numbers (defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$):

$$C_n^{[\bar{1}]} = [t^n]\frac{1-t}{1-t-t^2} = [t^{n+1}]\frac{t}{1-t-t^2} - [t^n]\frac{t}{1-t-t^2} = F_{n+1} - F_n = F_{n-1}.$$

In an analogous way, we can study compositions without summand 2, obtaining the function studied in [6]:

$$C^{[\bar{2}]}(t) = C^{[\bar{2}]}(t,1) = \frac{1-t}{1-2t+t^2-t^3}.$$

We can extract the $[t^n]$ coefficient from both sides of the equality $(1-2t+t^2-t^3)C^{[\bar{2}]}(t) = 1-t$ thus finding the following recurrence relation for the number $C_n^{[\bar{2}]}$ of compositions of n without summand 2 :

$$C_{n+3}^{[\bar{2}]} = 2C_{n+2}^{[\bar{2}]} - C_{n+1}^{[\bar{2}]} + C_n^{[\bar{2}]},$$

with $C_0^{[\bar{2}]} = C_1^{[\bar{2}]} = C_2^{[\bar{2}]} = 1$.

More generally, the total number of compositions of n without the summand i has generating function (see also [5]):

$$C^{[\bar{i}]}(t) = C^{[\bar{i}]}(t,1) = \frac{1-t}{1-2t+t^i-t^{i+1}}$$

and a recurrence relation for the number $C_n^{[\bar{i}]}$ of compositions without summand i can be found as in the case $i = 2$. We have:

$$C_{n+i+1}^{[\bar{i}]} = 2C_{n+i}^{[\bar{i}]} - C_{n+1}^{[\bar{i}]} + C_n^{[\bar{i}]}, \tag{4.10}$$

with initial conditions $C_j^{[\bar{i}]} = C_j = 2^{j-1}$, for $1 \leq j < i$ and $C_i^{[\bar{i}]} = C_i - 1 = 2^{i-1} - 1$ (since there is only a composition of i containing i). This recurrence relation can be explained from a combinatorial point of view by generalizing the result given for $i = 2$ in [6, Theorem 1]. In fact, the compositions of $n + i + 1$ without summand i can be generated recursively as follows:

1) by appending a one to the last summand of the compositions of $n + i$ without i;

2) by increasing by one the last summand of the compositions of $n + i$ without i;

3) By applying rule 2) to compositions of $n + i$ without i ending in $i - 1$ we obtain a composition of $n + i + 1$ ending in i which are forbidden. On the other hand, they can be generated by applying i times rule 2) to compositions of $n + 1$ without i, so we have to delete $C_{n+1}^{[\bar{i}]}$ terms;

4) by appending $i+1$ to the compositions of n without i; in fact compositions ending in $i+1$ cannot be generated from rules 1) and 2).

This interpretation cannot be applied in the case $i = 1$ because of rule 1). In this case, a composition of $n+2$ without 1 can be simply obtained by applying rule 2) and 4).

We have proved formula (4.10) by an algebraic point of view, and from this result, we have been able to give it a combinatorial interpretation obtaining the same result as in [5]. With a similar approach, in the case of palindromic compositions without summand i, we succeeded in finding the new result (4.13).

Moreover, if we want to know the number of m length compositions of n without terms $i_1, i_2, \ldots, i_k$, we have to extract the coefficient of $[t^n z^m]$ from the function:

$$\begin{aligned} C^{\overline{[i_1,i_2,\ldots,i_k]}}(t,z) &= \left[\hat{F}(t,z,w)\Big|\; w_{i_j} = 0,\; j = 1,\ldots,k,\; w_h = 1 \text{ otherwise}\right] = \\ &= \frac{1}{1 - z\left(\frac{t}{1-t} - (t^{i_1} + t^{i_2} \ldots + t^{i_k})\right)}, \end{aligned}$$

hence the total number of compositions not containing $i_1, i_2, \ldots, i_k$ can be obtained by the coefficient of the generating function:

$$C^{\overline{[i_1,i_2,\ldots,i_k]}}(t,1) = \frac{1-t}{1 - 2t + (1-t)(t^{i_1} + t^{i_2} + \ldots + t^{i_k})}.$$

In the previous examples, we examined compositions without particular summands; in [4] the authors study compositions only containing terms 1 and s. More generally, we can easily obtain analogous results for compositions only containing summands r and s, $r < s$. This can be done by substituting, in $\hat{F}(t,z,w)$, $w_r = w_s = 1$ and $w_j = 0$ otherwise. In particular, the number $C_{n,m}^{[r,s]}$ of m length compositions of the number n only containing summands r and s can be obtained as follows:

$$C_{n,m}^{[r,s]} = [t^n z^m]\frac{1}{1 - z(t^r + t^s)} = [t^n z^m]\sum_{j=0}^{\infty} z^j (t^r + t^s)^j = [t^n](t^r + t^s)^m =$$

$$= [t^n]t^{rm}(1 + t^{s-r})^m = [t^{n-mr}](1 + t^{s-r})^m = \binom{m}{\frac{n-rm}{s-r}}.$$

The lower value of the previous binomial coefficient must be an integer value, so we derive the following formula:

$$C_{n,m}^{[r,s]} = \begin{cases} \binom{m}{\frac{n-rm}{s-r}} & \text{if } n - rm = j(s-r), 0 \le j \le m \\ 0 & \text{otherwise.} \end{cases}$$

This result appears in [11], where the authors study a particular tiling problem related to a generalization of the Fibonacci numbers. In fact, from the previous generating function, we can easily derive that the number $C_n^{[r,s]}$ of compositions of the number n only containing summands r and s satisfies the recurrence relation shown in Table 4.4.

C_n	$=$	2^{n-1}
$C_{n,m}$	$=$	$\binom{n-1}{m-1}$
$C_{n,m}^{[\bar{1}]}$	$=$	$\binom{n-m-1}{m-1}$
$C_n^{[\bar{1}]}$	$=$	F_{n-1}
$C_{n,m}^{[\bar{i}]}$	$=$	$\sum_{l\geq\lceil\frac{mi-n}{i-1}\rceil}\binom{m}{l}(-1)^{m-l}\binom{n-1-i(m-l)}{l-1}$
$C_j^{[\bar{i}]}$ $C_i^{[\bar{i}]}$ $C_{n+i+1}^{[\bar{i}]}$	$=$ $=$ $=$	$C_j = 2^{j-1}$ if $1 \leq j < i$ $2^{i-1}-1$ $2C_{n+i}^{[\bar{i}]} - C_{n+1}^{[\bar{i}]} + C_n^{[\bar{i}]}$
$C_{n,m}^{[r,s]}$	$=$	$\begin{cases} \binom{m}{\frac{n-rm}{s-r}} & \text{if } n-rm=j(s-r),\ 0\leq j\leq m \\ 0 & \text{otherwise.} \end{cases}$
$C_i^{[r,s]}$ $C_i^{[r,s]}$ $C_n^{[r,s]}$	$=$ $=$ $=$	0 $0<i<s$, and $i\neq jr$ 1 $0<i<s$, and $i=jr$, $0\leq j\leq\lfloor\frac{s}{r}\rfloor$ $C_{n-r}^{[r,s]}+C_{n-s}^{[r,s]}$ otherwise.

Table 4.4: Summarizing results for compositions

4.2 Bit strings

Some interesting results on bit strings can be obtained using the bijection explained in Section 3. As an example, Bloom in [1] studies the number of singles in all the 2^n n-length bit strings, where a *single* is any isolated 1 or 0, i.e., any run of length 1. By using our generating function, we can easily compute the number of singles in all the 2^{n-1} n-length bit strings beginning with 0 :

$$R^{[1]}(t,x) \quad = \quad \left[\hat{F}(t,1,w)\ \middle|\ w_1=x,\ w_j=1,\ j\geq 2\right] = \frac{\frac{1-t}{1-t-t^2}}{1-xt\frac{1-t}{1-t-t^2}};$$

the coefficient $[t^nx^k]R^{[1]}(t,x)$ gives the number of n-length bit strings beginning with 0 and having k singles. The same function obviously counts the n-length bit strings beginning with 1 and having k singles (see Figure 3.2). As explained in the previous section, the resulting matrix can be viewed as a Riordan array, with

$$d(t) = h(t) = \frac{1-t}{1-t-t^2},$$

and obviously each element differs from the corresponding element in Bloom's array by a factor 2, apart from $n = 0$ and $k = 0$ (see Table 4.5).

n/k	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	1	0	1					
3	1	2	0	1				
4	2	2	3	0	1			
5	3	5	3	4	0	1		
6	5	8	9	4	5	0	1	
7	8	15	15	14	5	6	0	1

Table 4.5: The number of n length bit strings beginning with 0, having k singles

We can generalize the previous result, counting the number of i length runs in the n length bit strings beginning with 0, as follows:

$$R^{[i]}(t,x) = \left[\hat{F}(t,1,w) \;\middle|\; w_i = x,\ w_j = 1,\ j \neq i\right] = \frac{\dfrac{1-t}{1-2t+t^i-t^{i+1}}}{1 - xt\dfrac{t^{i-1}(1-t)}{1-2t+t^i-t^{i+1}}}; \qquad (4.11)$$

the coefficient $[t^n x^k]R^{[i]}(t,x)$ counts the n-length bit strings beginning with 0 containing k i length runs. In this case we have the Riordan array:

$$d(t) = \frac{1-t}{1-2t+t^i-t^{i+1}},\ h(t) = \frac{t^{i-1}(1-t)}{1-2t+t^i-t^{i+1}}.$$

Moreover we can compute the total number of i length runs contained in the n length bit string beginning with 0. To do this we simply have to compute the weighted row sum of the previous Riordan array. To obtain this number, we can differentiate (4.11) with respect to x or use formula (2.3) with $f_m = m$:

$$\left[\frac{\partial}{\partial x}R^{[i]}(t,x) \;\middle|\; x = 1\right] = \frac{td(t)h(t)}{(1-th(t))^2} = \frac{t^i(1-t)^2}{(1-2t)^2}.$$

The previous function represents the Riordan array $D = \left(\frac{(1-t)^2}{(1-2t)^2}, 1\right)$; this function is exactly the generating function (4.5) because, for the bijection illustrated in Section 3, the total number of i length runs contained in the n length bit strings beginning with 0 corresponds to the total number of summands i in the compositions of the number n.

4.3 Palindromic compositions

We can now compute some results on palindromic compositions of n. In [8], Grimaldi studies the palindromes of n without summand 1. Starting from Theorem 2.6, we can easily obtain the generating function for the number $P_{n,m}^{[\bar{1}]}$ of m length palindromic compositions of n without the summand 1; this is the $[t^n z^m]$ coefficient of the generating function:

$$P^{[\bar{1}]}(t,z) = \left[\hat{P}(t,z,w) \;\middle|\; w_1 = 0,\ w_j = 1, j \geq 2\right] = \frac{1 + \frac{zt^2}{1-t}}{1 - z^2 t^2 \frac{t^2}{1-t^2}}. \tag{4.12}$$

This coefficient can be extracted by observing that $P^{[\bar{1}]}(t,z)$ can be decomposed as follows:

$$P^{[\bar{1}]}(t,z) = \frac{1}{1 - z^2 t^2 \frac{t^2}{1-t^2}} + zt^2 \frac{(1-t)^{-1}}{1 - z^2 t \frac{t^3}{1-t^2}} = F^{[\bar{1}]}(t^2, z^2) + zt^2 G^{[\bar{1}]}(t, z^2),$$

where $F^{[\bar{1}]}(t,z)$ and $G^{[\bar{1}]}(t,z)$ are the bivariate generating functions corresponding to the Riordan arrays:

$$d_{F^{[\bar{1}]}}(t) = 1, \quad h_{F^{[\bar{1}]}}(t) = \frac{t}{1-t}; \quad d_{G^{[\bar{1}]}}(t) = \frac{1}{1-t}, \quad h_{G^{[\bar{1}]}}(t) = \frac{t^3}{1-t^2}.$$

The coefficients of $F^{[\bar{1}]}(t,z)$ and $G^{[\bar{1}]}(t,z)$ can be computed by using relation (2.1). For $F^{[\bar{1}]}(t,z)$ we have:

$$[t^n z^m] F^{[\bar{1}]}(t,z) = [t^n] d_{F^{[\bar{1}]}}(t) (t h_{F^{[\bar{1}]}}(t))^m = [t^n] \left(\frac{t^2}{1-t}\right)^m =$$

$$= [t^{n-2m}](1-t)^{-m} = \binom{-m}{n-2m}(-1)^{n-2m} = \binom{n-m-1}{n-2m} = \binom{n-m-1}{m-1}.$$

Analogously, for $G^{[\bar{1}]}(t,z)$ we obtain:

$$[t^n z^m] G^{[\bar{1}]}(t,z) = [t^n] d_{G^{[\bar{1}]}}(t) (t h_{G^{[\bar{1}]}}(t))^m = [t^n] \frac{1}{1-t} \left(\frac{t^4}{1-t^2}\right)^m = [t^{n-4m}] \frac{1+t}{(1-t^2)^{m+1}} =$$

$$= [t^{n-4m}] \frac{1}{(1-t^2)^{m+1}} + [t^{n-4m-1}] \frac{1}{(1-t^2)^{m+1}} = \binom{\lfloor n/2 \rfloor - m}{m},$$

where the last passage can be easily verified by distinguishing the cases n even and n odd and observing that in both cases one of the coefficients is equal to zero. Coming back to $P^{[\bar{1}]}(t,z)$ we have:

$$P_{n,m}^{[\bar{1}]} = [t^n z^m] F^{[\bar{1}]}(t^2, z^2) + [t^{n-2} z^{m-1}] G^{[\bar{1}]}(t, z^2)$$

and we can distinguish between even and odd values of n and m. In particular, if $n = 2k$ and $m = 2s$ we have:

$$[t^{2k} z^{2s}] F^{[\bar{1}]}(t^2, z^2) + [t^{2(k-1)} z^{2s-1}] G^{[\bar{1}]}(t, z^2) = [t^k z^s] F^{[\bar{1}]}(t,z) + 0 = \binom{k-s-1}{s-1};$$

if $n = 2k$ and $m = 2s+1$ we find:

$$[t^{2k}z^{2s+1}]F^{[\bar{1}]}(t^2,z^2) + [t^{2(k-1)}z^{2s}]G^{[\bar{1}]}(t,z^2) = 0 + [t^{2(k-1)}z^s]G^{[\bar{1}]}(t,z) = \binom{k-s-1}{s};$$

$n = 2k+1$ and $m = 2s+1$ yield:

$$[t^{2k+1}z^{2s+1}]F^{[\bar{1}]}(t^2,z^2) + [t^{2k-1}z^{2s}]G^{[\bar{1}]}(t,z^2) = 0 + [t^{2k-1}z^s]G^{[\bar{1}]}(t,z) = \binom{k-s-1}{s};$$

finally, if $n = 2k+1$ and $m = 2s$ we obtain a zero. Putting everything together we have the following formula for the m length palindromic compositions of number n without the summand 1:

$$P^{[\bar{1}]}_{n,m} = \begin{cases} \binom{k-s-1}{s-1} & \text{if } n = 2k, m = 2s \\ \binom{k-s-1}{s} & \text{if } n = 2k+1, m = 2s+1 \text{ and } n = 2k, m = 2s+1 \\ 0 & \text{if } n = 2k+1, m = 2s \end{cases}$$

This result coincides with the one presented in [8]. The total number $P^{[\bar{1}]}_n$ of palindromic compositions of n without the summand 1 can be found by summing with respect to m the previous values or, more simply, by putting $z = 1$ in (4.12) and then extracting the coefficient $[t^n]$:

$$P^{[\bar{1}]}_n = [t^n]P^{[\bar{1}]}(t,1) = [t^n]\frac{1+t^3}{1-t^2-t^4} = [t^n]\frac{1}{1-t^2-t^4} + [t^{n-3}]\frac{1}{1-t^2-t^4}.$$

When $n = 2k$ we have $[t^{2k}]1/(1-t^2-t^4)+[t^{2k-3}]1/(1-t^2-t^4) = [t^k](1/(1-t-t^2)+0 = F_{k+1}$ while, when $n = 2k+1$, we obtain $[t^{2k+1}]1/(1-t^2-t^4) + [t^{2(k-1)}]1/(1-t^2-t^4) = 0+[t^{k-1}]1/(1-t-t^2) = F_k$, being F_k the k^{th} Fibonacci number. So, we have the following formula for the total number of palindromic composition of n without the summand 1 (see also [8]):

$$P^{[\bar{1}]}_n = \begin{cases} F_{k+1} & \text{if } n = 2k \\ F_k & \text{if } n = 2k+1 \end{cases}$$

We can go on, and study the number $P^{[\bar{2}]}_{n,m}$ of m length palindromic compositions of n without the summand 2, as in [6]; in this case we have the following generating function:

$$P^{[\bar{2}]}(t,z) = \left[\hat{P}(t,z,w) \;\middle|\; w_2 = 0,\ w_j = 1, j \neq 2\right] = \frac{1 + zt\frac{1-t+t^2}{1-t}}{1 - z^2t^2\frac{1-t^2+t^4}{1-t^2}}.$$

This function can be expressed in terms of two Riordan arrays, as in the case of summand 1. In fact we have:

$$P^{[\bar{2}]}(t,z) = F^{[\bar{2}]}(t^2,z^2) + ztG^{[\bar{2}]}(t,z^2)$$

where $F^{[\bar{2}]}(t,z)$ and $G^{[\bar{2}]}(t,z)$ are the bivariate generating functions corresponding to the Riordan arrays:

$$d_{F^{[\bar{2}]}}(t) = 1, \quad h_{F^{[\bar{2}]}}(t) = \frac{1-t+t^2}{1-t}; \quad d_{G^{[\bar{2}]}}(t) = \frac{1-t+t^2}{1-t}, \quad h_{G^{[\bar{2}]}}(t) = \frac{t(1-t^2+t^4)}{1-t^2}.$$

We can proceed by extracting the coefficients of previous bivariate generating functions. We have

$$f^{[\bar{2}]}_{n,m} = [t^n z^m]F^{[\bar{2}]}(t,z) = [t^n]\left(\frac{t-t^2+t^3}{1-t}\right)^m = C^{[\bar{2}]}_{n,m}$$

as can be easily verified proceeding as in the case $C^{[\bar{i}]}(t,z)$.

For what concerns $G^{[\bar{2}]}(t,z)$ we have:

$$g^{[\bar{2}]}_{n,m} = [t^n z^m]G^{[\bar{2}]}(t,z) = [t^n]\frac{1-t+t^2}{1-t}\left(\frac{t^2(1-t^2+t^4)}{1-t^2}\right)^m =$$

$$= [t^{n-2m}]\left(\frac{1}{1-t}-t\right)\left(\frac{1}{1-t^2}-t^2\right)^m = [t^{n-2m}]\left(\frac{1+t}{1-t^2}-t\right)\sum_{l\geq 0}\binom{m}{l}\frac{t^{2(m-l)}}{(1-t^2)^l}(-1)^{m-l} =$$

$$= [t^{n-2m}]\sum_{l\geq 0}\binom{m}{l}\frac{t^{2(m-l)}}{(1-t^2)^{l+1}}(-1)^{m-l} + [t^{n-2m}]\sum_{l\geq 0}\binom{m}{l}\frac{t^{2(m-l)+1}}{(1-t^2)^{l+1}}(-1)^{m-l} +$$

$$-[t^{n-2m}]\sum_{l\geq 0}\binom{m}{l}\frac{t^{2(m-l)+1}}{(1-t^2)^{l}}(-1)^{m-l}.$$

The extraction of these coefficients can be obtained by distinguishing between even and odd values of n and m as shown in the case of summand 1; the final result is:

$$g^{[\bar{2}]}_{n,m} = \sum_{l=2m-k}^{m}\binom{m}{l}(-1)^{m-l}\binom{k-2m+2l}{l} - \gamma^{[\bar{2}]}_{n,m}$$

where

$$\gamma^{[\bar{2}]}_{n,m} = \begin{cases} 0 & \text{if } n=2k \\ \sum_{l=2m-k}^{m}\binom{m}{l}(-1)^{m-l}\binom{k-2m+2l-1}{l-1} & \text{if } n=2k+1 \end{cases}$$

Finally we have:

$$P^{[\bar{2}]}_{n,m} = [t^n z^m]F^{[\bar{2}]}(t^2,z^2) + [t^{n-1}z^{m-1}]G^{[\bar{2}]}(t,z^2) = \begin{cases} f^{[\bar{2}]}_{k,s} & \text{if } n=2k, m=2s \\ g^{[\bar{2}]}_{2k-1,s} & \text{if } n=2k, m=2s+1 \\ g^{[\bar{2}]}_{2k,s} & \text{if } n=2k+1, m=2s+1 \\ 0 & \text{if } n=2k+1, m=2s \end{cases}$$

For what concerns the total number $P^{[\bar{2}]}_n$ of palindromic compositions of n without the summand 2 (see [6, Thorem 10]), their generating function is given by

$$P^{[\bar{2}]}(t,1) = \frac{1+t}{1-t^2-t^3},$$

and we can extract the coefficient $[t^n]$ from both sides of the equality $(1-t^2-t^3)P^{[\bar{2}]}(t,1) = 1+t$ thus finding the following recurrence relation:

$$P^{[\bar{2}]}_{n+3} = P^{[\bar{2}]}_n + P^{[\bar{2}]}_{n+1}, \quad P^{[\bar{2}]}_0 = P^{[\bar{2}]}_1 = P^{[\bar{2}]}_2 = 1.$$

In general, the number $P^{[\bar{i}]}_{n,m}$ of m length palindromic compositions of n without summand i has the following generating function:

$$P^{[\bar{i}]}(t,z) = \left[\hat{P}(t,z,w) \;\middle|\; w_i = 0,\ w_j = 1, j \neq i\right] = \frac{1 + zt\frac{1-t^{i-1}+t^i}{1-t}}{1 - z^2t^2\frac{1-t^{2(i-1)}+t^{2i}}{1-t^2}}.$$

The computation of $P^{[\bar{i}]}_{n,m}$ is quite heavy even if the sketch of the proof is analogous to the one shown in the case $i = 2$, so we give only the final result in Table 4.6.

The total number $P^{[\bar{i}]}_n$ of palindromic compositions of n without the summand i corresponds to the generating function:

$$P^{[\bar{i}]}(t,1) = \frac{1 + t - t^i + t^{i+2}}{1 - 2t^2 + t^{2i} - t^{2(i+1)}}.$$

A recurrence relation for the coefficients of this function can be found as in the case of summand 2. In particular we find:

$$P^{[\bar{i}]}_{n+2(i+1)} = 2P^{[\bar{i}]}_{n+2i} - P^{[\bar{i}]}_{n+2} + P^{[\bar{i}]}_n \tag{4.13}$$

with $P^{[\bar{i}]}_j = P_j = 2^{\lfloor j/2 \rfloor}$ for $1 \leq j \leq 2i$ and $P^{[\bar{i}]}_{2i+1} = P_{2i+1} - 1 = 2^{\lfloor (2i+1)/2 \rfloor} - 1$. We can give a combinatorial interpretation to the above recurrence, as done for the recurrence relation involving compositions without summand i. Since we are dealing with palindromic compositions, their generation can use the same rules of general compositions but in this case we need to apply them to both symmetric sides.

In general, for all palindromes which do not contain the terms $i_1, i_2, \ldots, i_k$, we have the generating function:

$$P^{[\overline{i_1,i_2,\ldots,i_k}]}(t,z) = \left[\hat{P}(t,z,w) \;\middle|\; w_{i_j} = 0, j = 1,\ldots,k,\ w_h = 1 \text{ otherwise}\right] =$$

$$= \frac{1 + z\left(\frac{t}{1-t} - t^{i_1} - t^{i_2} - \ldots - t^{i_k}\right)}{1 - z^2\left(\frac{t^2}{1-t^2} - t^{2i_1} - t^{2i_2} - \ldots - t^{2i_k}\right)}.$$

The results obtained at the end of Section 4.1 can be extended to palindromic compositions. In particular, the number $P^{[r,s]}_{n,m}$ of m length palindromic compositions of the number n only containing summands r and s can be obtained by extracting the following coefficient:

$$P^{[r,s]}_{n,m} = [t^n z^m]\frac{1 + z(t^r + t^s)}{1 - z^2(t^{2r} + t^{2s})} =$$

$$= [t^n z^m]F^{[r,s]}(t^2,z^2) + [t^{n-r}z^{m-1}]F^{[r,s]}(t^2,z^2) + [t^{n-s}z^{m-1}]F^{[r,s]}(t^2,z^2),$$

where $F^{[r,s]}(t,z)$ corresponds to the Riordan array:

$$d^{[r,s]}(t) = 1, \qquad h^{[r,s]}(t) = t^{r-1} + t^{s-1}.$$

P_n	$= 2^{\lfloor n/2 \rfloor}$
$P_{n,m}$	$= \begin{cases} 0 & \text{if } n = 2k+1, m = 2s \\ \binom{k-1}{s-1} & \text{if } n = 2k, m = 2s \\ \binom{k-1}{s} & \text{if } n = 2k, m = 2s+1 \\ \binom{k}{s} & \text{if } n = 2k+1, m = 2s+1 \end{cases}$
$P^{[\bar{1}]}_{n,m}$	$= \begin{cases} 0 & \text{if } n = 2k+1, m = 2s \\ \binom{k-s-1}{s-1} & \text{if } n = 2k, m = 2s \\ \binom{k-s-1}{s} & \text{if } n = 2k+1, m = 2s+1 \text{ or } n = 2k, m = 2s+1 \end{cases}$
$P^{[\bar{1}]}_{n}$	$= \begin{cases} F_{k+1} & \text{if } n = 2k \\ F_k & \text{if } n = 2k+1 \end{cases}$
$P^{[\bar{i}]}_{n,m}$	$= \begin{cases} 0 & \text{if } n = 2k+1, m = 2s \\ C^{[\bar{i}]}_{k,s} & \text{if } n = 2k, m = 2s \\ \displaystyle\sum_{l \geq \lceil \frac{si-k+1}{i-1} \rceil} \binom{s}{l}(-1)^{s-l}\binom{k-1-i(s-l)}{l} - \begin{cases} 0 \text{ if } i \text{ odd, else} \\ C^{[\bar{i}]}_{k-i/2,s} \end{cases} & \text{if } n = 2k, m = 2s+1 \\ \displaystyle\sum_{l \geq \lceil \frac{si-k}{i-1} \rceil} \binom{s}{l}(-1)^{s-l}\binom{k-i(s-l)}{l} - \begin{cases} 0 \text{ if } i \text{ even, else} \\ C^{[\bar{i}]}_{k-\lfloor i/2 \rfloor,s} \end{cases} & \text{if } n = 2k+1, m = 2s+1 \end{cases}$
$P^{[\bar{i}]}_{j}$ $P^{[\bar{i}]}_{2i+1}$ $P^{[\bar{i}]}_{n+2i+2}$	$= 2^{\lfloor j/2 \rfloor} \quad \text{if } 1 \leq j \leq 2i$ $= 2^{\lfloor (2i+1)/2 \rfloor} - 1$ $= 2P^{[\bar{i}]}_{n+2i} - P^{[\bar{i}]}_{n+2} + P^{[\bar{i}]}_{n}$

Table 4.6: Summarizing results for palindromic compositions

This coefficient can be computed by using the same approach used for $P^{[\bar{2}]}_{n,m}$ and the results in [4] can be easily generalized.

Some more properties about palindromic compositions can be found by opportunely manipulating $\hat{P}(t, z, w)$. For example, the total number of summand i in the palindromic compositions of number n can be determined as in the case of general compositions by differentiating with respect to w_i and then substituting $w = 1$. The coefficient of the generating function obtained in this way can be extracted by applying rule (4.6).

5. Conclusions

In this paper we have found many multivariate generating functions related to the enumeration of compositions and, consequently, of the objects involved in the bijection discussed in Section 3. All the generating functions we found are rational, so, when we cannot find a closed form for their coefficients, we can always obtain an asymptotic approximation for the desired values (see, e.g., [13]). Alternatively, we can determine a recurrence relation, as done for example in formulas (4.10) and (4.13). From these algebraic results is often possible to deduce a combinatorial interpretation. Tables 4.4 and 4.6 summarize the new and already known results examined in this paper.

Results on average values and variance can also be easily obtained by using formula (2.3) or by differentiating opportunely the generating functions. For example, the average number of summands i in a composition of number n is $(n-i+3)/2^{i+1}$, as can be easily verified by dividing the coefficient of function (4.5) by the total number of compositions of n.

We believe the approach presented in this paper is quite general and allows to obtain many kinds of information about compositions of integers.

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