FINDING ALMOST SQUARES II

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Abstract

In this article, we study short intervals that contain "almost squares" of the type: any integer n which can be factored in two different ways $n = a_1 b_1 = a_2 b_2$ with a_1, a_2, b_1, b_2 close to $\sqrt{n}$.

1. Introduction

In [1], the author studied the problem of finding "almost squares" in short intervals, namely:

Question 1. *For $0 \leq \theta < 1/2$, what is the least $f(\theta)$ such that, for some constants $c_1, c_2 > 0$, any interval $[x - c_1 x^{f(\theta)}, x + c_1 x^{f(\theta)}]$ contains an integer n with $n = ab$, where a, b are integers in the interval $[x^{1/2} - c_2 x^\theta, x^{1/2} + c_2 x^\theta]$? Note: The constants c_1 and c_2 may depend on θ.*

A similar question is the following.

Question 2. *For $0 \leq \theta < 1/2$, what is the least $g(\theta)$ such that, for some constants $c_1, c_2 > 0$, any interval $[x - c_1 x^{g(\theta)}, x + c_1 x^{g(\theta)}]$ contains an integer n with $n = a_1 b_1 = a_2 b_2$, where $a_1 < a_2 \leq b_2 < b_1$ are integers in the interval $[x^{1/2} - c_2 x^\theta, x^{1/2} + c_2 x^\theta]$? Note: The constants c_1 and c_2 may depend on θ.*

Note: We first considered Question 2 and then turned to Question 1, which has connections to problems on the distribution of $n^2 \alpha \pmod 1$ and gaps between sums of two squares.

In [1], we showed that $f(\theta) = 1/2$ when $0 \leq \theta < 1/4$, $f(1/4) = 1/4$ and $f(\theta) \geq 1/2 - \theta$. We conjectured that $f(\theta) = 1/2 - \theta$ for $1/4 < \theta < 1/2$ and gave conditional result when $1/4 < \theta < 3/10$. For Question 2, we have the following result.

Theorem 1. *For $0 < \theta < 1/4$, $g(\theta)$ does not exist (i.e. all possible products of pairs of integers in $[x^{1/2} - c_2 x^\theta, x^{1/2} + c_2 x^\theta]$ are necessarily distinct for large x).*

Theorem 2. *For* $1/4 \le \theta < 1/2$, $g(\theta) \ge 1 - 2\theta$.

Theorem 3. *For* $1/4 \le \theta \le 1/3$, $g(\theta) \le 1 - \theta$.

We believe that the lower bound is closer to the truth.

Conjecture 1. *For* $1/4 \le \theta < 1/2$, $g(\theta) = 1 - 2\theta$.

2. Preliminaries and $0 \le \theta < 1/4$

Suppose $n = a_1 b_1 = a_2 b_2$ with $x^{1/2} - c_2 x^\theta \le a_1 < a_2 \le b_2 < b_1 \le x^{1/2} + c_2 x^\theta$. Let $d_1 = (a_1, a_2)$ and $d_2 = (b_1, b_2)$ be the greatest common divisors. Then we must have $d_1, d_2 > 1$. Otherwise, if $d_1 = 1$, then a_2 divides b_1 which implies $x^{1/2} + c_2 x^\theta \ge b_1 \ge 2a_2 \ge 2x^{1/2} - 2c_2 x^\theta$. This is impossible for large x as $\theta < 1/2$. Now, let $a_1 = d_1 e_1$, $a_2 = d_1 e_2$, $b_1 = d_2 f_1$ and $b_2 = d_2 f_2$. Here $(e_1, e_2) = 1 = (f_1, f_2)$. Then

$$n = d_1 e_1 d_2 f_1 = d_1 e_2 d_2 f_2 \quad \text{gives} \quad e_1 f_1 = e_2 f_2.$$

Due to co-primality, $e_2 = f_1$ and $e_1 = f_2$. Therefore,

$$n = (d_1 e_1)(d_2 e_2) = (d_1 e_2)(d_2 e_1) \tag{1}$$

with $1 < d_1 < d_2$, $e_1 < e_2$ and $(e_1, e_2) = 1$.

Now, from $a_2 - a_1 \le 2c_2 x^\theta$, $d_1 \le d_1 e_2 - d_1 e_1 \le 2c_2 x^\theta$. Similarly, one can deduce that $d_2, e_1, e_2 \le 2c_2 x^\theta$. Moreover, as $d_1 e_1 = a_1 \ge x^{1/2} - c_2 x^\theta$, we have $d_1, e_1 \ge \frac{1}{2c_2} x^{1/2-\theta} - \frac{1}{2}$. Similarly, $d_2, e_2 \ge \frac{1}{2c_2} x^{1/2-\theta} - \frac{1}{2}$. Summing up, we have

$$\frac{1}{2c_2} x^{1/2-\theta} - \frac{1}{2} \le d_1, d_2, e_1, e_2 \le 2c_2 x^\theta. \tag{2}$$

From (2), we see that no such n exists for $0 \le \theta < 1/4$ and hence Theorem 1 follows.

3. Lower bound for $g(\theta)$

From (1) and (2), we see that an integer $n = a_1 b_1 = a_2 b_2$, satisfying the conditions for a_1, a_2, b_1, b_2 in Question 2, must be of the form:

$$n = (d_1 e_1)(d_2 e_2) \text{ with } \frac{1}{2c_2} x^{1/2-\theta} - \frac{1}{2} \le d_1, d_2, e_1, e_2 \le 2c_2 x^\theta$$

and $x^{1/2} - c_2 x^\theta \le d_1 e_1 < d_1 e_2$, $d_2 e_1 < d_2 e_2 \le x^{1/2} + c_2 x^\theta$. In particular, $e_2 d_2 - e_2 d_1 \le 2c_2 x^\theta$ which implies $e_2 - e_1 \le 2c_2 x^\theta / d_2$. Similarly, $d_2 - d_1 \le 2c_2 x^\theta / e_2$. Thus, the number of such tuples (d_1, d_2, e_1, e_2) is bounded by

$$\ll \sum_{\substack{x^{1/2-\theta} \ll d_2, e_2 \ll x^\theta \\ x^{1/2} - c_2 x^\theta \le d_2 e_2 \le x^{1/2} + c_2 x^\theta}} \frac{x^\theta}{e_2} \frac{x^\theta}{d_2} \ll \frac{x^{2\theta}}{x^{1/2}} x^\theta x^\epsilon = x^{3\theta - 1/2 + \epsilon}$$

for any $\epsilon > 0$ as the number of divisor function $d(n) \ll n^\epsilon$. It follows that there are at most $\ll x^{3\theta-1/2+\epsilon}$ such integers n in the interval $[x - c_2 x^{1/2+\theta}/3, x + c_2 x^{1/2+\theta}/3]$. Therefore, there exist two consecutive such n's with difference

$$\gg \frac{x^{1/2+\theta}}{x^{3\theta-1/2+\epsilon}} = x^{1-2\theta-\epsilon}.$$

Pick y to be the midpoint between these two integers. Then, for some constant $c > 0$, the interval $[y - cy^{1-2\theta-\epsilon}, y + cy^{1-2\theta-\epsilon}]$ does not contain any integer $n = a_1 b_1 = a_2 b_2$ with $y^{1/2} - c_2 y^\theta/2 \le a_1 < a_2 \le b_2 < b_1 \le y^{1/2} + c_2 y^\theta/2$, as $x - c_2 x^{1/2+\theta}/3 \le y \le x + c_2 x^{1/2+\theta}/3$. Consequently, for any constants $c, c' > 0$, there is an arbitrarily large y such that the interval $[y - cy^{1-2\theta-2\epsilon}, y + cy^{1-2\theta-2\epsilon}]$ does not contain any integer $n = a_1 b_1 = a_2 b_2$ with $y^{1/2} - c' y^\theta \le a_1 < a_2 \le b_2 < b_1 \le y^{1/2} + c' y^\theta$. Therefore, $g(\theta) \ge 1 - 2\theta - 2\epsilon$ which gives Theorem 2 by letting $\epsilon \to 0$.

4. Upper bound for $g(\theta)$

In this section, we prove Theorem 3. For any large x, set $N = [x^{1/4}]$ and $\xi = \{x^{1/4}\}$, the integer part and fractional part of $x^{1/4}$ respectively. Based on (1), we choose, for $0 \le \epsilon \le 1/2$,

$$d_1 = qN + r_1,\ d_2 = qN + r_2,\ e_1 = \frac{N + s_1}{q},\ e_2 = \frac{N + s_2}{q} \tag{3}$$

for some $1 \le q \le N^\epsilon$, $0 \le r_1, r_2 < N$ and $s_1, s_2 \ll q$ with $N \equiv -s_1 \equiv -s_2 \pmod q$. Our goal is to make

$$\begin{aligned} x =&(N+\xi)^4 = N^4 + 4N^3\xi + O(N^2) \approx (qN + r_1)\frac{N+s_1}{q}(qN + r_2)\frac{N+s_2}{q} \\ =&\Big[N^2 + \Big(\frac{r_1}{q} + s_1\Big)N + \frac{r_1 s_1}{q}\Big]\Big[N^2 + \Big(\frac{r_2}{q} + s_2\Big)N + \frac{r_2 s_2}{q}\Big] \\ =&N^4 + \Big(\frac{r_1 + r_2}{q} + s_1 + s_2\Big)N^3 + \Big[\frac{r_1 s_1}{q} + \frac{r_2 s_2}{q} + \Big(\frac{r_1}{q} + s_1\Big)\Big(\frac{r_2}{q} + s_2\Big)\Big]N^2 \\ &+ \Big[\frac{r_1 s_1}{q}\Big(\frac{r_2}{q} + s_2\Big) + \frac{r_2 s_2}{q}\Big(\frac{r_1}{q} + s_1\Big)\Big]N + \frac{r_1 s_1 r_2 s_2}{q^2} \end{aligned} \tag{4}$$

By Dirichlet's Theorem on diophantine approximation, we can find an integer $1 \le q \le N^\epsilon$ such that

$$\Big|4\xi - \frac{p}{q}\Big| \le \frac{1}{qN^\epsilon}$$

for some integer p. Fix such a q. Then, pick $s_1 < s_2 < 0$ to be the largest two integers such that $N \equiv -s_1 \equiv -s_2 \pmod q$. Clearly, $s_1, s_2 \ll q$. Then, one simply picks some $0 < r_1 < r_2 \ll q^2$ such that $\frac{r_1+r_2}{q} + s_1 + s_2 = \frac{p}{q}$. With these values for q, r_1, r_2, s_1, s_2, (4) becomes

$$x \approx N^4 + 4N^3\xi + O(N^{3-\epsilon}) + O(q^2N^2) + O(q^3N) + O(q^4).$$

Hence, we have just constructed an integer $n = d_1e_1d_2e_2$ which is within $O(N^{3-\epsilon})+O(N^{2+2\epsilon}) = O(x^{3/4-\epsilon/4}) + O(x^{1/2+\epsilon/2}) = O(x^{3/4-\epsilon/4})$ from x if $\epsilon \leq 1/3$. One can easily check that $a_1 = d_1e_1$, $b_1 = d_2e_2$, $a_2 = d_1e_2$ and $b_2 = d_2e_1$ are in the interval $[x^{1/2} - Cx^{1/4+\epsilon/4}, x^{1/2} + Cx^{1/4+\epsilon/4}]$ for some constant $C > 0$. Set $\theta = 1/4 + \epsilon/4$. We have, for some $C' > 0$, $n = a_1b_1 = a_2b_2$ in the interval $[x - C'x^{1-\theta}, x + C'x^{1-\theta}]$ such that $a_1 < a_2, b_2 < b_1$ are integers in $[x^{1/2} - Cx^{\theta}, x^{1/2} + Cx^{\theta}]$, provided $1/4 \leq \theta \leq 1/4 + 1/12 = 1/3$. This proves Theorem 3.

5. Open questions

Conjecture 1 may be too hard to prove at the moment. As a possible starting point, one can attempt to show that $g(1/4) = 1/2$, or even just $g(1/4) < 3/4$. Another possibility is to try to obtain some conditional results, as in [1]. Also, one may consider $g(\theta)$ when θ is near 1/2. This leads to the problem about gaps between integers that have more than one representation as a sum of two squares.

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References

[1] Tsz Ho Chan, *Finding Almost Squares*, preprint, 2005, arXiv:math.NT/0502199.