A STUDY OF EULERIAN NUMBERS FOR PERMUTATIONS IN THE ALTERNATING GROUP

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Received: 5/9/06, Accepted: 10/15/06, Published: 10/20/06

Abstract

The cardinalities of the sets of even and odd permutations with a given ascent number are investigated by an operator that was introduced by the author. We will deduce the recurrence relations for such Eulerian numbers of even and odd permutations, and deduce divisibility properties by prime powers concerning them and some related numbers.

Keywords: Eulerian numbers; inversions; recurrence relations

1. Introduction

An *ascent* (or *descent*) of a permutation $a_1a_2\cdots a_n$ of $[n] = \{1, 2, \ldots, n\}$ is an adjacent pair such that $a_i < a_{i+1}$ (or $a_i > a_{i+1}$) for some i $(1 \le i \le n-1)$. Let $E(n, k)$ be the set of all permutations of $[n]$ with exactly k ascents, where $0 \le k \le n-1$. Its cardinality is the classical *Eulerian number*;

$$A_{n,k} = |E(n, k)|,$$

whose properties and identities can be found in [2-6], for example.

An *inversion* of a permutation $A = a_1a_2\cdots a_n$ is a pair (i, j) such that $1 \le i < j \le n$ and $a_i > a_j$. Let us denote by $\mathrm{inv}(A)$ the number of inversions in a permutation A, and by $E_{\mathrm{e}}(n, k)$ or $E_{\mathrm{o}}(n, k)$ the subsets of all permutations in $E(n, k)$ that have, respectively, even or odd numbers of inversions.

The aim of this paper is to investigate their cardinalities;

$$B_{n,k} = |E_{\mathrm{e}}(n, k)| \quad \text{and} \quad C_{n,k} = |E_{\mathrm{o}}(n, k)|.$$

Obviously we have $A_{n,k} = B_{n,k} + C_{n,k}$, while the differences $D_{n,k} = B_{n,k} - C_{n,k}$ are called *signed Eulerian numbers* in [1], where the descent number was considered instead of the ascent number. Therefore, the identities for $D_{n,k}$ presented here correspond to those in [1] that are obtained by replacing k with $n-k-1$.

In order to study these numbers, we make use of an operator on permutations in $[n]$, which was introduced in [9]. The operator σ is defined by adding one to all entries of a permutation and by changing $n+1$ into one. However, when n appears at either end of a permutation, it is removed and one is put at the other end. That is, for a permutation $a_1a_2\cdots a_n$ with $a_i = n$ for some i $(2 \le i \le n-1)$, we have

(i) $\sigma(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$,

where $b_i = a_i + 1$ for all i $(1 \le i \le n)$ and $n+1$ is replaced by one. However, for a permutation $a_1a_2\cdots a_{n-1}$ of $[n-1]$, we have:

(ii) $\sigma(a_1a_2\cdots a_{n-1}n) = 1b_1b_2\cdots b_{n-1}$;

(iii) $\sigma(na_1a_2\cdots a_{n-1}) = b_1b_2\cdots b_{n-1}1$,

where $b_i = a_i + 1$ for all i $(1 \le i \le n-1)$. We denote by $\sigma^\ell A$ the repeated ℓ applications of σ to a permutation A. It is obvious that the operator preserves the numbers of ascents and descents in a permutation, that is, $\sigma A \in E(n,k)$ if and only if $A \in E(n,k)$.

Let us observe the number of inversions of a permutation when σ is applied. When n appears at either end of a permutation $A = a_1a_2\cdots a_n$ as in (ii) or (iii), it is evident that

$$\operatorname{inv}(\sigma A) = \operatorname{inv}(A).$$

Next let us consider the case (i). When $a_i = n$ for some i $(2 \le i \le n-1)$, we get $\sigma(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$, where $b_i = 1$ is at the ith position. In this case, $n-i$ inversions $(i, i+1), \ldots, (i, n)$ of A vanish and, in turn, $i-1$ inversions $(1, i), \ldots, (i-1, i)$ of σA occur. Hence the difference between the numbers of inversions is

$$\operatorname{inv}(\sigma A) - \operatorname{inv}(A) = (i-1) - (n-i) = 2i - (n+1). \tag{1}$$

Therefore, if n is odd, the operator σ preserves the parity of all permutations of $[n]$. When n is even, however, each application of the operator changes the parity of permutations as long as n remains in the interior of permutations.

For convenience sake we denote by $E_{\mathrm{e}}^-(n,k)$ and $E_{\mathrm{e}}^+(n,k)$ the sets of permutations $a_1a_2\cdots a_n$ in $E_{\mathrm{e}}(n,k)$ with $a_1 < a_n$ and $a_1 > a_n$, respectively. Similarly, $E_{\mathrm{o}}^-(n,k)$ and $E_{\mathrm{o}}^+(n,k)$ denote those in $E_{\mathrm{o}}(n,k)$, respectively. In $E_{\mathrm{e}}^-(n,k)$ and $E_{\mathrm{o}}^-(n,k)$ *canonical* permutations are defined as those of the form $a_1a_2\cdots a_{n-1}n$, and in $E_{\mathrm{e}}^+(n,k)$ and $E_{\mathrm{o}}^+(n,k)$ as those of the form $na_1a_2\cdots a_{n-1}$, where $a_1a_2\cdots a_{n-1}$ is a permutation of $[n-1]$.

In [8] and the references therein, even and odd permutations were classified by the anti-excedance number, not by the ascent number. An anti-excedance in a permutation $a_1a_2\cdots a_n$ means an inequality $i \geq a_i$. Recurrence relations were given for the cardinalities $P_{n,k}$ and $Q_{n,k}$ of the sets of even and odd permutations, respectively, with anti-excedance number k. They hold for all natural integers n as follows:

$$\begin{aligned} P_{n,k} &= kQ_{n-1,k} + (n-k)Q_{n-1,k-1} + P_{n-1,k-1}; \\ Q_{n,k} &= kP_{n-1,k} + (n-k)P_{n-1,k-1} + Q_{n-1,k-1}. \end{aligned}$$

The recurrence relations for $B_{n,k}$ and $C_{n,k}$ seem not so simple, because they have different expressions according to the parity of n, as will be revealed in the following sections.

First we derive the recurrence relation for the signed Eulerian numbers $D_{n,k} = B_{n,k} - C_{n,k}$. The relation was conjectured in [7] and an analytic proof for it was given in [1]. In Section 4 we will derive it from a quite different point of view, based on the properties of the operator σ. Making use of it, the recurrence relations for $B_{n,k}$ and $C_{n,k}$ will be obtained.

In Section 5 it is shown that divisibility properties for $B_{n,k}$, $C_{n,k}$ and some related numbers by prime powers can be obtained by our approach.

2. The Numbers $B_{n,k}$ and $C_{n,k}$

The numbers $B_{n,k}$ and $C_{n,k}$ enjoy some symmetry properties according to the values of n. The permutation $n\cdots 21 \in E(n,0)$ has $n(n-1)/2$ inversions. Hence the values of $B_{n,0}$ and $C_{n,0}$ are given by

$$B_{n,0} = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod 4, \\ 0 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4, \end{cases}$$

and

$$C_{n,0} = \begin{cases} 0 & \text{if } n \equiv 0 \text{ or } 1 \pmod 4, \\ 1 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

For a permutation $A = a_1a_2\cdots a_n$ we define its reflection by $A^* = a_n\cdots a_2a_1$. Using reflected permutations and the parity of $n(n-1)/2$, the following symmetry properties between $B_{n,k}$ and $C_{n,k}$ are easily checked.

(i) $n \equiv 0$ or $1 \pmod 4$. In this case, $A \in E_{\mathrm{e}}(n,k)$ if and only if $A^* \in E_{\mathrm{e}}(n, n-k-1)$, and $A \in E_{\mathrm{o}}(n,k)$ if and only if $A^* \in E_{\mathrm{o}}(n, n-k-1)$, so we have

$$B_{n,k} = B_{n,n-k-1} \quad \text{and} \quad C_{n,k} = C_{n,n-k-1}.$$

(ii) $n \equiv 2$ or $3 \pmod 4$. In this case, $A \in E_{\mathrm{e}}(n,k)$ if and only if $A^* \in E_{\mathrm{o}}(n, n-k-1)$, and $A \in E_{\mathrm{o}}(n,k)$ if and only if $A^* \in E_{\mathrm{e}}(n, n-k-1)$, so we have

$$B_{n,k} = C_{n,n-k-1} \quad \text{and} \quad C_{n,k} = B_{n,n-k-1}.$$

The values of $B_{n,k}$ and $C_{n,k}$ for small n are shown in the next two tables. The integers in their top rows represent the values of k. In Section 4 a formula for calculating these numbers will be supplied by means of $A_{n,k}$ and $D_{n,k}$.

$B_{n,k}$	0	1	2	3	4	5	6	7	8	9
$n=2$	0	1								
$n=3$	0	2	1							
$n=4$	1	5	5	1						
$n=5$	1	14	30	14	1					
$n=6$	0	28	155	147	29	1				
$n=7$	0	56	605	1208	586	64	1			
$n=8$	1	127	2133	7819	7819	2133	127	1		
$n=9$	1	262	7288	44074	78190	44074	7288	262	1	
$n=10$	0	496	23947	227623	655039	655315	227569	23893	517	1

$C_{n,k}$	0	1	2	3	4	5	6	7	8	9
$n=2$	1	0								
$n=3$	1	2	0							
$n=4$	0	6	6	0						
$n=5$	0	12	36	12	0					
$n=6$	1	29	147	155	28	0				
$n=7$	1	64	586	1208	605	56	0			
$n=8$	0	120	2160	7800	7800	2160	120	0		
$n=9$	0	240	7320	44160	78000	44160	7320	240	0	
$n=10$	1	517	23893	227569	655315	655039	227623	23947	496	0

3. The Case of Odd n

Throughout this section we assume that n is an odd integer. It was shown in [9] that to each permutation A of $[n]$ there corresponds a smallest positive integer $\pi(A)$ such that $\sigma^{\pi(A)}A = A$, which is called the *period* of A. Its trace

$$\{\sigma A, \sigma^2 A, \ldots, \sigma^{\pi(A)}A = A\}$$

is called the *orbit* of A. The orbit of a permutation of $E_{\mathrm{e}}^{-}(n,k)$ under σ is entirely contained in $E_{\mathrm{e}}^{-}(n,k)$ and similarly for $E_{\mathrm{e}}^{+}(n,k)$, as is shown previously in the case of odd n. Here we mainly deal with the set $E_{\mathrm{e}}(n,k) = E_{\mathrm{e}}^{-}(n,k) \cup E_{\mathrm{e}}^{+}(n,k)$ and its cardinality $B_{n,k}$, since the same arguments can also be applied to $E_{\mathrm{o}}(n,k) = E_{\mathrm{o}}^{-}(n,k) \cup E_{\mathrm{o}}^{+}(n,k)$ and its cardinality $C_{n,k}$.

It was shown in [9] that the period satisfies the relation

$$\pi(A) = \begin{cases} (n-k)\gcd(n,\pi(A)) & \text{if } A \in E_{\mathrm{e}}^{-}(n,k) \cup E_{\mathrm{o}}^{-}(n,k), \\ (k+1)\gcd(n,\pi(A)) & \text{if } A \in E_{\mathrm{e}}^{+}(n,k) \cup E_{\mathrm{o}}^{+}(n,k). \end{cases} \tag{2}$$

It follows from (2) that the period of a permutation $A \in E(n,k)$ is either $d(n-k)$ or $d(k+1)$ for a positive divisor d of n, i.e., $d = \gcd(n,\pi(A))$, although there may be no permutations having such periods for some divisors. In this paper, divisors of n always mean positive divisors.

For a divisor d of n, we denote by α_d^k the number of orbits of period $d(n-k)$ in $E_{\mathrm{e}}^{-}(n,k)$ and by β_d^k that of orbits of period $d(k+1)$ in $E_{\mathrm{e}}^{+}(n,k)$. In the case of odd n the next theorem plays a fundamental role.

Theorem 3.1 *Let n be an odd integer and let k be an integer satisfying $1 \le k \le n-1$. Then it follows that*

$$B_{n-1,k-1} = \sum_{d|n} d\alpha_d^k, \tag{3}$$

$$B_{n-1,k} = \sum_{d|n} d\beta_d^k, \tag{4}$$

$$B_{n,k} = \sum_{d|n} d\{(n-k)\alpha_d^k + (k+1)\beta_d^k\}. \tag{5}$$

Proof. First let us consider permutations in $E_{\mathrm{e}}^{-}(n,k)$. In this case each orbit contains canonical permutations of the form $a_1a_2\cdots a_{n-1}n$ by (i) and (ii) of Section 1. It suffices to deal only with canonical ones in counting orbits. If $A = a_1a_2\cdots a_{n-1}n \in E_{\mathrm{e}}^{-}(n,k)$, we see that $a_1a_2\cdots a_{n-1} \in E_{\mathrm{e}}(n-1,k-1)$, since

$$\mathrm{inv}(a_1a_2\cdots a_{n-1}) = \mathrm{inv}(A)$$

and n is deleted. Therefore, there exist $B_{n-1,k-1}$ canonical permutations in $E_{\mathrm{e}}^{-}(n,k)$. It follows from (2) that the period of a permutation $A \in E_{\mathrm{e}}^{-}(n,k)$ is equal to $d(n-k)$ for a divisor d of n. There exist n canonical permutations in $\{\sigma A, \sigma^2 A, \ldots, \sigma^{n(n-k)}A = A\}$ due to [9, Corollary 2], and hence each orbit $\{\sigma A, \sigma^2 A, \ldots, \sigma^{d(n-k)}A = A\}$ of a permutation A with period $d(n-k)$ contains exactly d canonical permutations. This follows from the fact that the latter repeats itself n/d times in the former. Since there exist α_d^k orbits of period $d(n-k)$ for each divisor d of n, classifying all canonical permutations of $E_{\mathrm{e}}^{-}(n,k)$ into orbits leads us to (3).

The proof of (4) is similar. To do this we consider permutations in $E_{\mathrm{e}}^{+}(n,k)$. In this case each orbit contains canonical permutations of the form $na_1a_2\cdots a_{n-1}$ by (i) and (iii) of Section 1. If $A = na_1a_2\cdots a_{n-1} \in E_{\rm e}^{+}(n,k)$, we see that $a_1a_2\cdots a_{n-1} \in E_{\rm e}(n-1,k)$, since

$$\mathrm{inv}(a_1a_2\cdots a_{n-1}) = \mathrm{inv}(A) - (n-1)$$

and $n-1$ is an even number by assumption. Therefore, the set of all canonical permutations in $E_{\rm e}^{+}(n,k)$ has cardinality $B_{n-1,k}$. Again using (2), the period of a permutation $A \in E_{\rm e}^{+}(n,k)$ is equal to $d(k+1)$ for a divisor d of n. By [9, Corollary 2] there exist n canonical permutations in $\{\sigma A, \sigma^2 A, \ldots, \sigma^{n(k+1)}A = A\}$ and hence, as above, there exist exactly d such permutations in each orbit $\{\sigma A, \sigma^2 A, \ldots, \sigma^{d(k+1)}A = A\}$ of a permutation A with period $d(k+1)$. There exist β_d^k orbits of period $d(k+1)$ for each divisor d of n. Hence, we can obtain (4) by classifying all canonical permutations in $E_{\rm e}^{+}(n,k)$ into orbits.

Considering the numbers of orbits and periods, we see that the cardinalities of $E_{\rm e}^{\pm}(n,k)$ are obtained by

$$|E_{\rm e}^{-}(n,k)| = \sum_{d|n} d(n-k)\alpha_d^k \quad \text{and} \quad |E_{\rm e}^{+}(n,k)| = \sum_{d|n} d(k+1)\beta_d^k. \tag{6}$$

Since the set $E_{\rm e}(n,k)$ is a disjoint union of $E_{\rm e}^{-}(n,k)$ and $E_{\rm e}^{+}(n,k)$, we conclude that

$$B_{n,k} = |E_{\rm e}^{-}(n,k)| + |E_{\rm e}^{+}(n,k)| = \sum_{d|n} d\{(n-k)\alpha_d^k + (k+1)\beta_d^k\}, \tag{7}$$

which proves (5). This completes the proof. □

Let us denote by γ_d^k the number of orbits of period $d(n-k)$ in $E_{\rm o}^{-}(n,k)$ and by δ_d^k that of orbits of period $d(k+1)$ in $E_{\rm o}^{+}(n,k)$. When n is odd, analogous relations to (3)-(6) hold for $C_{n,k}$, γ_d^k and δ_d^k, since the orbit of a permutation of $E_{\rm o}^{\pm}(n,k)$ under σ is also contained in $E_{\rm o}^{\pm}(n,k)$.

We state them for the sake of completeness;

$$C_{n-1,k-1} = \sum_{d|n} d\gamma_d^k, \quad C_{n-1,k} = \sum_{d|n} d\delta_d^k,$$

and

$$|E_{\rm o}^{-}(n,k)| = \sum_{d|n} d(n-k)\gamma_d^k, \quad |E_{\rm o}^{+}(n,k)| = \sum_{d|n} d(k+1)\delta_d^k.$$

Since the set $E_{\rm o}(n,k)$ is a disjoint union of $E_{\rm o}^{-}(n,k)$ and $E_{\rm o}^{+}(n,k)$, we obtain

$$C_{n,k} = |E_{\rm o}^{-}(n,k)| + |E_{\rm o}^{+}(n,k)| = \sum_{d|n} d\{(n-k)\gamma_d^k + (k+1)\delta_d^k\}, \tag{8}$$

Making use of (3) and (4), we see that both cardinalities in (6) can be written simply by the notation $B_{n,k}$ and their counterparts for $E_{\rm o}(n,k)$ also follow from the above relations in a similar manner.

Corollary 3.2 *When n is odd, the cardinalities of $E_{\rm e}^{\pm}(n,k)$ and $E_{\rm o}^{\pm}(n,k)$ are given by*

(i) $|E_{\rm e}^{-}(n,k)| = (n-k)B_{n-1,k-1}$ *and* $|E_{\rm o}^{-}(n,k)| = (n-k)C_{n-1,k-1}$ $(1 \le k \le n-1)$,

(ii) $|E_{\rm e}^{+}(n,k)| = (k+1)B_{n-1,k}$ *and* $|E_{\rm o}^{+}(n,k)| = (k+1)C_{n-1,k}$ $(0 \le k \le n-2)$.

Using (7), (8) and this corollary, we can obtain the following two corollaries. The relations in Corollary 3.3 have the same form as the recurrence relation for classical Eulerian numbers $A_{n,k}$ in [3];

$$A_{n,k} = (n-k)A_{n-1,k-1} + (k+1)A_{n-1,k}. \tag{9}$$

Corollary 3.3 *When n is odd, the following relations hold for $B_{n,k}$ and $C_{n,k}$:*

$$B_{n,k} = (n-k)B_{n-1,k-1} + (k+1)B_{n-1,k}; \tag{10}$$
$$C_{n,k} = (n-k)C_{n-1,k-1} + (k+1)C_{n-1,k}. \tag{11}$$

Corollary 3.4 *When n is odd, the following relations hold:*

(i) $|E_{\rm e}^{-}(n,k)| - |E_{\rm o}^{-}(n,k)| = (n-k)D_{n-1,k-1}$ $(1 \le k \le n-1)$;

(ii) $|E_{\rm e}^{+}(n,k)| - |E_{\rm o}^{+}(n,k)| = (k+1)D_{n-1,k}$ $(0 \le k \le n-2)$.

4. Recurrences for $B_{n,k}$ and $C_{n,k}$

When n is even, neither equality (10) nor (11) holds, as is seen from the tables of Section 2. For example, an odd integer $C_{10,4}$ cannot be written as a linear sum of $C_{9,k}$'s or $B_{9,k}$'s $(1 \le k \le 7)$ with integral coefficients, since they are all even. Therefore, neither (10) nor (11) provide a recurrence relation of the numbers $B_{n,k}$ or $C_{n,k}$.

As for the differences $D_{n,k} = B_{n,k} - C_{n,k}$, however, their recurrence relation was conjectured in [7] and an analytic proof for it was given in [1]. In our notation it is described as the next theorem, for which we provide another proof from a combinatorial point of view. Notice that there is a different flavor in the case of even n.

Theorem 4.1 *The recurrence relation for $D_{n,k}$ is given by*

$$D_{n,k} = \begin{cases} (n-k)D_{n-1,k-1} + (k+1)D_{n-1,k}, & \textit{if } n \textit{ is odd}, \\ D_{n-1,k-1} - D_{n-1,k}, & \textit{if } n \textit{ is even}. \end{cases} \tag{12}$$

Proof. The first part of this relation follows immediately from (10) and (11) of Corollary 3.3. Assuming that n is even, we show the second part by means of the operator σ.

Recall that when n is even, the operator may change the parity of permutations of $E(n,k)$, but it is a bijection on $E_{\mathrm{e}}^-(n,k) \cup E_{\mathrm{o}}^-(n,k)$ and on $E_{\mathrm{e}}^+(n,k) \cup E_{\mathrm{o}}^+(n,k)$, respectively.

First let us consider permutations $A = a_1a_2\cdots a_n$ in $E_{\mathrm{e}}^-(n,k) \cup E_{\mathrm{o}}^-(n,k)$ and divide all permutations in $E_{\mathrm{e}}^-(n,k) \cup E_{\mathrm{o}}^-(n,k)$ into the following two types:

(i) $A = a_1a_2\cdots a_{n-1}n$, where $a_1a_2\cdots a_{n-1}$ is a permutation of $[n-1]$;

(ii) $A = a_1a_2\cdots a_n$ with $a_1 < a_n$, where $a_i = n$ for some i $(2 \le i \le n-1)$.

Suppose $A \in E_{\mathrm{e}}^-(n,k)$. If A is of type (i), then σA remains an even permutation, since $\mathrm{inv}(\sigma A) = \mathrm{inv}(A)$. We see that the cardinality of permutations of type (i) is $B_{n-1,k-1}$, since A is even and n is the last entry. However, if $A \in E_{\mathrm{e}}^-(n,k)$ is of type (ii), then we have $\sigma A \in E_{\mathrm{o}}^-(n,k)$ by (1), since $n+1$ is an odd integer. Therefore, the cardinality of permutations of type (ii) in $E_{\mathrm{e}}^-(n,k)$ is

$$|E_{\mathrm{e}}^-(n,k)| - B_{n-1,k-1}, \tag{13}$$

and precisely so many permutations change the parity from even to odd under σ.

Similarly, suppose $A \in E_{\mathrm{o}}^-(n,k)$. If A is of type (i), then σA remains an odd permutation. We see that the cardinality of permutations of type (i) is $C_{n-1,k-1}$. If $A \in E_{\mathrm{o}}^-(n,k)$ is of type (ii), then we have $\sigma A \in E_{\mathrm{e}}^-(n,k)$ by (1). The cardinality of permutations of type (ii) in $E_{\mathrm{o}}^-(n,k)$ is

$$|E_{\mathrm{o}}^-(n,k)| - C_{n-1,k-1}, \tag{14}$$

and precisely so many permutations change the parity from odd to even under σ.

Since σ is a bijection on $E_{\mathrm{e}}^-(n,k) \cup E_{\mathrm{o}}^-(n,k)$, both cardinalities given by (13) and (14) must be equal. Hence we obtain

$$|E_{\mathrm{e}}^-(n,k)| - |E_{\mathrm{o}}^-(n,k)| = B_{n-1,k-1} - C_{n-1,k-1} = D_{n-1,k-1}. \tag{15}$$

Next let us consider permutations $A = a_1a_2\cdots a_n$ in $E_{\mathrm{e}}^+(n,k) \cup E_{\mathrm{o}}^+(n,k)$ and divide all permutations in $E_{\mathrm{e}}^+(n,k) \cup E_{\mathrm{o}}^+(n,k)$ into the following two types:

(iii) $A = na_1a_2\cdots a_{n-1}$, where $a_1a_2\cdots a_{n-1}$ is a permutation of $[n-1]$;

(iv) $A = a_1a_2\cdots a_n$ with $a_1 > a_n$, where $a_i = n$ for some i $(2 \le i \le n-1)$.

If $A \in E_e^+(n,k)$ is of type (iii), then σA remains an even permutation. We see that the cardinality of permutations of type (iii) is $C_{n-1,k}$, since $\mathrm{inv}(A) - \mathrm{inv}(a_1 a_2 \cdots a_{n-1}) = n-1$ and $n-1$ is odd. However, if $A \in E_e^+(n,k)$ is of type (iv), then we have $\sigma A \in E_o^+(n,k)$ by (1). The cardinality of permutations of type (iv) in $E_e^+(n,k)$ is

$$|E_e^+(n,k)| - C_{n-1,k}, \tag{16}$$

and precisely so many permutations change the parity from even to odd under σ.

Similarly, if $A \in E_o^+(n,k)$ is of type (iii), then σA remains an odd permutation. We see that the cardinality of permutations of type (iii) is $B_{n-1,k}$ as above. If $A \in E_o^+(n,k)$ is of type (iv), then we have $\sigma A \in E_e^+(n,k)$ by (1). The cardinality of permutations of type (iv) in $E_o^+(n,k)$ is

$$|E_o^+(n,k)| - B_{n-1,k}, \tag{17}$$

and precisely so many permutations change the parity from odd to even under σ.

Since σ is a bijection on $E_e^+(n,k) \cup E_o^+(n,k)$, both cardinalities given by (16) and (17) must be equal. Hence we obtain

$$|E_e^+(n,k)| - |E_o^+(n,k)| = -B_{n-1,k} + C_{n-1,k} = -D_{n-1,k}. \tag{18}$$

From (7) and (8), adding (15) and (18) yields $B_{n,k} - C_{n,k} = D_{n,k} = D_{n-1,k-1} - D_{n-1,k}$, which is the required relation. This completes the proof. □

Symmetry properties for $D_{n,k}$ follow from the relations presented in Section 2:

(i) For $n \equiv 0$ or 1 (mod 4), $D_{n,k} = D_{n,n-k-1}$;

(ii) For $n \equiv 2$ or 3 (mod 4), $D_{n,k} = -D_{n,n-k-1}$.

The values of $D_{n,k}$ for small n is given below.

$D_{n,k}$	0	1	2	3	4	5	6	7	8	9
$n=2$	−1	1								
$n=3$	−1	0	1							
$n=4$	1	−1	−1	1						
$n=5$	1	2	−6	2	1					
$n=6$	−1	−1	8	−8	1	1				
$n=7$	−1	−8	19	0	−19	8	1			
$n=8$	1	7	−27	19	19	−27	7	1		
$n=9$	1	22	−32	−86	190	−86	−32	22	1	
$n=10$	−1	−21	54	54	−276	276	−54	−54	21	1

Thus the values of $B_{n,k}$ and $C_{n,k}$ can be known through

$$B_{n,k} = \frac{A_{n,k} + D_{n,k}}{2}, \quad C_{n,k} = \frac{A_{n,k} - D_{n,k}}{2},$$

using $A_{n,k}$ and $D_{n,k}$ that are calculated according to the respective recurrence relations (9) and (12). From these equalities, we can obtain the expressions of $B_{n,k}$ and $C_{n,k}$ by means of $B_{n-1,k}$'s and $C_{n-1,k}$'s in the case of even n, which constitute recurrence relations for $B_{n,k}$ and $C_{n,k}$ together with Corollary 3.3.

Corollary 4.2 *When n is even, the following relations hold for $B_{n,k}$ and $C_{n,k}$:*

$$\begin{aligned} 2B_{n,k} &= (n-k+1)B_{n-1,k-1} + kB_{n-1,k} + (n-k-1)C_{n-1,k-1} + (k+2)C_{n-1,k}; \\ 2C_{n,k} &= (n-k+1)C_{n-1,k-1} + kC_{n-1,k} + (n-k-1)B_{n-1,k-1} + (k+2)B_{n-1,k}. \end{aligned}$$

From (15) and (18) we get a counterpart of Corollary 3.4.

Corollary 4.3 *When n is even, the following relations hold:*

(i) $|E_{\mathrm{e}}^{-}(n,k)| - |E_{\mathrm{o}}^{-}(n,k)| = D_{n-1,k-1} \quad (1 \le k \le n-1)$;

(ii) $|E_{\mathrm{e}}^{+}(n,k)| - |E_{\mathrm{o}}^{+}(n,k)| = -D_{n-1,k} \quad (0 \le k \le n-2)$.

5. Orbits and their Applications

Again assume that n is an odd integer. We examine the numbers of orbits of particular types and from them deduce divisibility properties for $B_{n,k}$, $C_{n,k}$ and some related numbers by prime powers.

For a positive integer ℓ with $\gcd(\ell, n) = 1$, a canonical permutation of $[n]$ of the form

$$P_n^\ell = 1(1+\ell)(1+2\ell)\cdots(1+(n-1)\ell)$$

can be defined, where $\ell, 2\ell, \ldots, (n-1)\ell$ represent numbers modulo n. According to whether P_n^ℓ is an even or odd permutation, let us put

$$\epsilon_n^\ell = \begin{cases} 1 & \text{if } P_n^\ell \text{ is even,} \\ 0 & \text{if } P_n^\ell \text{ is odd.} \end{cases}$$

Theorem 5.1 *Let n be an odd integer and let k be an integer such that $1 \le k \le n-1$.*

(i) *If a divisor d of n satisfies $\gcd(k, n/d) > 1$, then $\alpha_d^k = \gamma_d^k = 0$.*

(ii) *If $\gcd(k, n) = 1$, then $\alpha_1^k = \epsilon_n^{n-k}$ and $\gamma_1^k = 1 - \epsilon_n^{n-k}$.*

Proof. In order to prove (i), suppose A is a permutation that belongs to $E_{\rm e}^-(n,k)$. From (2) its period $\pi(A)$ satisfies $\pi(A) = (n-k)\gcd(n,\pi(A))$. Then, putting $d = \gcd(n,\pi(A))$, we have $\pi(A) = d(n-k)$ and $d = \gcd(n, d(n-k))$, which implies $\gcd(n-k, n/d) = 1$ or $\gcd(k, n/d) = 1$. Consequently, we see that there exist no permutations of period $d(n-k)$, i.e., $\alpha_d^k = 0$, if a divisor d of n satisfies $\gcd(k, n/d) > 1$.

The same arguments can be applied to permutations in $E_{\rm o}^-(n,k)$ and we obtain the assertion that $\gamma_d^k = 0$ if d satisfies $\gcd(k, n/d) > 1$.

Next, in order to prove (ii), suppose $\gcd(k,n) = 1$. In case of $d = 1$, due to [9, Theorem 7], there exists a unique orbit of period $n-k$ in $E_{\rm e}^-(n,k) \cup E_{\rm o}^-(n,k)$, which contains only one canonical permutation P_n^{n-k}. Hence, if it is an even permutation, then we have $\alpha_1^k = 1$ and $\gamma_1^k = 0$. Otherwise, $\alpha_1^k = 0$ and $\gamma_1^k = 1$. This completes the proof. □

From Theorem 5.1 we can derive a criterion under which $B_{n-1,k-1}$, $C_{n-1,k-1}$ and $D_{n-1,k-1}$ are divisible by a prime power.

Corollary 5.2 *Suppose that p is a prime and that an odd integer n is divisible by p^m for a positive integer m. If k is divisible by p, then $B_{n-1,k-1}$, $C_{n-1,k-1}$ and $D_{n-1,k-1}$ are also divisible by p^m.*

Proof. Without loss of generality we can assume that m is the largest integer for which p^m divides n. Suppose k is a multiple of p. In Theorem 5.1 (i) we have seen that $\alpha_d^k = 0$ for a divisor d of n such that $\gcd(k, n/d) > 1$. On the other hand, a divisor d for which $\gcd(k, n/d) = 1$ must be a multiple of p^m, since k is a multiple of p. Therefore, equality (3) of Theorem 3.1 implies that $B_{n-1,k-1}$ is divisible by p^m. By the same arguments we see that $C_{n-1,k-1}$ and $D_{n-1,k-1}$ are also divisible by p^m, if k is a multiple of p. This completes the proof. □

When k is not a multiple of a prime p in Corollary 5.2, $B_{n-1,k-1}$ and $C_{n-1,k-1}$ are not necessarily divisible by p. For example, the next result summarizes the case where n is a prime power.

Corollary 5.3 *Let p be an odd prime and m a positive integer. Then*

$$B_{p^m-1,k-1} \equiv \begin{cases} \epsilon_{p^m}^{p^m-k} \pmod{p}, & \text{if } \gcd(p,k) = 1, \\ 0 \pmod{p^m}, & \text{if } \gcd(p,k) = p, \end{cases}$$

$$C_{p^m-1,k-1} \equiv \begin{cases} 1 - \epsilon_{p^m}^{p^m-k} \pmod{p}, & \text{if } \gcd(p,k) = 1, \\ 0 \pmod{p^m}, & \text{if } \gcd(p,k) = p. \end{cases}$$

Proof. Remarking Corollary 5.2, it suffices to treat the case where $\gcd(p,k) = 1$. From equality (3) we get

$$B_{p^m-1,k-1} = \sum_{d|p^m} d\alpha_d^k,$$

and we know that $\alpha_1^k = \epsilon_{p^m}^{p^m-k}$ by Theorem 5.1 (ii). Thus we get the first part. Similarly, the second one follows from the equality

$$C_{p^m-1,k-1} = \sum_{d|p^m} d\gamma_d^k,$$

together with $\gamma_1^k = 1 - \epsilon_{p^m}^{p^m-k}$. □

The final corollary easily follows from Corollaries 3.2 and 5.2.

Corollary 5.4 *Under the same assumptions as Corollary 5.2 the following hold.*

(i) *If k is divisible by p^i for some i $(1 \leq i \leq m)$, then $|E_{\mathrm{e}}^-(n,k)|$ and $|E_{\mathrm{o}}^-(n,k)|$ are divisible by p^{m+i}.*

(ii) *If $k+1$ is divisible by p^i for some i $(i \geq 1)$, then $|E_{\mathrm{e}}^+(n,k)|$ and $|E_{\mathrm{o}}^+(n,k)|$ are divisible by p^{m+i}.*

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