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A DELETION GAME ON GRAPHS: "LE PIC ARÊTE"

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Abstract

This paper presents a deletion game on graphs, called "Le Pic Arête." Two players alternately remove an edge of a given graph. A player gets one point each time he isolates a vertex (so that he gets at most two points by edge deletion). When no edge remains, the player with the maximum number of points is declared the winner. For instance, if we consider this game on a grid, we have a variant of the famous "Dots and Boxes" game.

Our study consists of finding an approximation of the value of any game configuration, i.e., the difference between the numbers of points attainable by both players (when each player is supposed to play his best strategy).

1. Presentation

1.1 The Original Game (or How to Fight Against Students' Boredom)...

Time spent in lecture theatres often seems too long. The following game may help students kill time: Find a sheet of squared paper and draw a grid (the game zone) by darkening some segments (generally we draw a rectangular outline, and darken some segments inside).

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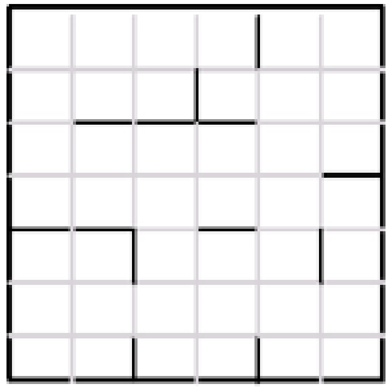


Then two players alternately darken one segment of the grid. A player gets one point each time he darkens the fourth side of a square. The game ends when all the segments have been darkened, the winner being the player with the highest score.

When playing any game, the natural question is: "Does there exist a strategy that leads me to victory, or at least, that helps me obtain the best score. And in that case, which one?" Actually this game can be considered as a variant of the "Dots and Boxes" game (studied by Berlekamp in [1]). In that game, players are required to move again after winning points.

1.2 ... and its Generalization: A Deletion Game on Graphs

We consider a generalization of this student game: we play on a simple graph with no loop. Both players alternately remove an edge, and a player gets one point each time the deletion of an edge isolates a vertex (so that he can get at most two points by edge deletion). The goal remains the same – getting the most points.

Other deletion games on graphs have already been studied in the past. In [2] for example, the game is played on an hypergraph, both players deleting either a vertex or an edge. We could cite the game "Geography" (in [3,4]), where the deletion of an edge (or a vertex) depends on the edge (or vertex) deleted before. Combinatorial games are often defined with the rule "last player that moves wins". That is the case in examples cited before, unlike "Le Pic Arête", where the winner is defined by the number of points he gets. In terms of winning strategy, our study consists in finding the maximum number of points attainable by each player (his opponent is supposed to play according to his best strategy). The computation of these values enables us to decide whether a given game configuration is a first or a second-player win. Note that our results are an extension of the work of Meyniel and Roudneff (see [5]).

2. Classifying the Configurations

Let C be a game configuration. Note that the sum of the final scores of both players is always equal to n, where n is the number of vertices of C. We define g_1 as the maximum score attainable by the first player, when the second player tries to minimize this score. Denote by g_2 the value $n - g_1$. The previous remark ensures that g_2 is the maximum score attainable by the second player, when the first player tries to minimize it.

We define the **value** of C, $w(C)$, as follows:

$$w(C) = g_1 - g_2.$$

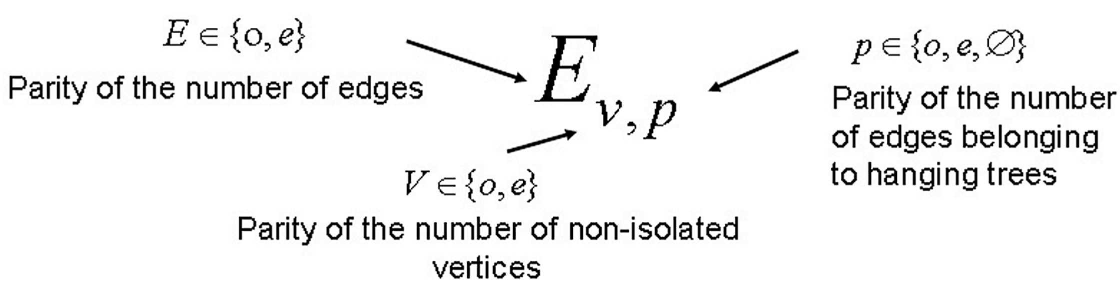
This means that a configuration with a positive value is a first-player win. In a zero value configuration, no matter who begins, as no one has a winning strategy.

For example, if you play on a triangle, you should prefer being the first player as this configuration has a value equal to 1. Conversely, a square is a second-player win configuration, as its value is equal to -2.

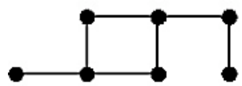
The problem now consists of finding which parameters of a given configuration play a role in the computation of its value. The study of small configurations gets us to consider the following parameters:

- number of edges
- number of non-isolated vertices
- number of edges belonging to hanging trees
- number of connected graphs constituting the configuration

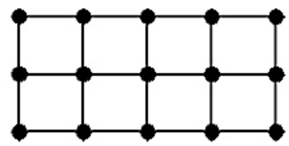
We first define a classification of the connected graphs:



This classification defines 12 types of connected graphs. We add an extra type noted $I.E$, corresponding to the case of an isolated edge. Indeed we do not consider an isolated edge as a graph of type $O_{e,o}$, since it is the only case where a player wins two points by deleting an edge. Indices E, V and P have two possible values, denoted by o and e (for "odd" and "even"). There exists an additional value for the index P, denoted by $\emptyset$. That corresponds to the configurations without a hanging edge (on such configurations, there exists no immediate deletion that gives points). We use the symbol $*$ to describe that an index takes any value of its definition set (for instance, $E_{*,*}$ defines connected graphs with an even number of edges). The following diagrams show examples of connected graphs of types $O_{o,o}$ and $E_{o,\emptyset}$:



$O_{o,o}$

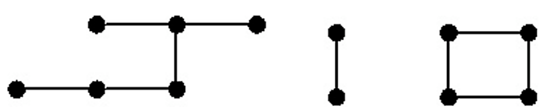


$E_{o,\emptyset}$

We now define a classification of the game configurations. By playing on small graphs, it seems that the type of a configuration (i.e. winning or losing for the first player) is linked to the parity of the number of "odd connected components" (i.e. graphs with an odd number of edges). Taking this into account, our classification is defined over three bits (p_1, p_2, p_3), where:

- the parity of the number of isolated edges is denoted by p_1
- the parity of the number of components of type $O_{o,\emptyset}$ or $O_{o,e}$ is denoted by p_2
- the parity of the number of the other "odd components", i.e., the set $\{O_{o,o}, O_{e,\emptyset}, O_{e,o}, O_{e,e}\}$, is denoted by p_3

Hence we define a partition of the configuration set into eight subsets. For instance, the following configuration is of type (101):



3. Finding an Approximation of w

In this section, we give upper and lower bounds on $w(C)$ for any configuration C. These bounds will help us decide whether a given configuration is a first or second-player win. In some cases, they will lead to a good strategy of play.

The configuration set can be split into two groups: the configurations with an odd number of odd connected graphs (i.e., configurations of types (001), (010), (100) and (111)), and those with an even number of odd connected graphs (i.e., (000), (011), (101), (110)). Note that this split satisfies the following property: removing an edge from a configuration belonging to one of these two groups leads to a configuration belonging to the other group.

Given a game configuration C, denote by $|E_{v,p}|$ the number of graphs of type $E_{v,p}$ composing C. We define $f(C)$ as:

$$f(C) = |O_{o,\emptyset}| + |O_{o,e}| + 2|O_{o,o}| + |O_{e,*}| + |E_{o,*}| + 2|E_{e,\emptyset}| + 2|E_{e,e}| + 2\lfloor \frac{|E_{e,o}|}{2} \rfloor$$

Besides, denote by $w(p_1, p_2, p_3)$ the value of $w(C)$ when C is of type (p_1, p_2, p_3). Before stating the main theorem, we give a useful lemma.

Lemma 1 On a connected graph G with an odd number of edges there exists a "maximal odd chain", i.e. a chain extracted from G with an odd number of edges, with inner vertices of degree 2 and both extremities are vertices of degree different from 2. This chain may be a cycle.

Proof. Label each edge of G with the value 1. Then apply the following algorithm: For each remaining vertex of degree 2, delete it and replace both adjacent edges (labeled a and b for example) with a single edge labeled $a + b$, as shown below:

At the end, we get a labeled graph (not necessarily simple and without loop) where each edge corresponds to a maximal chain. The label of an edge in the resulting graph is equal to the number of corresponding edges in the initial graph. As G has initially an odd number of edges, there exists at least an edge with an odd label in the resulting graph. $\square$

Theorem 2 Let C be a game configuration. The value $w(C)$ satisfies the following inequalities:

$$\left\{\begin{array}{lll} \text{w(001)} & \geq & \text{f(C)+1} \\ \text{w(010)} & \geq & \text{f(C)} \quad \text{if } |E_{e,o}| \equiv 0(2) \\ & \geq & \text{f(C)+2} \quad \text{if } |E_{e,o}| \equiv 1(2) \\ \text{w(100)} & \geq & \text{f(C)+2} \\ \text{w(111)} & \geq & \text{f(C)+1} \quad \text{if } |E_{e,o}| \equiv 0(2) \\ & \geq & \text{f(C)+3} \quad \text{if } |E_{e,o}| \equiv 1(2) \end{array}\right. \qquad \left\{\begin{array}{lll} \text{w(000)} & \leq & \text{-f(C)} \\ \text{w(011)} & \leq & \text{-f(C)+1} \quad \text{if } |E_{e,o}| \equiv 0(2) \\ & \leq & \text{-f(C)-1} \quad \text{if } |E_{e,o}| \equiv 1(2) \\ \text{w(101)} & \leq & \text{-f(C)+1} \\ \text{w(110)} & \leq & \text{-f(C)+2} \quad \text{if } |E_{e,o}| \equiv 0(2) \\ & \leq & \text{-f(C)} \quad \text{if } |E_{e,o}| \equiv 1(2) \end{array}\right.$$

Proof. By way of contradiction, assume there exists a configuration C for which one of these inequalities is not true, and choose C as the "smallest" counter-example in terms of number of edges. We distinguish two cases about C, according to the group of configurations to which it belongs.

C contains an even number of odd connected graphs: As a counter-example, C satisfies an inequality of the type $w(C) > -f(C) + \varepsilon$, where $\varepsilon \in \{-1, 0, 1, 2\}$. This means that if $C \neq \emptyset$, there exists a move from C to a resulting configuration C', giving g points, and such that $w(C') < f(C) - \varepsilon + g$. We will get a contradiction by showing that each legal move from C cannot lead to such an inequality. For each of the 13 types of connected graphs, we need to know to which types of resulting graphs a move can lead. The following table contains all the moves playable from a connected graph G:

Initial component	Resulting components after one move		
	Giving points	Non-disconnecting	Disconnecting (2 resulting graphs)
I.E.	$\emptyset$(2 pts)	-	-
$O_{o,\emptyset}$	-	$E_{o,*}$	$O_{o,*} + O_{e,*}$, $E_{o,*} + E_{e,*}$
$O_{o,e}$	$E_{e,o}$(1 pt)	$E_{o,e/o}$	$O_{o,*} + O_{e,o/e}$, $O_{o,o/e} + O_{e,\emptyset}$, $O_{o,*} + I.E.$, $E_{o,*} + E_{e,o/e}$, $E_{o,o/e} + E_{e,\emptyset}$
$O_{o,o}$	$E_{e,e/\emptyset}$(1 pt)	$E_{o,e/o}$	$O_{o,*} + O_{e,o/e}$, $O_{o,o/e} + O_{e,\emptyset}$, $O_{o,*} + I.E.$, $E_{o,*} + E_{e,o/e}$, $E_{o,o/e} + E_{e,\emptyset}$
$O_{e,\emptyset}$	-	$E_{e,*}$	$O_{o,*}+O_{o,*}$, $O_{e,*}+O_{e,*}$, $E_{o,*}+E_{o,*}$, $E_{e,*}+E_{e,*}$
$O_{e,e}$	$E_{o,o}$(1 pt)	$E_{e,o/e}$	$O_{o,*}+O_{o,o/e}$, $O_{o,o/e}+O_{o,\emptyset}$, $O_{e,*}+O_{e,o/e}$, $O_{e,o/e}+O_{e,\emptyset}$, $O_{e,*}+I.E.$, $E_{o,*}+E_{o,o/e}$, $E_{o,o/e}+E_{o,\emptyset}$, $E_{e,*}+E_{e,o/e}$, $E_{e,o/e}+E_{e,\emptyset}$
$O_{e,o}$	$E_{o,e/\emptyset}$(1 pt)	$E_{e,o/e}$	$O_{o,*}+O_{o,o/e}$, $O_{o,o/e}+O_{o,\emptyset}$, $O_{e,*}+O_{e,o/e}$, $O_{e,o/e}+O_{e,\emptyset}$, $O_{e,*}+I.E.$, $E_{o,*}+E_{o,o/e}$, $E_{o,o/e}+E_{o,\emptyset}$, $E_{e,*}+E_{e,o/e}$, $E_{e,o/e}+E_{e,\emptyset}$
$E_{o,\emptyset}$	-	$O_{o,*}$	$O_{o,*} + E_{e,*}$, $E_{i,*} + O_{e,*}$
$E_{o,e}$	$I.E.$, $O_{e,o}$(1 pt)	$O_{o,o/e}$	$O_{o,*}+E_{e,o/e}$, $O_{o,o/e}+E_{e,\emptyset}$, $E_{i,*}+O_{e,o/e}$, $E_{i,o/e}+O_{e,\emptyset}$, $E_{i,*}+I.E.$
$E_{o,o}$	$O_{e,e/\emptyset}$(1 pt)	$O_{o,o/e}$	$O_{o,*}+E_{e,o/e}$, $O_{o,o/e}+E_{e,\emptyset}$, $E_{i,*}+O_{e,o/e}$, $E_{i,o/e}+O_{e,\emptyset}$, $E_{i,*}+I.E.$
$E_{e,\emptyset}$	-	$I.E.$, $O_{e,*}$	$O_{o,*} + E_{o,*}$, $O_{e,*} + E_{e,*}$
$E_{e,e}$	$O_{o,o}$(1 pt)	$O_{e,o/e}$	$O_{o,*}+E_{o,o/e}$, $O_{o,o/e}+E_{o,\emptyset}$, $O_{e,*}+E_{e,o/e}$, $O_{e,o/e}+E_{e,\emptyset}$, $E_{e,*}+I.E.$
$E_{e,o}$	$O_{o,e/\emptyset}$(1 pt)	$O_{e,o/e}$	$O_{o,*}+E_{o,o/e}$, $O_{o,o/e}+E_{o,\emptyset}$, $O_{e,*}+E_{e,o/e}$, $O_{e,o/e}+E_{e,\emptyset}$, $E_{e,*}+I.E.$

Suppose C of the type (000), with $|E_{e,o}| \equiv 0(2)$, and such that $w(C) > -f(C)$. If there exists no move from C, then C does not contain any edge, and thus $w(C) = f(C) = 0$, which yields a contradiction. Otherwise, there exists a move from C to some C', giving g points, and such that $w(C') < f(C) + g$ (*). We consider each move from C appearing in the table above and show that (*) can never be satisfied. For example, consider the removing of an edge from an $O_{o,e}$ component belonging to C. There are several possible resulting sets:

- Deleting a hanging edge (which gives one point) leaves an $E_{e,o}$ component to the other player. The resulting configuration is of the type (010) with $|E_{e,o}| \equiv 1(2)$. As C' has less edges than C and by minimality of C, we have $w(C') \geq f(C') + 2$. Besides we have $f(C') = f(C) - 1$. This is in contradiction with (*).

- Playing from $O_{o,e}$ to $E_{o,o}$ (with a gain $g = 0$). The resulting configuration is (010), with $|E_{e,o}| \equiv 0(2)$. Then we have by minimality of C: $w(C') \geq f(C') = f(C)$, ensuring a contradiction.

- Removing a disconnecting edge from an $O_{o,e}$ can lead to $O_{o,\emptyset}+O_{e,e}$ for example ($g = 0$). The (001) resulting configuration satisfies $w(C') \geq f(C') + 1 = f(C) + 2$, and ensures a contradiction with (*).

- For each other move from $O_{o,e}$, and from each of the 12 other types, we can show similarly that we get a contradiction with (*).

This implies that C is not of the type (000) with $|E_{e,o}| \equiv 0(2)$. Then we suppose that C is of each other type of this group (i.e., (011), (101), and (110), with the $|E_{e,o}|$ parity equal to 0 or 1). Like previously, a careful examination (that we do not detail in this paper) of all the moves listed in the table proves that all the inequalities of the type $w(C) > -f(C) + \varepsilon$ are contradicted, so that the minimal counter-example C does not belong to this group of configurations.

C **contains an odd number of odd connected graphs:** The second part of the proof is linked to the strategy of play when the configuration contains an odd number of "odd connected graphs". In this case, the counter-example C satisfies an inequality of the type $w(C) < f(C) + \varepsilon$, $\varepsilon \in \{0, 1, 2, 3\}$. This means that for every move from C to some C' giving g points, we have $w(C') > -f(C) - \varepsilon + g$. We will get a contradiction by finding a move for which this inequality is not true. Consider all different cases about C:

- C is of the type (100) and satisfies $w(C) < f(C) + 2$. For every move from C to C' with a gain g, we have $w(C') > -f(C) - 2 + g$. As $p_1 = 1$ in this type of configuration, we choose the deletion of an isolated edge, giving two points. The resulting configuration C' is (000) and satisfies $w(C') \leq -f(C') = -f(C)$ as C is minimal. We get a contradiction with the previous inequality.

- C is of the type (111). We choose the same kind of move, i.e. the deletion of an isolated edge. No matter the parity of $|E_{e,o}|$, we get a contradiction.

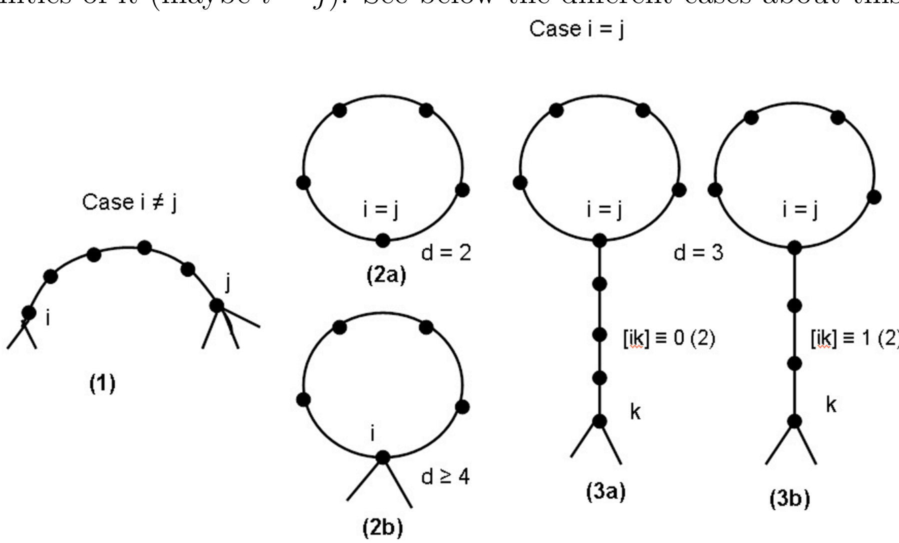
- C is of the type (010). The parity of $|E_{e,o}|$ defines two cases:
  - $|E_{e,o}| \equiv 1(2)$. C satisfies $w(C) < f(C) + 2$ and for every move from C to C' (with a gain g), we have $w(C') > -f(C) - 2 + g$. We choose to remove a hanging edge from a $E_{e,o}$ component belonging to C, which gives one point and creates a $O_{o,\emptyset/e}$. Hence the resulting configuration C' is (000), and satisfies $w(C') \leq -f(C') = -f(C) - 1$, which yields a contradiction.

   - $|E_{e,o}| \equiv 0(2)$. We have $w(C) < f(C)$, implying that for every move from C to C' giving g points: $w(C') > -f(C) + g$. In a (010) configuration, there exists a $O_{o,\emptyset}$ or a $O_{o,e}$ component. If there exists a $O_{o,\emptyset}$ component, we remove a non-disconnecting edge from it. Indeed, such an edge exists: if all the edges were disconnecting ones, the component would be a tree, contradicting $p = \emptyset$ in a $O_{o,\emptyset}$. This move gives no point and leaves a resulting component $E_{o,*}$ in a (000) configuration C', satisfying $w(C') \leq -f(C') = -f(C)$. Otherwise, there exists a $O_{o,e}$ component, and we can remove a hanging edge to leave a $E_{e,o}$ in a (000) configuration, winning one point. The resulting configuration C' satisfies $w(C') \leq -f(C') = -f(C) + 1$. In both cases, the contradiction is guaranteed.

- C is of the type (001). If $|E_{e,o}| \equiv 1(2)$, we choose the same move as previously to get the contradiction, i.e. the deletion of a hanging edge from a $E_{e,o}$ component. If $|E_{e,o}| \equiv 0(2)$, we only know that C contains a component in the form $O_{o,o}$, $O_{e,e}$, $O_{e,o}$ or $O_{e,\emptyset}$. Moreover, we know that $w(C) < f(C) + 1$. This means that for every move from C to C' winning g points, we have $w(C') > -f(C) - 1 + g$. Two cases are considered:

   - There exists a component with at least a hanging edge (i.e. $O_{o,o}$, $O_{e,e}$ or $O_{e,o}$). As previously, the deletion of a hanging edge gives a (000) resulting set with one point won, and leads to a contradiction.

   - If C does not contain such a component, we remove an edge belonging to a $O_{e,\emptyset}$. Almost each move from an $O_{e,\emptyset}$ component (appearing in the table) to a resulting configuration C' leads to an inequality of the form $w(C') \leq -f(C) + 1$, which is a contradiction of the initial inequality of this case. The only case without contradiction is the move that creates a single $E_{e,o}$. We have thus to prove that it is always possible to find a move different from this one.
     First of all, we use lemma 1, which states that there exists a maximal odd chain taken from the $O_{e,\emptyset}$ component. This chain may be a cycle. We note i and j both extremities of it (maybe $i = j$). See below the different cases about this chain:



     * The first case is $i \neq j$. As a $O_{e,\emptyset}$ component does not contain vertices of

degree one, i and j are vertices of degrees at least 3. As every inner vertex has degree 2, if we remove an edge from the chain, 1 or 2 hanging trees will appear, whose total number of edges is even. We cannot create a single $E_{e,o}$.

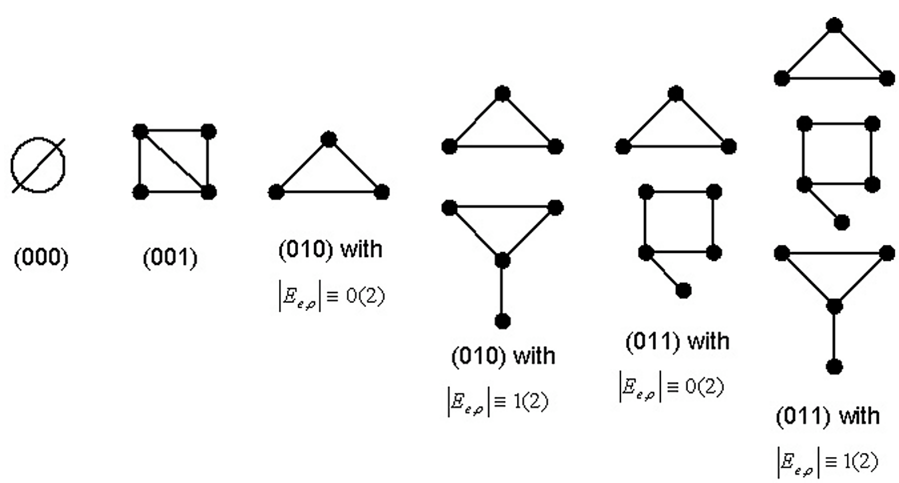
* Consider the case where the chain is a cycle and note i a vertex of maximum degree of this cycle. Figures ($2a$) and ($2b$) present cycles where respectively $deg(i) = 2$ and $deg(i) \geq 4$. In both cases, we remove an edge of this cycle, giving a resulting graph where the hanging trees created have an even total number of edges. If $deg(i) = 3$, we consider the length l of the chain between i and k (where k is the first vertex of this chain with a degree upper than 2). If l is even (case ($3a$)), we can play on the cycle, as the resulting component contains a hanging tree with an even number of edges. Otherwise, as we cannot play on the cycle (or a single $E_{e,o}$ would appear), we remove an edge from the chain between i and k (a such edge exists as the chain has an odd length). The graph is disconnected, so that a single $E_{e,o}$ cannot appear. □

This result leads to the following corollary, about winning positions of this game.

Corollary 3 Configurations with an odd number of "odd connected components" are winning for the first player. Configurations of the types (000) and (101) are non-winning. Those of the types (011) and (110) when $|E_{e,o}| \equiv 1(2)$ are losing ones.

Proof. As $f(C) \geq 0$, it seems clear that configurations with an odd number of "odd connected components" have a value strictly greater than 0 (note that $f(010) \geq 1$).
If a configuration C is of the type (101), then $f(C) \leq 1$, so that we can assert that $w(000)$ and $w(101)$ are lower or equal to zero, which means that they are not winning.
Concerning the other configurations, we have $f(011) \geq 2$ and $f(110) \geq 1$. This leads to the conclusion. □

Note that the configurations of the type (110) with an even number of $E_{e,o}$ components are the only ones of their group that may be winning. According to this choice of parameters, the bounds given by the theorem are the best ones that we can get. For each type of configuration, we have found an instance whose value is equal to the announced bound:



The examples of configurations of the type (1..) whose value is equal to the bound are obtained by adding an isolated edge to each corresponding example of the type (0..) presented above.

For each winning configuration, we cannot decide what the best move is, but we have a strategy that wins at least the number of points announced by the lower bound of the game value w. This is enough to win the game.

Conversely, we have no strategy of play when the configuration has an even number of "odd connected components". In most of these cases, it would consist in finding a way to lose "with honors". For example, it seems conceivable to remove an isolated edge when the configuration is of type (101) or (110). What about both other types of configurations?

Finding lower bounds for this group of configurations would help us to answer this question. However, the configuration coding choice over 3 bits is not good enough to get such bounds. Actually this problem can be expressed with a linear system, from which we have extracted a small sub-system with no solution. A larger coding system (over 13 bits at the most) may lead us to find both lower and upper bounds of the value w, but it would give a too large number of configuration types, making the proof not very exciting. Besides, another coding for the set of the connected graphs could be envisaged to improve these results.

Remark Online, at `http://www-leibniz.imag.fr/LAVALISE/PicArete/index.html`, "Le Pic Arête" can be played.

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