ON THE IRRATIONALITY OF A DIVISOR FUNCTION SERIES

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Abstract

Here, we show, unconditionally for $k = 3$, and on the prime k-tuples conjecture for $k \geq 4$, that $\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$ is irrational, where $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n.

1. Introduction

For a positive integer n put

$$\sigma_k(n) = \sum_{d \mid n} d^k$$

for the sum of the kth powers of the (positive) divisors of n. In [4], Erdős and Kac showed that the series

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$$

is irrational for both $k = 1$ and $k = 2$. The problem is mentioned also in [5], wherein it is stated that the method does not seem to extend to $k \geq 3$, and it appears as B14 in [6]. Let

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us quickly give proofs of the irrationality of these series when $k = 0, 1, 2$. Assume that the given series is A/B, where A and B are positive integers. Multiplying by $(n-1)!$, where $n > B$, we get that

$$\sum_{j=1}^{n-1} \sigma_k(j) \frac{(n-1)!}{j!} + \frac{\sigma_k(n)}{n} + \frac{\sigma_k(n+1)}{n(n+1)} + \sum_{j \geq 2} \frac{\sigma_k(n+j)}{n(n+1)\cdots(n+j)} = \frac{A(n-1)!}{B}.$$

The first of the above sums is an integer and the right hand side is also an integer. The second sum is positive and, for $k \leq 2$, its size is

$$\leq \frac{\sigma_2(n+2)}{n(n+1)(n+2)} + \sum_{j \geq 3} \frac{\sigma_2(n+j)}{n(n+1)\cdots(n+j)} \ll \frac{1}{n} + \sum_{m \geq n} \frac{1}{m^2} \ll \frac{1}{n},$$

so that this term belongs to the interval $[0, c/n]$, where c is an absolute constant. We shall take n to be a prime $n \equiv 1 \pmod 4$ such that $(n+1)/2$ has no prime factor $< y$, where y is a large positive real number. Such primes exist by Dirichlet's theorem on primes in arithmetical progressions. Then

$$\frac{\sigma_1(n+1)}{n(n+1)} \leq \frac{c_1 \log\log n}{n}$$

for some constant c_1, and

$$\begin{aligned} \frac{5}{4} &\leq \frac{\sigma_2(n+1)}{n(n+1)} \leq \frac{5(n+1)}{4n} \prod_{q \geq y} \left(1 + \frac{1}{q^2} + \cdots\right) \\ &\leq \frac{5(n+1)}{4n} \exp\left(\sum_{q \geq y} \frac{1}{q(q-1)}\right) = \frac{5(n+1)}{4n} e^{O(1/y)} \\ &= \frac{5}{4} + O\left(\frac{1}{y}\right) \end{aligned} \tag{1}$$

whenever $n > y$. Thus, when $k = 1$, we get that $\sigma_1(n)/n = 1 + 1/n$, and we conclude that there is an integer in the interval $[1/n, (c + 1 + c_1 \log\log n)/n]$, while when $k = 2$ we get that $\sigma_2(n)/n = n + 1/n$, and so there exists an integer in the interval $[1/4, 1/4 + c/n + c_2/y]$, where c_2 is the constant implied by the O symbol in (1). Choosing n (and y) to be sufficiently large, we get a contradiction in the case $k = 1$ (and $k = 2$), which completes the proof.

In this paper, we make a modest contribution to the problem, proving two results about the irrationality of the above series for $k \geq 3$.

Theorem 1. *The sum of the series*

$$\sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n!} \tag{2}$$

is irrational.

We recall the statement of the *Prime k-tuples Conjecture* (see [3, 7, 9]), which is due to Dickson.

Conjecture 1. *For any $k \geq 2$, let $a_1, \dots, a_k$ and $b_1, \dots, b_k$ be integers with $a_i > 0$ for each $i = 1, \dots, k$. Suppose that for every prime number p there exists an integer n such that $\prod_{i=1}^{k}(a_i n + b_i)$ is not a multiple of p. Then there exist infinitely many positive integers n such that $p_i = a_i n + b_i$ is prime for all $i = 1, \dots, k$.*

We have the following result for any positive integer k.

Theorem 2. *The Prime k-tuples Conjecture implies that*

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$$

is irrational.

Throughout this paper, we use the Landau symbols O and o as well as Vinogradov's symbols $\gg$, $\ll$ and $\asymp$ with their regular meaning. The constants implied by these symbols depend at most on k. We use p and q to denote prime numbers.

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2. Proof of Theorem 1

Our main tool is the following well-known theorem of Chen (see [1], [2]).

Theorem 3. *Let a be an integer. There exists x_a such that for $x > x_a$, the interval $[x/2, x]$ contains $\gg xa/\varphi(a)^2(\log x)^2$ prime numbers $p \equiv 1 \pmod{a}$ such that $q = (p+1)/2$ is either a prime or a product of two primes each one of which exceeds $x^{1/10}$.*

A proof containing the basic ideas for this theorem appears, for example, in Chapter 11 of [8]. Chen actually proved that for large even integers N there are $\gg N/(\log N)^2$ primes p such that $N - p$ is either a prime or a product of two large primes. Easy and well-known modifications of Chen's argument yield the above theorem.

To prove our Theorem 1 we assume that the sum of the series shown at (2) is rational and deduce a contradiction. We write it, as we did for $k = 0$, 1 and 2 as A/B, multiply across by $(n-1)!$ for some large n (with $n > B$), and obtain

$$\frac{\sigma_3(n)}{n}+\frac{\sigma_3(n+1)}{n(n+1)}+\frac{\sigma_3(n+2)}{n(n+1)(n+2)}+\sum_{j\geq 3}\frac{\sigma_3(n+j)}{n(n+1)\cdots(n+k)}=(n-1)!\left(\frac{A}{B}-\sum_{j=1}^{n-1}\frac{\sigma_k(j)}{j!}\right), \tag{3}$$

where the right hand side is an integer. In what follows, we shall exploit the above relation. Since $\sigma_3(n) \ll n^3$, we have

$$\sum_{j\geq 3}\frac{\sigma_3(n+j)}{n(n+1)\cdots(n+j)} \ll \frac{1}{n}+\sum_{m\geq n}\frac{1}{m^2} \ll \frac{1}{n}.$$

Furthermore, the sum appearing on the left hand side of formula (3) above is a positive integer. We now let y be a large positive real number which we shall fix later, and take $a = 72\prod_{5\leq p\leq y} p$. Note that, by Chebyshev's bounds, this means that $y \asymp \log a$. We let x be large compared to a and $m \in [x/2a, x/a]$ be such that $p = am+1$ is prime and $q = (p+1)/2 = (am+2)/2$ is either a prime or a product of two primes q_1q_2 each exceeding $x^{1/10}$. Choose n to be the prime $n = p$. We then note that

$$\frac{\sigma_3(n)}{n} = n^2+\frac{1}{n}.$$

Furthermore, writing $n+2 = am+3 = 3t$, we have that t is coprime to all primes $q \leq y$, and so obtain the inequalities

$$\begin{aligned}\frac{28}{27} < \frac{\sigma_3(n+2)}{n(n+1)(n+2)} &= \frac{28}{27}\frac{\sigma_3(t)}{t^3}\left(1+O\left(\frac{1}{n}\right)\right)\\ &< \frac{28}{27}\prod_{p>y}\left(1-\frac{1}{p^3}\right)^{-1}\left(1+O\left(\frac{1}{n}\right)\right)\\ &< 1+\frac{1}{27}+O\left(\frac{1}{y^2}\right),\end{aligned}$$

since $y < \sqrt{x}$. Combining this with our estimate for the tail of the series we have, for n a prime as above,

$$\frac{\sigma_3(n)}{n}+\frac{\sigma_3(n+2)}{n(n+1)(n+2)}+\sum_{j\geq 3}\frac{\sigma_3(n+j)}{n(n+1)\cdots(n+j)} = A_n+\frac{1}{27}+O\left(\frac{1}{y^2}\right), \tag{4}$$

where A_n is a positive integer.

We now consider the remaining term $\sigma_3(n+1)/(n(n+1))$.

Assume first that for some large x there is a prime $p = am+1 \in [x/2, x]$ such that $q = (p+1)/2 = (a/2)m+1$ is prime. Then

$$\begin{aligned}\frac{\sigma_3(n+1)}{n(n+1)} &= \frac{\sigma_3(2q)}{2q(2q-1)} = \frac{9(q^3+1)}{2q(2q-1)}\\ &= \frac{9}{4}q+\frac{9}{8}+\frac{9q-4}{8q(2q-1)} = B_n+\frac{3}{8}+O\left(\frac{1}{n}\right)\end{aligned}$$

with some integer B_n, where we have used the fact that $q = (a/2)m + 1 \equiv 1 \pmod 4$, because $8 \mid a$. Summing up everything, we find that

$$\frac{1}{27} + \frac{3}{8} + O\left(\frac{1}{y^2}\right) \in \mathbb{Z},$$

which is impossible if y is chosen to be sufficiently large.

So, we are left with the more interesting part of the problem where $q = q_1 q_2$, where $q_1 > q_2 > x^{1/10}$ holds for all the $\gg xa/\varphi(a)^2(\log x)^2$ choices of primes $p = am + 1 \in [x/2, x]$ guaranteed by Chen's Theorem 3.

We put

$$M = \left\lfloor \frac{\sigma_3(n+1)}{n(n+1)} \right\rfloor = \left\lfloor \frac{9\sigma_3(q)}{2q(2q-1)} \right\rfloor.$$

Note that since $q_2 \le q^{1/2}$, we have that

$$\frac{\sigma_3(q)}{2q(2q-1)} = \frac{q^3 + q_1^3 + q_2^3 + 1}{2q(2q-1)} = \frac{q^3 + q_1^3 + O(q^{3/2})}{2q(2q-1)} = \frac{q^3 + q_1^3}{2q(2q-1)} + O\left(\frac{1}{q^{1/2}}\right).$$

Thus, using also the previous calculations of fractional parts (see (4)), we find that for large x,

$$M + \frac{26}{27} - \frac{c_0}{y^2} \le \frac{9q^3 + 9q_1^3}{2q(2q-1)} \le M + \frac{26}{27} + \frac{c_0}{y^2}$$

holds for all large x with some constant $c_0 > 0$, an inequality which can be rewritten as

$$\begin{aligned} & q^2\left(4M + 3 - 9q + \frac{23}{27} - \frac{2q(m + O(1))}{q^2} - \frac{4c_0}{y^2}\right) \le 9q_1^3 \\ \le\ & q^2\left(4M + 3 - 9q + \frac{23}{27} - \frac{2q(m + O(1))}{q^2} + \frac{4c_0}{y^2}\right). \end{aligned}$$

From this inequality and the fact that $q_1 < qx^{-1/10}$, we find that

$$\frac{M}{q} = \frac{9(q^3 + q_1^3)}{2q^2(2q-1)} + O\left(\frac{1}{q}\right) = \frac{9}{4} + O\left(\frac{1}{x^{3/10}}\right)$$

so

$$-\frac{2q(M + O(1))}{q^2} = -\frac{9}{2} + O\left(\frac{1}{x^{3/10}}\right).$$

Hence, we deduce that

$$4M + 3 - 9q + \frac{23}{27} - \frac{2q(M + O(1))}{q^2} = L + \frac{19}{54} + O\left(\frac{1}{x^{3/10}}\right)$$

holds with some non-negative integer L. Thus, for large x, provided that $y < x^{3/20}$, we get that

$$\frac{q^2}{9}\left(L + \frac{19}{54} - \frac{5c_0}{y^2}\right) < q_1^3 < \frac{q^2}{9}\left(L + \frac{19}{54} + \frac{5c_0}{y^2}\right),$$

which is equivalent to

$$\frac{q_2^2}{9}\left(L+\frac{19}{54}-\frac{5c_0}{y^2}\right) < q_1 < \frac{q_2^2}{9}\left(L+\frac{19}{54}+\frac{5c_0}{y^2}\right). \tag{5}$$

The above inequalities certainly tell us that $L \ge 0$, provided that y is sufficiently large. Further, the left hand side of the above inequality is $> q_2^2/27$ if $y^2 > 220c_0$, so if y satisfies the above inequality and x is large, then $q_2^2/27 \le q_1 \le x/q_2$, and therefore $q_2 \le 3x^{1/3}$. Now fix q_2. Then the left hand side of the above inequality is $\ge (L+1/3)q_2^2/27$ and the middle term satisfies $q_1 \le x/q_2$. Thus, $L+1/3 \le 27x/q_2^3$. Since $L \ge 0$ is an integer, it follows that the number of possibilities for L is

$$1+\left\lfloor\frac{27x}{q_2^3}\right\rfloor \le \frac{81x}{q_2^3}. \tag{6}$$

We now fix also $L \ge 0$. Then q_1 is a prime in the interval shown at (5) above such that $q_1q_2 \equiv 1 \pmod a$ and $2q_1q_2 - 1 = p$ is a prime. By the Brun sieve, the number of such primes is

$$\ll \frac{c_0q_2^2}{y^2}\cdot\left(\frac{a}{\varphi(a)^2}\right)\cdot\frac{1}{\left(\log(10c_0q_2^2/(9y^2a))\right)^2}.$$

Since $q_2 > x^{1/10}$, it follows that for large x we have $10c_0q_2^2/(9y^2a) > x^{1/6}$. Thus, the number of possibilities for q_1 once q_2 and L are fixed is

$$\ll \frac{1}{y^2}\cdot\frac{a}{\varphi(a)^2}\cdot\frac{q_2^2}{(\log x)^2}.$$

Summing up over all the possibilities for L shown at (6), we get that the total number of possibilities for q_1 when q_2 is fixed is

$$\ll \frac{1}{y^2}\cdot\frac{a}{\varphi(a)^2}\cdot\frac{x}{(\log x)^2}\cdot\frac{1}{q_2},$$

and now summing up the above bound over all primes $q_2 \in [x^{1/10}, 3x^{1/3}]$, we get that the total number of possibilities for p is

$$\ll \frac{1}{y^2}\frac{a}{\varphi(a)^2}\frac{x}{(\log x)^2}\sum_{x^{1/10}\le q_2\le 3x^{1/3}}\frac{1}{q_2} \ll \frac{1}{y^2}\frac{a}{\varphi(a)^2}\frac{x}{(\log x)^2}, \tag{7}$$

where the fact that the last sum above is $O(1)$ follows from Mertens's estimate for the summatory function of the reciprocal of the primes. Let c_1 be the constant implied in the Vinogradov symbol from Chen's Theorem 3 and c_2 be the constant implied in the last Vinogradov symbol in (7). Comparing the estimates for the number of primes p under scrutiny we get

$$\frac{c_1ax}{\varphi(a)^2(\log x)^2} \le \frac{c_2ax}{\varphi(a)^2y^2(\log x)^2}$$

leading to $y^2 < c_3$, where $c_3 = c_2/c_1$. Choosing y larger than this we complete the proof of Theorem 1.

3. Proof of Theorem 2

Let $k \geq 4$. For $i = 1, \ldots, k$ we let $Q_i(X),\ R_i(X) \in \mathbb{Z}[X]$ be the polynomials given by the division algorithm:

$$(X+i)^k = Q_i(X)(X+1)\cdots(X+i) + R_i(X) \qquad i = 1, \ldots, k,$$

where $\deg Q_i(X) = k - i$ and $\deg R_i(X) \leq i - 1$. Note that when $i = k$ we have that $Q_k(X) = 1$. For each of those $i = 1, \ldots, k$ for which $Q_i(-i) \neq 0$ choose distinct primes $p_i > k$ such that $p_i \nmid \sigma_k(i)Q_i(-i)$. (For any other i, for notational purposes, take $p_i = 1$.) As we have seen, $Q_k(-k) = 1 \neq 0$, so that $p_k \nmid \sigma_k(k)Q_k(-k)$ can be arranged simply by choosing $p_k > \sigma_k(k)$.

Let $P = \prod_{i=1}^k p_i$ and let m be a positive integer such that

$$\frac{(k!)^2}{i} m + 1 \equiv p_i \pmod{p_i^2} \qquad i = 1, \ldots, k.$$

By the Chinese Remainder Theorem, there are infinitely many such positive integers m and they form the arithmetic progression $m_0 \pmod{P^2}$, where m_0 is the first positive integer in this progression. Given such an m, we choose $n = (k!)^2 m$ and write $m = m_0 + \ell P^2$ with some nonnegative integer ℓ. Then

$$n + i = i\left(\frac{(k!)^2}{i} m + 1\right) = ip_i\left(\frac{(k!)^2}{i}\frac{P^2}{p_i}\ell + \frac{(k!)^2 m_0/i + 1}{p_i}\right).$$

Put

$$A_i = \frac{(k!)^2}{i}\frac{P^2}{p_i} \quad \text{and} \quad B_i = \frac{(k!)^2 m_0/i + 1}{p_i}$$

for $i = 1, \ldots, k$.

One checks easily that we can apply the Prime k-tuples Conjecture 1 to the linear polynomials $A_i\ell + B_i$. Indeed, the prime numbers dividing A_i are exactly the primes $p \leq k$ together with the primes p_i. Since $B_i \mid (k!)^2 m_0/i + 1$, we see that B_i is coprime to all primes $p \leq k$. Further, $B_i \equiv 1 \pmod{p_i}$ from the way m_0 was chosen. To see that B_i is coprime to p_j if $j \neq i$ assume otherwise. Then $p_j \mid ((k!)^2/j)m_0 + 1$ and $p_j \mid B_i \mid ((k!)^2/i)m_0 + 1$. Hence, $p_j \mid (k!)^2 m_0 + j$ and $p_j \mid (k!)^2 m_0 + i$, so $p_j \mid (j - i)$, but this is false because $p_j > k$. Thus, the conditions for the applicability of the Prime k-tuples Conjecture 1 are fulfilled and we can let ℓ be large and such that $A_i\ell + B_i = q_i$ are all primes for $i = 1, \ldots, k$. Put $a_i = ip_i$ and note that if $m = m_0 + \ell P^2$, then $n + i = a_i q_i$ for $i = 1, \ldots, k$.

Now assume that the series

$$\sum_{j=1}^{\infty} \frac{\sigma_k(j)}{j!} = \frac{A}{B}$$

with A and B coprime integers. Suppose n is sufficiently large, and in particular $n > B$. Multiplying across by $n!$, we get

$$\sum_{j\leq n}\sigma_k(j)\frac{n!}{j!}+\sum_{i=1}^{k}\frac{\sigma_k(n+i)}{(n+1)\cdots(n+i)}+\sum_{j\geq k+1}\frac{\sigma_k(n+j)}{(n+1)\cdots(n+j)}\in\mathbb{Z}. \tag{8}$$

The first sum above is an integer, while the last sum is positive and, since $\sigma_k(n) \ll n^k$,

$$\begin{aligned}\sum_{j\geq k+1}\frac{\sigma_k(n+j)}{(n+1)\cdots(n+j)} &= \frac{\sigma_k(n+k+1)}{(n+1)\cdots(n+k+1)}+\sum_{j\geq k+2}\frac{\sigma_k(n+j)}{(n+1)\cdots(n+j)}\\ &\ll \frac{1}{n}+\sum_{j\geq k+2}\frac{1}{(n+j)^2}\ll\frac{1}{n}.\end{aligned}$$

As for the intermediate terms, since q_i is prime we have

$$\begin{aligned}\frac{\sigma_k(n+i)}{(n+1)\cdots(n+i)} &= \frac{\sigma_k(a_i)(q_i^k+1)}{(n+1)\cdots(n+i)}\\ &= \frac{\sigma_k(a_i)}{a_i^k}\left(\frac{(n+i)^k+a_i^k}{(n+1)\cdots(n+i)}\right)\\ &= \frac{\sigma_k(a_i)}{a_i^k}\left(Q_i(n)+\frac{R_i(n)+a_i^k}{(n+1)\cdots(n+i)}\right)\\ &= \frac{\sigma_k(a_i)}{a_i^k}Q_i(n)+O\left(\frac{1}{n}\right),\end{aligned}$$

where for the above error term we used the fact that deg $R_i \leq i-1$. Since $n \equiv -i \pmod{p_i}$, we get that $Q_i(n) \equiv Q_i(-i) \pmod{p_i}$ so that $Q_i(n) = Q_i(-i) + p_i\ell_i(n)$, for some integers $\ell_i(n)$. Thus,

$$\frac{\sigma_k(n+i)}{(n+1)\cdots(n+i)}=\frac{\sigma_k(ip_i)}{(ip_i)^k}\left(Q_i(-i)+\ell_i(n)p_i\right)+O\left(\frac{1}{n}\right).$$

We add everything together, multiply formula (8) by $(k!)^kP^{k-1}$ and get

$$\sum_{i=1}^{k}P_i^{k-1}\sigma_k(ip_i)\frac{(k!)^k}{i^k}\frac{1}{p_i}\left(Q_i(-i)+p_i\ell_i(n)\right)+O\left(\frac{1}{n}\right)\in\mathbb{Z},$$

where

$$P_i = Pp_i^{-1}=\prod_{\substack{1\leq j\leq k\\ j\neq i}}p_j\ .$$

From this it follows that

$$\sum_{i=1}^{k}P_i^{k-1}\frac{(k!)^k}{i^k}\frac{\sigma_k(ip_i)Q_i(-i)}{p_i}+O\left(\frac{1}{n}\right)\in\mathbb{Z}.$$

At this moment, we observe that the sum on the left does not depend on n and so

$$\sum_{1\le i\le k} P_i^{k-1}\frac{(k!)^k}{i^k}\frac{\sigma_k(ip_i)}{p_i}Q_i(-i)\in\mathbb{Z}.$$

Using also the fact that $\sigma_k(ip_i)=\sigma_k(i)\sigma_k(p_i)=\sigma_k(i)(p_i^k+1)$, we get that the above relation implies

$$\sum_{1\le i\le k} P_i^{k-1}\frac{(k!)^k}{i^k}\frac{\sigma_k(i)}{p_i}Q_i(-i)\in\mathbb{Z}.$$

This is a non-empty (since $Q_k(-k)=1$) sum of non-zero rational numbers whose reduced denominators are distinct primes. Thus, the above sum cannot be an integer. The proof of Theorem 2 is complete.

Note Added: March 2007. Shortly after this paper was submitted, we learned about the recent appearance of the paper [10], where the same two results as the present ones have been obtained. The proof of the case $k=3$ in [10] also uses sieve methods but is rather different, whereas the conditional proof for larger k's is similar to ours. We thank Professor Igor Shparlinski for pointing out this reference to us.

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