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RANDOM B_h SETS AND ADDITIVE BASES IN $\mathbb{Z}_N$

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Abstract

We determine a threshold function for B_h and additive basis properties in $\mathbb{Z}_n$.

1. Introduction

We use the following notations: $\mathbb{Z}$ denotes the integers $0, \pm 1, \pm 2, \dots$; $\mathbb{N}$ is the set of positive integers; $\mathbb{Z}_n$ is the additive cyclic group of order n. Members of a set S are referred to as $\{s_1, s_2, \dots\}$. The cardinality of a finite set S is denoted by $|S|$. A multiset $\mathbf{q} = \{q_1, \dots, q_k\}_m$ can be formally defined as a pair (Q, m), where Q is the set of distinct elements of $\mathbf{q}$ and $m : Q \to \mathbb{N}$, where $m(q)$ is the multiplicity of $q \in \mathbf{q}$ for each $q \in Q$. The number of distinct elements of $\mathbf{q}$ is denoted by $|\mathbf{q}|_d$. The usual set operations such as union, intersection and Cartesian product can be easily generalized for multisets. In this paper we use the intersection: suppose that (A, m) and (B, n) are multisets, then the intersection can be defined as $(A \cap B, f)$, where $f(x) = \min\{m(x), n(x)\}$.

For a given $S \subset \mathbb{Z}_n$ and $x \in \mathbb{Z}_n$ denote by $r_{S,h}(x)$ the number of different representations $x = s_1 + \cdots + s_h$ with $s_i \in S$, that is

$$r_{S,h}(x) = |\{\{s_1, \dots, s_h\}_m : s_1 + \cdots + s_h = x, \quad s_i \in S\}|.$$

A set $S \subset \mathbb{Z}_n$ is called B_h set if the number of distinct representation of x as $s_1 + \cdots + s_h$, $s_i \in S$ is at most 1, that is $r_{S,h}(x) \le 1$ for all $x \in \mathbb{Z}_n$. A set $S \subset \mathbb{Z}_n$ is called additive h-basis if every element in $\mathbb{Z}_n$ can be represented as the sum of not necessarily distinct h elements of the set S, that is $r_{S,h}(x) \ge 1$ for every $x \in \mathbb{Z}_n$.

For n a positive integer, let $0 \le p_n \le 1$. The random subset $S(n, p_n)$ is a probabilistic space over the set of subsets of $\mathbb{Z}_n$ determined by $Pr(k \in S_n) = p_n$ for every $k \in \mathbb{Z}_n$,

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with these events being mutually independent. This model is often used for proving the existence of certain sequences. Given any combinatorial number theoretic property P, there is a probability that $S(n, p_n)$ satisfies P, which we write $Pr\{S(n, p_n) \models P\}$. The function $r(n)$ is called a threshold function for a combinatorial number theoretic property P if

(i) When $p_n = o(r(n))$, $\lim_{n\to\infty} Pr\{S(n, p_n) \models P\} = 0$,

(ii) When $r(n) = o(p(n))$, $\lim_{n\to\infty} Pr\{S(n, p_n) \models P\} = 1$,

or visa versa.

The goal of this paper is to determine a threshold function for B_h sets and additive h-bases in $\mathbb{Z}_n$. We use the typical notation $\exp(x) = e^x$

Theorem 1.1. *Let $c > 0$ be arbitrary. Let us suppose that $p_n = \frac{c}{n^{\frac{2h-1}{2h}}}$ and the random set $A_n \subset \mathbb{Z}_n$ is defined the following way: For every $k \in \mathbb{Z}_n$ we have $Pr(k \in A_n) = p_n$. Then $\lim\limits_{n\to\infty} Pr\{A_n \text{ is a } B_h \text{ set}\} = \exp\left(-\frac{c^{2h}}{2(h!)^2}\right)$.*

Theorem 1.2. *Let c be an arbitrary real number. Suppose that $p_n = \frac{(h! n logn)^{1/h}(1+\frac{c}{hlogn})}{n}$ and the random set $A_n \subset \mathbb{Z}_n$ is defined the following way: For every $k \in \mathbb{Z}_n$ we have $Pr\{k \in A_n\} = p_n$. Then $\lim\limits_{n\to\infty} Pr(A_n \text{ is an additive } h\text{-basis}) = \exp(-\exp(-c))$.*

2. Proofs

In order to prove the theorems we need two lemmas from probability theory (see e.g. [1] p. 41, 95-98.). In many instances, we would like to bound the probability that none of the bad events B_i, $i \in I$, occur. If the events are mutually independent, then $Pr(\cap_{i\in I}\overline{B_i}) = \prod_{i\in I} Pr(\overline{B_i})$. When the B_i are "mostly" independent, the Janson's inequality allows us, sometimes, to say that these two quantities are "nearly" equal. Let Ω be a finite universal set and R be a random subset of Ω given by $Pr(r \in R) = p_r$, these events being mutually independent over $r \in \Omega$. Let E_i, $i \in I$ be subsets of Ω, where I a finite index set. Let B_i be the event $E_i \subset R$. Let X_i be the indicator random variable for B_i and $X = \sum_{i\in I} X_i$ be the number of E_is contained in R. The event $\cap_{i\in I}\overline{B_i}$ and $X = 0$ are then identical. For $i, j \in I$, we write $i \sim j$ if $i \neq j$ and $E_i \cap E_j \neq \emptyset$. We define $\Delta = \sum_{i\sim j} Pr(B_i \cap B_j)$, here the sum is over ordered pairs. We set $M = \prod_{i\in I} Pr(\overline{B_i})$.

Lemma 1.3 (Janson's inequality). *Let $B_i, i \in I, \Delta, M$ be as above and assume that $Pr(B_i) \leq \epsilon$ for all i. Then*

$$M \leq Pr\left(\bigcap_{i\in I} \overline{B}_i\right) \leq M \exp\left(\frac{1}{1-\epsilon} \cdot \frac{\Delta}{2}\right).$$

The more traditional approach to the Poisson paradigm is called Brun's sieve, for its use by the number theorist T. Brun. Let $F_1, \dots, F_m$ be events, X_i the indicator random variable for F_i, and $X = X_1 + \cdots + X_m$ the number of B_i that hold. Let there be a hidden parameter n (so that actually $m = m(n), B_i = B_i^{(n)}, X = X^{(n)}$) which will define our O notations. Define

$$S^{(r)} = \sum Pr\{B_{i_1} \wedge \cdots \wedge B_{i_r}\},$$

where the sum is over all sets $\{i_1, \dots, i_r\} \subseteq \{1, \dots, m\}$. The inclusion-exclusion principle gives that $Pr\{X = 0\} = Pr\{\overline{B}_1 \wedge \cdots \wedge \overline{B}_m\} = 1 - S^{(1)} + S^{(2)} - \cdots + (-1)^r S^{(r)} \cdots$.

Lemma 1.4. *Suppose there is a constant μ so that $E(X) = S^{(1)} \to \mu$ and such that for every fixed r,*

$$E\left(\frac{X^{(r)}}{r!}\right) = S^{(r)} \to \frac{\mu^r}{r!}.$$

Then $Pr\{X = 0\} \to \exp(-\mu)$ and, for every t, we have $Pr(X = t) \to \dfrac{\mu^t}{t!}\exp(-\mu)$.

In order to prove the theorems we need two lemmas. In the sequel, for the sake of brevity, we write $\mathbf{u} = \{u_1, \dots, u_h\}_m$ and $\mathbf{v} = \{v_1, \dots, v_h\}_m$ with $\mathbf{u} \neq \mathbf{v}$. For every $a \in \mathbb{Z}_n$ and $h, t \in \mathbb{N}$, $0 < t \leq h$, let

$$S_{a,h,t} = |\{\mathbf{u} : \quad u_i \in \mathbb{Z}_n \quad \sum_{i=1}^{h} u_i = a, \quad |\mathbf{u}|_d = t\}|$$

and for every $a_1, a_2 \in \mathbb{Z}_n$ and $h, t, s, k \in \mathbb{N}$ with $0 < k \leq \min\{s, t\}$ let

$$C_{a_1,a_2,h,t,s,k} = \left|\{\{\mathbf{u}, \mathbf{v}\} : \quad \sum_{i=1}^{h} u_i = a_1, \sum_{i=1}^{h} v_i = a_2, |\mathbf{u}|_d = s, |\mathbf{v}|_d = t, |\mathbf{u} \cap \mathbf{v}|_d = k\}\right|.$$

Lemma 1.5. *For every $a \in \mathbb{Z}_n$ and $h \geq 2$ we have*

1. $S_{a,h,h} = \frac{n^{h-1}}{h!} + O_h(n^{h-2})$;
2. $S_{a,h,t} = O_h(n^{t-1})$ *for* $1 \leq t \leq h - 1$.

Proof. Case (1): By the definition of $S_{a,h,h}$

$$h!S_{a,h,h} = \left|\{(u_1, \dots, u_h) : \quad u_i \in \mathbb{Z}_n, \quad \sum_{i=1}^{h} u_i = a, \quad and \quad u_i \neq u_j \quad for \quad i \neq j\}\right|. \quad (1)$$

An upper bound for (1) is $n(n-1)\dots(n-h+2)$ and a lower bound is $n(n-1)\dots(n-h+3)(n-(h-2)-(h-2)-2)$ because we have $n(n-1)\dots(n-(h-3))$ possibilities for $u_1, \dots, u_{h-2}$ and the conditions $u_{h-1} \neq u_i$, $u_h \neq u_i$ for $1 \leq i \leq h-2$ and $u_{h-1} \neq u_h$ exclude at most $h - 2 + h - 2 + 2$ choices for u_{h-1}.

Case (2): The condition $|\mathbf{u}|_d = t$ implies that there is a partition $\{1,\dots,h\} = \bigcup_{i=1}^{t} A_i$ such that $u_i = u_j$ if and only if $1 \le i, j \le h$ are in the same A_l. Fix such a partition. Then there are n choices for the elements $u_i, i \in A_1$, then $(n-1)$ possibilities for the elements $u_i, i \in A_2$ etc. and finally $(n-(t-2))$ choices for the elements $u_i, i \in A_{t-1}$. It follows from this that if we have already chosen the elements u_i, $i \in \bigcup_{i=1}^{t-1} A_i$ then we have at most $t \le h$ possibilities for the elements $u_i, i \in A_t$. In order to finish the proof we mention that the number of suitable partitions is $O_h(1)$. □

Lemma 1.6. *For every $a_1, a_2 \in \mathbb{Z}_n$ and $h \ge 2$ we have*

1. $C_{a_1,a_2,h,h,h,0} = \frac{n^{2h-2}}{(h!)^2 2} + O_h(n^{2h-3})$;
2. $C_{a_1,a_2,h,t,s,k} = O_h(n^{t+s-k-2})$ *for $t \ge s$ and $t > k \ge 0$;*
3. $C_{a_1,a_2,h,s,s,s} = O_h(n^{s-2})$ *for every $2 \le s < h$.*

Proof. Case (1): By the definition of $C_{a_1,a_2,h,h,h,0}$

$$2(h!)^2 C_{a_1,a_2,h,h,h,0} = \Big|\{\big((u_1,\dots,u_h),(v_1,\dots,v_h)\big) : \quad u_i \ne u_j, v_i \ne v_j, u_i \ne v_j, \sum_{i=1}^{h} u_i = a_1, \sum_{i=1}^{h} v_i = a_2\}\Big|. \quad (2)$$

An upper bound for (2) is $n^{h-1}n^{h-1}$ and a lower bound for (2) is $n(n-1)\dots(n-(h-3))(n-(h-2)-(h-2)-2)(n-h)(n-(h+1))\dots(n-h-(h-3))(n-(2h-2)-(2h-2)-2)$, because we have $n(n-1)\dots(n-(h-3))$ choices for $u_1,\dots,u_{h-2}$. After choosing $u_1,\dots,u_{h-2}$ there are at least $n-(h-2)-(h-2)-2$ possibilities left for u_{h-1} because $u_{h-1} \ne u_j$ and $u_h \ne u_j$ for $1 \le j \le h-2$ and $u_{h-1} \ne u_h$. After fixing $u_1,\dots,u_h$ we have $(n-h)\dots(n-(2h-2))$ choices for $v_1,\dots,v_{h-2}$. Finally, we have at least $n-2h-(2h-4)-2$ choices for v_{h-1} because $v_{h-1} \ne u_j$, $v_h \ne u_j$, for $1 \le j \le h$, $v_{h-1} \ne v_j$, $v_h \ne v_j$ for $1 \le j \le h-2$ and $v_{h-1} \ne v_h$.

Case (2): Obviously,

$$C_{a_1,a_2,h,t,s,k} \le \Big|\{((u_1,\dots,u_h),(v_1,\dots,v_h)) : \sum_{i=1}^{h} u_i = a_1, \sum_{i=1}^{h} v_i = a_2, |\mathbf{u}|_d = t, |\mathbf{v}|_d = s, |\mathbf{u}\cap\mathbf{v}|_d = k\}\Big|. \quad (3)$$

By the conditions $|u|_d = s$, $|v|_d = t$ there are partitions $\{1,\dots,h\} = \cup_{i=1}^{t} A_i = \bigcup_{i=1}^{s} B_i$ such that $u_i = u_j$ if and only if there exists an $1 \le l \le t$ such that $i,j \in A_l$, and $v_i = v_j$ if and only if there exists an $1 \le l \le s$ such that $i,j \in B_l$. We have at most hn^{s-1} choices for $(v_1,\dots,v_h)$ with $\sum_{i=1}^{h} v_i = a_2$. The condition $|\mathbf{u}\cap\mathbf{v}|_d = k$ implies that there are injections $\chi_u : \{1,\dots,k\} \to \{1,\dots,t\}$ and $\chi_v : \{1,\dots,k\} \to \{1,\dots,s\}$ such that $u_i = v_j$ if and only if there exists a $1 \le l \le k$ such that $u_i \in A_{\chi_u(l)}$ and $v_j \in B_{\chi_v(l)}$. Hence we get that there are at most hn^{t-k-1} choices for the v_is, $i \in \{1,\dots,h\} \setminus \bigcup_{i=1}^k B_{\chi_v(i)}$. Since the numbers of partitions and injections are $O_h(1)$, the proof is completed.

Case (3): Evidently,

$$C_{a_1,a_2,h,s,s,s} \le \Bigg|\Big\{((u_1,\dots,u_h),(v_1,\dots,v_h)) : \sum_{i=1}^h u_i = a_1, \sum_{i=1}^h v_i = a_2, \mathbf{u} \ne \mathbf{v}, |\mathbf{u}|_d = s, |\mathbf{v}|_d = s, |\mathbf{u}\cap\mathbf{v}|_d = s\Big\}\Bigg|. \quad (4)$$

By the conditions $|u|_d = s$, $|v|_d = s$ there are partitions $\{1,\dots,h\} = \bigcup_{i=1}^s A_i = \bigcup_{i=1}^s B_i$ such that $u_i = u_j$ if and only if there exists an $1 \le l \le s$ such that $i,j \in A_l$ and $v_i = v_j$ if and only if there exists an $1 \le m \le s$ such that $i,j \in B_m$. The condition $|\mathbf{u}\cap\mathbf{v}|_d = k$ implies that there is a bijection $\chi : \{1,\dots,s\} \to \{1,\dots,s\}$ such that $u_i = v_j$ if and only if there exists a $1 \le l \le s$ such that $i \in A_l$ and $j \in B_{\chi(l)}$. Since $\mathbf{u} \ne \mathbf{v}$, therefore there exists a $1 \le l \le s$ such that $|A_l| \ne |B_{\chi(l)}|$. Fix such an l. Then there exists a $1 \le k \le s$ such that $\frac{|A_k|}{|B_{\chi(k)}|} \ne \frac{|A_l|}{|B_{\chi(l)}|}$, because otherwise $|A_k| = |B_{\chi(k)}|\frac{|A_l|}{|B_{\chi(l)}|}$ for every $1 \le k \le s$, but

$$h = \sum_{k=1}^s |A_k| = \frac{|A_l|}{|B_{\chi(l)}|}\sum_{k=1}^s |B_{\chi(k)}| = \frac{|A_l|}{|B_{\chi(l)}|}h,$$

which is a contradiction. Fix such a k. Let $\{i_1,\dots,i_{s-2}\} = \{1,\dots,s\} \setminus \{k,l\}$. We have $n(n-1)\cdots(n-(s-3))$ choices for the elements u_i, $i \in \bigcup_{j=1}^{s-2} A_{i_j}$. After fixing the elements u_i, $i \in \bigcup_{j=1}^{s-2} A_{i_j}$ let $\sum_{j=1}^{s-2}\sum_{m\in A_{i_j}} u_m = U$ and $\sum_{j=1}^{s-2}\sum_{m\in B_{\chi(i_j)}} v_m = V$. Then we need $x, y \in \mathbb{Z}_n$ such that $U + |A_k|x + |A_l|y = a_1$ and $V + |B_{\chi(k)}|x + |B_{\chi(l)}|y = a_2$. Hence,

$$(|A_l||B_{\chi(k)}| - |A_k||B_{\chi(l)}|)y = a_1|B_{\chi(k)}| + V|A_k| - U|B_{\chi(k)}| - a_2|A_k|. \quad (5)$$

After fixing $1 \le k, l \le s$ and the elements u_i, $i \in \displaystyle\bigcup_{j=1}^{s-2} A_{i_j}$, the elements U and V are determined, therefore the right-hand side in (3) is unique. Since $0 < ||A_l||B_{\chi(k)}| - |A_k||B_{\chi(l)}|| \le h^2$, therefore the number of possible y's is at most h^2 and after fixing y we have at most h choices for x. Finally we mention that we have got $O_h(1)$ choices for the partitions and bijection. □

Proof of Theorem 1. For each unordered, different $u_1,\dots,u_h \in \mathbb{Z}_n$ and $v_1,\dots,v_h \in \mathbb{Z}_n$ with $\sum_{i=1}^h u_i = \sum_{i=1}^h v_i$. Let $B_{\mathbf{u},\mathbf{v}}$ be the event that $u_1,\dots,u_h,v_1,\dots,v_h \in A_n$. In the following we suppose that $\sum_{i=1}^h u_i = \sum_{i=1}^h v_i$. If we prove $\Delta = \sum_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}\cap\mathbf{v}|_d>0} \Pr\{B_{\mathbf{u},\mathbf{v}}\} =$

$o(1)$, then by the Janson inequality we have

$$\begin{aligned}
\Pr\{A_n \text{ is } B_h \text{ set}\} &= (1+o(1)) \prod_{\{\mathbf{u},\mathbf{v}\}} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \\
&= (1+o(1)) \left(\prod_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=h,|\mathbf{v}|_d=h,|\mathbf{u}\cap\mathbf{v}|_d=0} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \right) \\
&\quad \times \left(\prod_{k=1}^{h-1} \prod_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=h,|\mathbf{v}|_d=h,|\mathbf{u}\cap\mathbf{v}|_d=k} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \right) \\
&\quad \times \left(\prod_{s=2}^{h-1} \prod_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=s,|\mathbf{u}\cap\mathbf{v}|_d=s} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \right) \\
&\quad \times \left(\prod_{s=1}^{h-1} \prod_{k=0}^{s-1} \prod_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=s,|\mathbf{u}\cap\mathbf{v}|_d=k} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \right) \\
&\quad \times \left(\prod_{s=1}^{h-1} \prod_{t=s+1}^{h} \prod_{k=0}^{s} \prod_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=t,|\mathbf{u}\cap\mathbf{v}|_d=k} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \right) \\
&= P_1P_2P_3P_4P_5,
\end{aligned}$$

where, by Lemma 1.6.1,

$$\begin{aligned}
P_1 &= \prod_{a\in\mathbb{Z}_n} \prod_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=h,|\mathbf{v}|_d=h,|\mathbf{u}\cap\mathbf{v}|_d=0,\sum_{i=1}^h u_i \sum_{i=1}^h v_i=a} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \\
&= \left(1-\frac{c^{2h}}{n^{2h-1}}\right)^{\frac{n^{2h-1}}{2(h!)^2}\left(1+O_h\left(\frac{1}{n}\right)\right)} \\
&= (1+o(1)) \exp\left(-\frac{c^{2h}}{2(h!)^2}\right),
\end{aligned}$$

by Lemma 1.6.2,

$$\begin{aligned}
P_2 &= \prod_{a\in\mathbb{Z}_n} \prod_{k=1}^{h-1} \prod_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=h,|\mathbf{v}|_d=h,|\mathbf{u}\cap\mathbf{v}|_d=k,\sum_{i=1}^h u_i=\sum_{i=1}^h v_i=a} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \\
&= \prod_{k=1}^{h-1} (1-p_n^{2h-k})^{O_h(n^{2h-k-1})} \\
&= \prod_{k=1}^{h-1} \exp\left((p_n n)^{2h-k} O_h\left(\frac{1}{n}\right)\right) \\
&= \exp\left(o(1)\right),
\end{aligned}$$

by Lemma 1.6.3,

$$\begin{aligned} P_3 &= \prod_{a\in\mathbb{Z}_n} \prod_{s=2}^{h-1} \prod_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=s,|\mathbf{u}\cap\mathbf{v}|_d=s,\sum_{i=1}^h u_i=\sum_{i=1}^h v_i=a} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \\ &= \prod_{s=2}^{h-1}(1-p_n^s)^{O_h(n^{s-1})} \\ &= \prod_{k=1}^{h} \exp\left((-p_n n)^k O_h\left(\frac{1}{n}\right)\right) \\ &= \exp\left(o(1)\right), \end{aligned}$$

by Lemma 1.6.3,

$$\begin{aligned} P_4 &= \prod_{a\in\mathbb{Z}_n} \prod_{s=1}^{h-1} \prod_{k=0}^{s-1} \prod_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=s,|\mathbf{u}\cap\mathbf{v}|_d=k,\sum_{i=1}^h u_i=\sum_{i=1}^h v_i=a} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \\ &= \prod_{s=1}^{h}\prod_{k=0}^{s-1}(1-p_n^{2s-k})^{O_h(n^{2s-k-1})} \\ &= \prod_{s=1}^{h}\prod_{k=0}^{s-1} \exp\left(-(p_n n)^{2s-k} O_h(\frac{1}{n})\right) \\ &= \exp\left(o(1)\right), \end{aligned}$$

and, by Lemma 1.6.2,

$$\begin{aligned} P_5 &= \prod_{a\in\mathbb{Z}_n} \prod_{s=1}^{h-1} \prod_{t=s+1}^{h} \prod_{k=0}^{s} \prod_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=t,|\mathbf{u}\cap\mathbf{v}|_d=k,\sum_{i=1}^h u_i=\sum_{i=1}^h v_i=a} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \\ &= \prod_{s=1}^{h-1}\prod_{t=s+1}^{h}\prod_{k=0}^{s}(1-p_n^{s+t-k})^{O(n^{s+t-k-1})} = \exp\left(o(1)\right). \end{aligned}$$

Hence, it remains to prove that $\Delta = o(1)$. In order to prove $\Delta = o(1)$ we partition Δ as

$$\begin{aligned} \Delta &= \sum_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}\cap\mathbf{v}|_d>0} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \\ &= \sum_{s=1}^{h-1} \sum_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=s,|\mathbf{u}\cap\mathbf{v}|_d=s} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \\ &\quad + \sum_{s=2}^{h}\sum_{k=1}^{s-1} \sum_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=s,|\mathbf{u}\cap\mathbf{v}|_d=k} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \\ &\quad + \sum_{s=1}^{h-1}\sum_{t=s+1}^{h}\sum_{k=0}^{s} \sum_{\{\mathbf{u},\mathbf{v}\},|\mathbf{u}|_d=s,|\mathbf{v}|_d=t,|\mathbf{u}\cap\mathbf{v}|_d=k} \Pr\{B_{\mathbf{u},\mathbf{v}}\} \\ &= \sum_1 + \sum_2 + \sum_3. \end{aligned}$$

By Lemma 1.6.3,

$$\begin{aligned}\sum_1 &= \sum_{a\in\mathbb{Z}_n}\sum_{s=1}^{h-1}\sum_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=s,|\mathbf{u}\cap\mathbf{v}|_d=s,\sum_{i=1}^h u_i=\sum_{i=1}^h v_i=a}\Pr\{B_{\mathbf{u},\mathbf{v}}\}\\ &= \sum_{s=2}^{h-1} O_h(n^{s-1})p_n^s\\ &= O_h\left(\frac{1}{n}\sum_{s=2}^{h-1}(p_n n)^s\right)=o(1),\end{aligned}$$

by Lemma 1.6.2,

$$\begin{aligned}\sum_2 &= \sum_{a\in\mathbb{Z}_n}\sum_{s=2}^{h}\sum_{k=1}^{s-1}\sum_{\{\mathbf{u},\mathbf{v}\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=s,|\mathbf{u}\cap\mathbf{v}|_d=k,\sum_{i=1}^h u_i=\sum_{i=1}^h v_i=a}\Pr\{B_{\mathbf{u},\mathbf{v}}\}\\ &= \sum_{s=2}^{h}\sum_{k=1}^{s-1} O_h(n^{2s-k-1})p_n^{2s-k}\\ &= O_h\left(\frac{1}{n}\sum_{s=2}^{h}\sum_{k=1}^{s-1}(p_n n)^{2s-k}\right)=o(1),\end{aligned}$$

and by Lemma 1.6.2,

$$\begin{aligned}\sum_3 &= \sum_{a\in\mathbb{Z}_n}\sum_{s=1}^{h-1}\sum_{t=s+1}^{h}\sum_{k=0}^{s}\sum_{\{\mathbf{u},\mathbf{v}\},|\mathbf{u}|_d=s,|\mathbf{v}|_d=t,|\mathbf{u}\cap\mathbf{v}|_d=k,\sum_{i=1}^h u_i=\sum_{i=1}^h v_i=a}\Pr\{B_{\mathbf{u},\mathbf{v}}\}\\ &= \sum_{s=1}^{h-1}\sum_{t=s+1}^{h}\sum_{k=1}^{s} O_h(n^{t+s-k-1})p_n^{t+s-k}\\ &= O_h\left(\frac{1}{n}\sum_{s=1}^{h-1}\sum_{t=s+1}^{h}\sum_{k=1}^{s}(p_n n)^{t+s-k}\right)=o(1),\end{aligned}$$

which completes the proof. □

Proof of Theorem 2. For a fixed $x \in \mathbb{Z}_n$ and $y_1, \dots, y_h \in \mathbb{Z}_n$ with $\sum_{i=1}^h y_i = x$ let $\mathbf{y} = \{y_1, \dots, y_h\}$ and let $B_{\mathbf{y},x}$ be the event $y_1, \dots, y_h \in A_n$. For a fixed $x \in \mathbb{Z}_n$ let $C_x = \cap_{\mathbf{y},\sum_{i=1}^h y_i=x}\overline{B}_{\mathbf{y},x}$. Obviously,

$$\Pr\{A_n \text{ is an h-basis}\} = \Pr(\cap_{x\in\mathbb{Z}_n}\overline{C_x}).$$

By Lemma 1.4 it is sufficient to show that for every fixed positive integer r we have

$$\sum_{\{x_1,\dots,x_r\}:x_i\in\mathbb{Z}_n,x_i\neq x_j}\Pr\{C_{x_1}\cap\cdots\cap C_{x_r}\}\to\frac{\exp(-rc)}{r!}.$$

In order to estimate

$$\sum_{\{x_1,\dots,x_r\}:x_i\in\mathbf{Z}_n,x_i\neq x_j}\Pr\{C_{x_1}\cap\cdots\cap C_{x_r}\} = \sum_{\{x_1,\dots,x_r\}:x_i\in\mathbf{Z}_n,x_i\neq x_j}\Pr\{\cap_{1\le i\le r}\cap_{\mathbf{y}:\sum_{j=1}^h y_j=x_i}\overline{B}_{\mathbf{y},x_i}\}$$

we use Janson's inequality. Obviously, $\Pr\{B_{\mathbf{y},x_i}\}=o(1)$. If we prove $\Delta=o(1)$, then by Lemmas 1.3 and 1.5, and the definition of p_n

$$\begin{aligned}
&\sum_{\{x_1,\dots,x_r\}:x_i\in\mathbf{Z}_n,x_i\neq x_j}\Pr\left\{\bigcap_{1\le i\le r\cap}\ \bigcap_{\mathbf{y}:\sum_{j=1}^h y_j=x_i}\overline{B}_{\mathbf{y},x_i}\right\}\\
&= (1+o(1))\prod_{i=1}^{r}\prod_{\mathbf{y}:\sum_{j=1}^h y_j=x_i}\Pr\{\overline{B}_{\mathbf{y},x_i}\}\\
&= (1+o(1))\prod_{i=1}^{r}\prod_{k=1}^{h}\prod_{\mathbf{y}:y_1+\cdots+y_h=x_i,|\mathbf{u}|_d=k}(1-p_n^k)\\
&= (1+o(1))\prod_{i=1}^{r}\prod_{k=1}^{h-1}\left((1-p_n^k)^{O_h(n^{k-1})}\right)(1-p_n^k)^{\frac{n^{h-1}}{h!}\left(1+O_h\left(\frac{1}{n}\right)\right)}\\
&= (1+o(1))\prod_{i=1}^{r}\left[\left(\exp\left\{-O_h\left(\frac{1}{n}\right)\sum_{1\le k\le h-1}(p_nn)^k\right\}\right)\right.\\
&\qquad\left.\times\left(\exp\left\{-\frac{(p_nn)^h}{h!}\left(1+O_h(p_n^h)\right)\left(\frac{1}{n}+O_h\left(\frac{1}{n^2}\right)\right)\right\}\right)\right]\\
&= (1+o(1))\left(\exp\left\{-r\frac{h!n\log n(1+\frac{c}{\log n})(1+O_{h,c}(\frac{1}{\log^2 n}))}{h!}\frac{1}{n}\right\}\right)\\
&= (1+o(1))\frac{\exp(-cr)}{n^r}.
\end{aligned}$$

Therefore,

$$\sum_{\{x_1,\dots,x_r\},x_i\in\mathbb{Z}_n x_i\neq x_j}\Pr\{C_{x_1}\cap\cdots\cap C_{x_r}\} = (1+o(1))\binom{n}{r}\frac{\exp(-cr)}{n^r} = (1+o(1))\frac{\exp(-cr)}{r!}.$$

Let $\mathbf{u}=\{u_1,\dots,u_h\}$ with $u_1+\cdots+u_h=x_i$ and $\mathbf{v}=\{v_1,\dots,v_h\}$ with $v_1+\cdots+v_h=x_j$.

In order to finish the proof, we separate Δ as

$$\begin{aligned}
\Delta &= \sum_{1\le i,j\le r} \sum_{\{\mathbf{u},x_i\},\{\mathbf{v},x_j\}:|\mathbf{u}\cap\mathbf{v}|_d>0} \Pr\{B_{\mathbf{u},x_i}\cap B_{\mathbf{v},x_j}\} \\
&= \sum_{1\le i,j\le r} \sum_{s=2}^{h-1} \sum_{\{\mathbf{u},x_i\},\{\mathbf{v},x_j\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=s,|\mathbf{u}\cap\mathbf{v}|_d=s} p_n^s \\
&\quad + \sum_{1\le i,j\le r} \sum_{s=2}^{h} \sum_{k=1}^{s-1} \sum_{\{\mathbf{u},x_i\},\{\mathbf{v},x_j\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=s,|\mathbf{u}\cap\mathbf{v}|_d=k} p_n^{2s-k} \\
&\quad + \sum_{1\le i,j\le r} \sum_{s=1}^{h-1} \sum_{t=s+1}^{h} \sum_{k=1}^{s} \sum_{\{\mathbf{u},x_i\},\{\mathbf{v},x_j\}:|\mathbf{u}|_d=s,|\mathbf{v}|_d=t,|\mathbf{u}\cap\{v_1\ldots,v_r\}|_d=k} p_n^{s+t-k} \\
&= \sum_1 + \sum_2 + \sum_3,
\end{aligned}$$

where, by Lemma 1.6.3,

$$\sum_1 \le r^2 \sum_{s=2}^{h-1} p_n^s O_h(n^{s-2}) = O_{h,r}\left(\frac{1}{n^2}\sum_{s=2}^{h-1}(p_n n)^s\right) = o(1),$$

by Lemma 1.6.2,

$$\sum_2 \le r^2 \sum_{s=2}^{h}\sum_{k=1}^{s-1} p_n^{2s-k} O_h(n^{2s-k-2}) = O_{h,r}\left(\frac{1}{n^2}\sum_{s=2}^{h}\sum_{k=1}^{s-1}(p_n n)^{2s-k}\right) = o(1),$$

and, by Lemma 1.6.2,

$$\sum_3 \le r^2 \sum_{s=1}^{h-1}\sum_{t=s+1}^{h}\sum_{k=1}^{s} p_n^{t+s-k} O_h(n^{t+s-k}) = O_{h,r}\left(\frac{1}{n^2}\sum_{s=1}^{h-1}\sum_{t=s+1}^{h}\sum_{k=1}^{s}(p_n n)^{t+s-k}\right) = o(1)$$

which completes the proof. □

References

[1] N. Alon, and J. Spencer, *The Probabilistic Method*, Wiley-Interscience, Series in Discrete Math. and Optimization, 1992.