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COMBINATORIAL PROPERTIES OF THE ANTICHAINS OF A GARLAND

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Abstract

In this paper we study some combinatorial properties of the antichains of a *garland*, or *double fence*. Specifically, we encode the order ideals and the antichains in terms of words of a regular language, we obtain several enumerative properties (such as generating series, recurrences and explicit formulae), we consider some statistics leading to Riordan matrices, we study the relations between the lattice of ideals and the semilattice of antichains, and finally we give a combinatorial interpretation of the antichains as lattice paths with no peaks and no valleys.

1. Introduction

Fences (or *zigzag posets*), *crowns*, *garlands* (or *double fences*) and several of their generalizations are posets that are very well-known in combinatorics [1, 6, 7, 8, 9, 11, 14, 15, 19, 20, 21, 26] and in the theory of Ockham algebras [2, 3, 4]. Here we are interested in garlands, which can be considered as an extension of fences of even order and of crowns. More precisely, the *garland* of order n is the partially ordered set $\mathcal{G}_n$ defined as follows: $\mathcal{G}_0$ is the empty poset, $\mathcal{G}_1$ is the chain of length 1 and, for any other $n \geq 2$, $\mathcal{G}_n$ is the poset on $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ with cover relations $x_1 < y_1$, $x_1 < y_2$, $x_i < y_{i-1}$, $x_i < y_i$, $x_i < y_{i+1}$ for $i = 2, \ldots, n-1$ and $x_n < y_{n-1}$, $x_n < y_n$. See Figure 1 for an example.

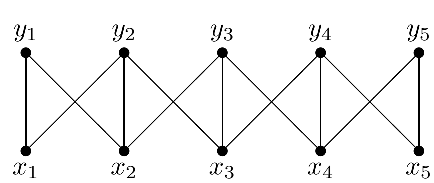

Figure 1: The garland $\mathcal{G}_5$.

In [19] we studied some enumerative properties of the order ideals of a garland. In particular, we proved that the numbers g_n of all ideals of $\mathcal{G}_n$ satisfy the recurrence $g_{n+2} = 2g_{n+1} + g_n$ with the initial values $g_0 = 1$ and $g_1 = 3$. These numbers appears in [25] as sequence A078057 (essentially the same as A001333) and admit several combinatorial interpretations. For instance, as proved in [26], they enumerate all n-step self-avoiding paths starting from $(0, 0)$ with steps of type $(1, 0)$, $(-1, 0)$ and $(0, 1)$.

As is well-known, the maximal elements of an ideal of a poset P is an antichain, and this establishes a bijective correspondence between all ideals of P and all antichains of P. However, ideals and antichains have several different properties either from an enumerative point of view or from an order theoretical point of view. In this paper we will study some combinatorial properties of the antichains of a garland. Specifically, the paper is organized as follows.

- First we define an encoding for ideals and antichains by means of words of a same regular language. Using such an encoding we obtain several enumerative properties (such as generating series, recurrences and explicit formulas) for the numbers of antichains and for the antichain polynomials. In particular, we show that the antichain polynomials can be expressed in terms of the Chebyshev polynomials.
- We also consider the square matrix and the cubical matrix generated by certain natural statistics on antichains and we show that from such matrices it is possible to extract some *Riordan matrices* [24, 18]. In particular, we enumerate all antichains equidistributed on the two levels of the garland, proving that they are equinumerous as the central ideals (i.e., ideals whose size is half the size of the garland).
- Then we study the meet-semilattice obtained by the set of all antichains of a garland ordered by inclusion. We show that there exists a simple but non-standard bijection between antichains and ideals (that can also be defined in terms of our previous encoding). Then, using such a bijection, we give an explicit correspondence between equidistributed antichains and central ideals, and we prove that the simple graph underling the Hasse diagram of the antichain semilattice is isomorphic to the simple graph underling the Hasse diagram of the distributive lattice of order ideals, extending a similar result for fences [20] and crowns [29]. Finally we show that these semilattices are rank unimodal and rank log-concave.
- Finally, we show an interpretation of the antichains of a garland as trinomial paths with no peaks and no valleys. In particular, we interpret the equidistributed antichains as those paths of this kind ending on the x-axis.

2. An Encoding for Ideals and Antichains

The ideals and the antichains of a garland can be represented as words of a regular language. Let $\mathcal{J}(\mathcal{G}_n)$ be the set of all ideals of $\mathcal{G}_n$, $\mathcal{W}$ be the language on the alphabet

$\Sigma = \{a, b, c\}$ formed of all words in which ac and ca never appear as factors and $\mathcal{W}_n$ the subset of $\mathcal{W}$ formed of all words of length n. Then let $\psi_1 : \mathcal{J}(\mathcal{G}_n) \to \mathcal{W}_n$ be the map defined, for every ideal I of $\mathcal{G}_n$, by setting $\psi_1(I) = w_1 \cdots w_n$, where

$$w_k = \begin{cases} a & \text{if } x_k \notin I \text{ and } y_k \notin I, \\ b & \text{if } x_k \in I \text{ and } y_k \notin I, \\ c & \text{if } x_k \in I \text{ and } y_k \in I, \end{cases} \qquad (1 \le k \le n).$$

For instance, the ideal $I = \{x_2, x_3, x_4, x_5, x_6, y_3, y_4\}$ of $\mathcal{G}_6$ corresponds to the word $\psi_1(I) = abccbb$ (see Figure 2). It is easy to see that the map ψ_1 is a bijection. Notice that such an encoding is essentially the same used in [19] for the n-step self-avoiding paths starting from $(0,0)$ with steps of type $(1,0)$, $(-1,0)$ and $(0,1)$.

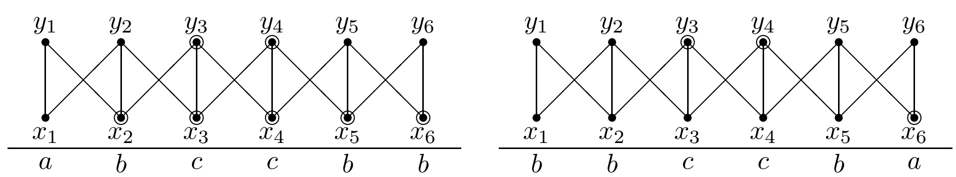

Figure 2: Encoding of an ideal (given by the circled points in the first picture) and of the corresponding antichain (given by the circled points in the second picture) of the garland $\mathcal{G}_6$.

Now, let $\mathcal{A}(\mathcal{G}_n)$ be the set of all antichains of $\mathcal{G}_n$. Then let $\psi_2 : \mathcal{A}(\mathcal{G}_n) \to \mathcal{W}_n$ be the map defined, for every antichain A of $\mathcal{G}_n$, by setting $\psi_2(A) = w'_1 \cdots w'_n$, where

$$w'_k = \begin{cases} a & \text{if } x_k \in A, \\ b & \text{if } x_k, y_k \notin A, \\ c & \text{if } y_k \in A, \end{cases} \qquad (1 \le k \le n).$$

For instance, the antichain $A = \{y_3, y_4, x_6\}$ of $\mathcal{G}_6$ corresponds to the word $\psi_2(A) =$ $bbccba$ (see Figure 2). Also this time, it is easy to see that ψ_2 is a bijection.

The order relation in $J(\mathcal{G}_n)$, given by set-inclusion, can be easily recovered from the words in $\mathcal{W}$. Indeed, from the meaning of the letters a, b and c in the encoding of the ideals of garlands it is natural to define a total order relation $\le$ on the alphabet Σ by setting $a < b < c$, and then to extend it to a partial order relation $\le$ on all words of $\mathcal{W}_n$ by setting $w_1 \cdots w_n \le w'_1 \cdots w'_n$ if and only if $w_1 \le w'_1, \ldots, w_n \le w'_n$. With this definition ψ_1 is an order-preserving bijection; that is, for every ideals I and J of $\mathcal{G}_n$ we have $I \subseteq J$ if and only if $\psi_1(I) \le \psi_1(J)$. Moreover the rank of the lattice $\mathcal{J}(\mathcal{G}_n)$, which is given by $r(I) = |I|$, in terms of the words in $\mathcal{W}$ becomes $r(w) = \omega_b(w) + 2\omega_c(w)$, where $\omega_x(w)$ denotes the number of occurrences of the letter x in the word w.

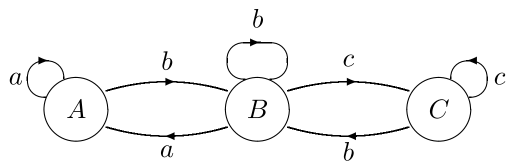

Figure 3: Automaton recognizing the language $\mathcal{W}$.

Similarly, if we define the partial order relation $\leq$ on the alphabet Σ by setting $b < a$ and $b < c$, and then we extend it to $\mathcal{W}_n$, we have that also ψ_2 is an order-preserving bijection. This time the rank is given by $r(w) = \omega_a(w) + \omega_c(w)$.

So we have that the words of $\mathcal{W}$ encode, in different ways, either the ideals or the antichains of garlands. Now, the regular language $\mathcal{W}$ can be described in terms of the deterministic automaton [12] in which the set of states is $S = \{A, B, C\}$, the alphabet is Σ, every state is both initial and final, and the transition function is defined according to the characterization of $\mathcal{W}$: in a word, the symbol $y \in \Sigma$ determines a transition from a state X to a state Y if and only if $xy \neq ac$ and $xy \neq ca$ (see Figure 3). It is easy to prove that such an automaton recognizes all and only the words in $\mathcal{W}$ and that the language $\mathcal{W}$ is defined by the unambiguous regular expression:

$$\mathcal{W} = (\varepsilon + a^+ + c^+)\left(b^+(a^+ + c^+)\right)^* b^* \tag{1}$$

where ε is the empty word. Moreover, it is easy to see that every word in $\mathcal{W}$ can be univocally written as the product of the following factors: $a^k b$ $(k \geq 1)$, b and $c^k b$ $(k \geq 1)$. This implies that $\mathcal{W}$ is also defined by the expression:

$$\mathcal{W} = (a^+ b + b + c^+ b)^*(\varepsilon + a^+ + c^+). \tag{2}$$

From the regular expressions (1) and (2), using the symbolic method [23], it follows that the generating series of the language $\mathcal{W}$ is

$$f(x, y, z) = \frac{1 - xz}{1 - x - y - z + xz + xyz} \tag{3}$$

where x, y and z mark the occurrences of a, b and c, respectively. In particular, from (3) we re-obtain the generating series

$$g(x, t) = f(t, xt, x^2 t) = \frac{1 - x^2 t^2}{1 - (1 + x + x^2)t + x^2 t^2 + x^3 t^3} \tag{4}$$

for the rank polynomials $g_n(x) = \sum_{k \geq 0} g_{nk} x^k$ of the lattices $\mathcal{J}(\mathcal{G}_n)$, already obtained in [19] in a different way. Moreover, again from (3), we can obtain the generating series

$$\begin{aligned} a(x, y, t) &= \sum_{n,i,j \geq 0} a_{n,i,j} x^i y^j t^n = f(xt, t, yt) \\ &= \frac{1 - xyt^2}{1 - (1 + x + y)t + xyt^2 + xyt^3} \end{aligned} \tag{5}$$

where the coefficients $a_{n,i,j}$ give the numbers of all antichains of $\mathcal{G}_n$ with i elements of rank 0 and j elements of rank 1. In particular, from Series (5) we immediately have the recurrence

$$a_{n+3,i+1,j+1} = a_{n+2,i+1,j+1} + a_{n+2,i,j+1} + a_{n+2,i+1,j} - a_{n+1,i,j} - a_{n,i,j}. \tag{6}$$

Moreover, expanding series (5) as follows

$$a(x,y,t) = \frac{1-xyt^2}{(1-xt)(1-yt)-(1-xyt^2)t} = \sum_{k\geq 0} \frac{(1-xyt^2)^{k+1}t^k}{(1-xt)^{k+1}(1-yt)^{k+1}},$$

we can obtain the identity

$$a_{n,i,j} = \sum_{k=0}^{m} \binom{n-i-j+1}{k}\binom{n-i-k}{j-k}\binom{n-j-k}{i-k}(-1)^k \tag{7}$$

where $m = \min(i,j,n-i,n-j)$.

3. An Explicit Formula for the Number of Antichains

In this section we give an explicit formula for the number g_n of all antichains (or ideals) of $\mathcal{G}_n$. From (4) we have that the generating series for the numbers g_n can be written as

$$g(t) = g(1,t) = \frac{1+t}{1-2t-t^2} = \frac{1+t}{(1-at)(1-bt)} = \frac{a+1}{a-b}\frac{1}{1-at} - \frac{b+1}{a-b}\frac{1}{1-bt}$$

where $a = 1+\sqrt{2}$ and $b = 1-\sqrt{2}$. Hence it follows that

$$g_n = \frac{a^{n+1}+b^{n+1}}{2} = \frac{(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}}{2} \tag{8}$$

and consequently, expanding the powers, we have the identity

$$g_n = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1}{2k} 2^k. \tag{9}$$

This identity can also be obtained using the following combinatorial argument on the words of $\mathcal{W}$. Every word in $\mathcal{W}_n$ admits a unique decomposition of the form

$$\beta_1\xi_1(x_1b)\beta_2\xi_2(x_2b)\cdots\beta_k\xi_k(x_kb)\beta_{k+1}\xi_{k+1}$$

where $\beta_i \in b^*$ for every $i = 1,2,\dots,k+1$, $\xi_i \in x_i^*$ for every $i = 1,2,\dots,k$, $\xi_{k+1} \in a^*$ or $\xi_{k+1} \in c^*$ and $x_i = a$ or $x_i = c$ for every $i = 1,2,\dots,k$. To generate all such words with a given number k of blocks (xb), we consider the following two cases.

(i) The last block is empty, i.e., $\xi_{k+1} = \varepsilon$. In this case we first displace the k blocks $(x_i b)$ and then we choose the letters x_i. In total we have used $2k$ letters which can be chosen in 2^k different ways. Now, it remains to distribute $n-2k$ letters in $2k+1$ places. Notice that at this point the nature of the letters is completely determined by the block in which they are placed. Then we have $\left(\!\!\binom{2k+1}{n-2k}\!\!\right)2^k$ different words, where $\left(\!\!\binom{n}{k}\!\!\right) = n(n+1)\cdots(n+k-1)/k!$ is the number of all multisets of order k on a set of size n, [5, 26].

(ii) The last block is non-empty, i.e., $\xi_{k+1} \neq \varepsilon$. In this case we first displace the k blocks $(x_i b)$ and a letter x_{k+1} in the last block ξ_{k+1} which has to be non-empty. Then we choose the letters x_i, in all 2^{k+1} possible ways. Now, it remains to distribuite $n-2k-1$ letters in $2k+2$ places. So this time we have $\left(\!\!\binom{2k+2}{n-2k-1}\!\!\right)2^{k+1}$ different words.

Hence we obtain the identity

$$g_n = \sum_{k\geq 0} \left(\!\!\binom{2k+1}{n-2k}\!\!\right) 2^k + \sum_{k\geq 0} \left(\!\!\binom{2k+2}{n-2k-1}\!\!\right) 2^{k+1}$$

which is essentially the same as (9). Indeed,

$$\begin{aligned} g_n &= \sum_{k\geq 0} \binom{n}{2k} 2^k + \sum_{k\geq 0} \binom{n}{2k+1} 2^{k+1} \\ &= \sum_{k\geq 0} \binom{n}{2k} 2^k + \sum_{k\geq 1} \binom{n}{2k-1} 2^k = \sum_{k\geq 0} \binom{n+1}{2k} 2^k. \end{aligned}$$

4. Antichain Polynomials

In this section we will study the *antichain polynomials*

$$a_n(x) = \sum_{k=0}^{n} a_{nk} x^k$$

where the coefficients a_{nk} give the number of all antichains of size k in the garland $\mathcal{G}_n$. Since the largest size of an antichain in $\mathcal{G}_n$is n, $a_n(x)$ has degree n. From (5) we have that the generating series

$$a(x,t) = \sum_{n\geq 0} a_n(x) t^n = \sum_{n,k\geq 0} a_{nk} x^k t^n$$

is given by

$$a(x,t) = a(x,x,t) = \frac{1 - x^2t^2}{1-(1+2x)t + x^2t^2 + x^2t^3} = \frac{1+xt}{1-(1+x)t - xt^2}. \tag{10}$$

In particular, from (10), we have the linear recurrence

$$a_{n+2,k+1} = a_{n+1,k+1} + a_{n+1,k} + a_{nk}, \tag{11}$$

from (7) we have the explicit formula

$$\begin{aligned} a_{nk} &= \sum_{i+j=k} a_{n,i,j} \\ &= \sum_{i=0}^{k} \sum_{j=0}^{\min(i,k-i)} \binom{n-k+1}{j} \binom{n-i-j}{n-k} \binom{n-k+i-j}{n-k} (-1)^j \end{aligned} \tag{12}$$

and finally from (10) we can obtain the identity

$$a_{nk} = \sum_{j=0}^{k} \binom{n-k+1}{j} \left(\!\!\binom{j}{k-j}\!\!\right) 2^j.$$

4.1. Recurrences

From Series (10), we immediately obtain the following recurrence for the antichain polynomials:

$$a_{n+2}(x) = (1+x)a_{n+1}(x) + x a_n(x) \tag{13}$$

and then, from (13), it is easy to obtain the following identities:

$$\begin{aligned} a_{n+2}(x)a_n(x) - a_{n+1}(x)^2 &= (-1)^{n+1} 2x^{n+2} \\ a_{m+1}(x)a_{n+1}(x) + x a_m(x) a_n(x) &= a_{m+n+2}(x) + x a_{m+n+1}(x) \\ a_{n+1}(x)^2 + x a_n(x)^2 &= a_{2n+2}(x) + x a_{2n+1}(x). \end{aligned}$$

We can also obtain another recurrence with the following combinatorial argument.

Theorem 1 *The antichain polynomials $a_n(x)$ satisfy the recurrence*

$$a_{n+2}(x) = a_{n+1}(x) + 2\sum_{k=0}^{n} x^{n-k+1} a_k(x) + 2x^{n+2}. \tag{14}$$

Proof. For every antichain A of $\mathcal{G}_{n+2}$ there are the following three cases.

(i) If A does not contain the last elements x_{n+2} and y_{n+2}, then it is equivalent to an antichain of $\mathcal{G}_{n+1}$.

(ii) If A does not contain the element x_k (resp. y_k) but contains $x_{k+1}, \ldots, x_{n+2}$ (resp. $y_{k+1}, \ldots, y_{n+2}$), then it is equivalent to an antichain of $\mathcal{G}_{k-1}$ (where $1 \le k \le n+1$).

(iii) If A contains all the elements $x_1, \ldots, x_{n+2}$ (resp. $y_1, \ldots, y_{n+2}$), it cannot contain other elements.

This implies (14). $\square$

4.2. Explicit Forms

The antichain polynomials can be expressed in terms of Chebyshev polynomials. Indeed

$$a(x,t)=\frac{1+3x}{1+x}U\left(\frac{1+x}{2\mathrm{i}\sqrt{x}},\mathrm{i}\sqrt{x}t\right)-\frac{2x}{1+x}T\left(\frac{1+x}{2\mathrm{i}\sqrt{x}},\mathrm{i}\sqrt{x}t\right)$$

where i is the imaginary unit and

$$T(x;t)=\sum_{n\geq 0}T_n(x)t^n=\frac{1-xt}{1-2xt+t^2}$$

and

$$U(x;t)=\sum_{n\geq 0}U_n(x)t^n=\frac{1}{1-2xt+t^2}$$

are the generating series for the Chebyshev polynomials of the first and the second kind, respectively. So

$$a_n(x)=\frac{1+3x}{1+x}U_n\left(\frac{1+x}{2\mathrm{i}\sqrt{x}}\right)(\mathrm{i}\sqrt{x}t)^n-\frac{2x}{1+x}T_n\left(\frac{1+x}{2\mathrm{i}\sqrt{x}}\right)(\mathrm{i}\sqrt{x}t)^n.$$

Now, using the following expansions of the Chebyshev polynomials [22]:

$$\begin{aligned}T_n(x) &= \frac{1}{2}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\frac{n}{n-k}(-1)^k(2x)^{n-2k},\\ U_n(x) &= \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}(-1)^k(2x)^{n-2k},\end{aligned}$$

we can obtain, for every $n\geq 1$, the expression

$$a_n(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\frac{(2n-3k)x+n-k}{n-k}(1+x)^{n-2k-1}x^k.$$

In particular, since $g_n=a_n(1)$, we have

$$g_n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\frac{3n-4k}{n-k}2^{n-2k-1}.$$

We can also obtain another expression for $a_n(x)$ as follows. Since

$$a(x,t)=\frac{1+xt}{(1-\alpha t)(1-\beta t)}=\frac{\alpha+x}{\alpha-\beta}\frac{1}{1-\alpha t}-\frac{\beta+x}{\alpha-\beta}\frac{1}{1-\beta t}$$

where $\alpha, \beta = (1 + x \pm \sqrt{1 + 6x + x^2})/2$ are the roots of the equation $\xi^2 - (1+x)\xi - x = 0$, the identity

$$a_n(x) = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{1 + 6x + x^2}} + \frac{x(\alpha^n - \beta^n)}{\sqrt{1 + 6x + x^2}}$$

follows, and hence

$$a_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{n + 1 + (3n - 4k + 1)x}{2k + 1} (1 + x)^{n-2k-1}(1 + 6x + x^2)^k.$$

Again, since $g_n = a_n(1)$, we have

$$g_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{2n - 2k + 1}{2k + 1} 2^k.$$

4.3. Euler Transform

The *Euler transform* of a formal series $F(x) = \sum_{n \geq 0} F_n x^n$ is defined as

$$\mathcal{T}^\alpha[F(x)] = \frac{1}{1 - \alpha x} F\left(\frac{x}{1 - \alpha x}\right) = \sum_{n \geq 0} \left[\sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} F_k\right] x^n.$$

The operator $\mathcal{T}^\alpha$ is always invertible and in particular $(\mathcal{T}^\alpha)^{-1} = \mathcal{T}^{-\alpha}$.

Now, considering $a(x, t)$ as a series in t, we have

$$\mathcal{T}^x[a(x, t)] = \frac{1}{1 - (1 + 3x)t + 2x^2t^2} = U\left(\frac{1 + 3x}{2\sqrt{2}x}; \sqrt{2}xt\right)$$

and consequently

$$a(x, t) = \mathcal{T}^{-x}\left[U\left(\frac{1 + 3x}{2\sqrt{2}x}; \sqrt{2}xt\right)\right].$$

Hence

$$a_n(x) = x^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sqrt{2^k} U_k\left(\frac{1 + 3x}{2\sqrt{2}x}\right).$$

5. Equidistributed Antichains

We say that an *antichain* is *equidistributed* in a garland $\mathcal{G}_n$ when the number of elements at level 0 is equal to the number of elements at level 1. From identity (7) it follows immediately that the number of equidistributed antichains of size $2k$ of $\mathcal{G}_n$ is

$$e_{nk} = a_{n,k,k} = \sum_{j=0}^{\min(k,n-k)} \binom{n - 2k + 1}{j} \binom{n - k - j}{k - j}^2 (-1)^j.$$

Moreover, their generating series $e(x,t)$ is the diagonal of Series (5), considered as a double series in x and y. By Cauchy's integral theorem [5, 13, 27] we have

$$e(x,t) = \frac{1}{2\pi \mathrm{i}} \oint a(z, x/z, t) \frac{\mathrm{d}z}{z} = \frac{1}{2\pi \mathrm{i}} \oint \frac{1 - xt^2}{-tz^2 + (1 - t + xt^2 + xt^3)z - tx} \mathrm{d}z$$

where the integral is taken over a simple contour containing all the singularities $s(t)$ of the series such that $s(t) \to 0$ as $t \to 0$. The polynomial appearing at the denominator has roots

$$z^{\pm} = \frac{1 - t + xt^2 + xt^3 \pm \sqrt{(1 - t + xt^2 + xt^3)^2 - 4t^2 x}}{2t}.$$

Since only $z^- \to 0$ as $t \to 0$, by the residue theorem we have

$$e(x,t) = \lim_{z \to z^-} \frac{1 - xt^2}{-t(z - z^+)} = \frac{1 - xt^2}{t(z^+ - z^-)};$$

that is,

$$e(x,t) = \frac{1 - xt^2}{\sqrt{(1 - t + xt^2 + xt^3)^2 - 4xt^2}} = \sqrt{\frac{1 - xt^2}{1 - 2t + t^2 - xt^2 - 2xt^3 - xt^4}}.$$

Finally, the generating series for the coefficients e_n giving the total number of equidistributed antichains is given by

$$\begin{aligned} e(x) = e(1,x) \quad &= \frac{1 - x^2}{\sqrt{1 - 2x - x^2 - x^4 + 2x^5 + x^6}} \\ &= \sqrt{\frac{1 - x^2}{(1 + x^2)(1 - 2x - x^2)}}. \end{aligned} \tag{15}$$

The coefficients of this series appear in [25] as sequence A136029. From identity (7), we have

$$e_n = \sum_{k=0}^{n} a_{n,k,k} = \sum_{k=0}^{n} \sum_{j=0}^{\min(k,n-k)} \binom{n - 2k + 1}{j} \binom{n - k - j}{k - j}^2 (-1)^j.$$

Remark. Series (15) coincides with the generating series for the *central ideals* of garlands obtained in [19]. This immediately implies that the antichains and the central ideals (i.e., ideals of size n) of $\mathcal{G}_n$ are equinumerous. In Section 7 we will give a bijective proof of this result.

6. Riordan Matrices Generated by Antichains

A Riordan matrix [24, 18] is an infinite lower triangular matrix $R = [r_{nk}]_{n,k \geq 0} = (g(x), f(x))$ whose columns have generating series $r_k(x) = g(x) f(x)^k$, where $g(x)$ and $f(x)$ are given series with $g_0 = 1$, $f_0 = 0$ and $f_1 \neq 0$. Riordan matrices appear very often in enumerative combinatorics and several times they are contained in some other matrices. In this section we will prove that from the matrix

$$A = [a_{n,k}]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & & & & & & \\ 1 & 2 & & & & & & & & & \\ 1 & 4 & 2 & & & & & & & & \\ 1 & 6 & 8 & 2 & & & & & & & \\ 1 & 8 & 18 & 12 & 2 & & & & & & \\ 1 & 10 & 32 & 38 & 16 & 2 & & & & & \\ 1 & 12 & 50 & 88 & 66 & 20 & 2 & & & & \\ 1 & 14 & 72 & 170 & 192 & 102 & 24 & 2 & & & \\ 1 & 16 & 98 & 292 & 450 & 360 & 146 & 28 & 2 & & \\ 1 & 18 & 128 & 462 & 912 & 1002 & 608 & 198 & 32 & 2 & \\ 1 & 20 & 162 & 688 & 1666 & 2364 & 1970 & 952 & 258 & 36 & 2 \\ \dots & & & & & & & & & & \end{bmatrix}$$

(which is not Riordan and which appears in [25] as sequence A035607) and from the cubic matrix $[a_{n,i,j}]_{n,i,j\geq 0}$ it is possible to extract some Riordan matrices.

6.1. First Matrix

First we consider the matrix

$$R = [r_{n,k}]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & & & & \\ 4 & 1 & & & & & & & \\ 18 & 8 & 1 & & & & & & \\ 88 & 50 & 12 & 1 & & & & & \\ 450 & 292 & 98 & 16 & 1 & & & & \\ 2364 & 1666 & 688 & 162 & 20 & 1 & & & \\ 12642 & 9424 & 4482 & 1340 & 242 & 24 & 1 & & \\ 68464 & 53154 & 28004 & 9922 & 2312 & 338 & 28 & 1 & \\ 374274 & 299660 & 170610 & 68664 & 19266 & 3668 & 450 & 32 & 1 \\ \dots & & & & & & & & \end{bmatrix}$$

where $r_{n,k} = a_{2n,n-k}$. Recurrence (11) implies a recurrence also for the coefficients $r_{n,k}$. Indeed, applying (11), we have

$$\begin{aligned} r_{n+2,k+1} &= a_{2n+4,n-k+1} = a_{2n+3,n-k+1} + a_{2n+3,n-k} + a_{2n+2,n-k} \\ &= a_{2n+2,n-k+1} + a_{2n+2,n-k} + a_{2n+1,n-k} \\ &\quad + a_{2n+2,n-k} + a_{2n+2,n-k-1} + a_{2n+1,n-k-1} + r_{n+1,k+1} \\ &= r_{n+1,k} + 3r_{n+1,k+1} + r_{n+1,k+2} + a_{2n+1,n-k} + a_{2n+1,n-k-1}. \end{aligned}$$

Since from (11) we have

$$a_{2n+1,n-k} + a_{2n+1,n-k-1} = a_{2n+2,n-k} - a_{2n,n-k-1} = r_{n+1,k+1} - r_{n,k+1},$$

then we obtain the recurrence

$$r_{n+2,k+1} = r_{n+1,k} + 4r_{n+1,k+1} + r_{n+1,k+2} - r_{n,k+1}. \tag{16}$$

Now, from (16) it follows that

$$xr_{k+2}(x) - (1 - 4x + x^2)r_{k+1}(x) + xr_k(x) = 0 \tag{17}$$

for the generating series $r_k(x) = \sum_{n\geq k} r_{n,k}x^n$ of the columns of R. If there exist two series $c(x)$ and $f(x)$ such that $r_k(x) = c(x)f(x)^k$ for every $k \in \mathbb{N}$, then

$$c(x) = r_0(x) = \sum_{n\geq 0} a_{2n,n}x^n$$

is the generating series of the central coefficients of the matrix A, and hence it is the diagonal of the double series

$$\begin{aligned} c(x,t) = \sum_{n,k\geq 0} a_{2n,k}x^k t^n &= \frac{a(x,-\sqrt{t}) + a(x,-\sqrt{t})}{2} \\ &= \frac{1 + x^2 t}{1 - (1 + 4x + x^2)t + x^2t^2}. \end{aligned}$$

By Cauchy's integral theorem we have

$$c(x) = \frac{1}{2\pi \mathrm{i}} \oint c(z; x/z)\frac{\mathrm{d}z}{z} = \frac{1}{2\pi \mathrm{i}} \oint \frac{1 + xz}{-xz^2 + (1 - 4x + x^2)z - x}\mathrm{d}z.$$

The polynomial appearing in the denominator has roots

$$z^{\pm} = \frac{1 - 4x + x^2 \pm (1 - x)\sqrt{1 - 6x + x^2}}{2x}$$

and only $z^- \to 0$ as $t \to 0$. Hence, by the residue theorem, we have

$$c(x) = \lim_{z\to z^-} \frac{1 - xz}{-z(z - z^+)} = \frac{1 - xz^-}{x(z^+ - z^-)} = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2\sqrt{1 - 6x + x^2}}. \tag{18}$$

The coefficients c_n of this series appear in [25] as sequence A050146. In particular, it is easy to see that $c_n = (3D_n - D_{n-1})/2$ (for $n \geq 1$), where the D_n's are the *central Delannoy numbers* [25, A001850].

Moreover, from recurrence (17) we obtain $xf(x)^2 - (1-4x+x^2)f(x) + x = 0$ and hence the series

$$f(x) = \frac{1-4x+x^2-(1-x)\sqrt{1-6x+x^2}}{2x}, \tag{19}$$

whose coefficients f_n appear in [25] as sequence A006319.

Since $c_1 = 1$, $f_0 = 0$ and $f_1 \neq 0$, it follows that R is indeed a Riordan matrix. Specifically, we have $R = (c(x), f(x))$, where $c(x)$ is series (18) and $f(x)$ is series (19).

From identity (12) we immediately obtain the explicit formula

$$r_{n,k} = \sum_{i=0}^{k} \sum_{j=0}^{\min(i,k-i)} \binom{n+k+1}{j}\binom{2n-i-j}{n+k}\binom{n+k+i-j}{n+k}(-1)^j.$$

6.2. Second Matrix

Now we consider the matrix

$$S = [s_{n,k}]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & & & & \\ 6 & 1 & & & & & & & \\ 32 & 10 & 1 & & & & & & \\ 170 & 72 & 14 & 1 & & & & & \\ 912 & 462 & 128 & 18 & 1 & & & & \\ 4942 & 2816 & 978 & 200 & 22 & 1 & & & \\ 27008 & 16722 & 6800 & 1782 & 288 & 26 & 1 & & \\ 148626 & 97880 & 44726 & 14016 & 2938 & 392 & 30 & 1 & \\ 822560 & 568150 & 284000 & 101946 & 25872 & 4510 & 512 & 34 & 1 \\ \cdots & & & & & & & & \end{bmatrix}$$

where $s_{n,k} = a_{2n+1,n-k}$. Exactly as in the previous case, recurrence (11) implies the linear recurrence

$$s_{n+2,k+1} = s_{n+1,k} + 4s_{n+1,k+1} + s_{n+1,k+2} - s_{n,k+1}, \tag{20}$$

which is the same as (16). Then the generating series $s_k(x) = \sum_{n\geq k} s_{n,k}x^n$ of the columns of S satisfy the recurrence $xs_{k+2}(x) - (1-4x+x^2)s_{k+1}(x) + xs_k(x) = 0$. Consequently $s_k(x) = d(x)f(x)^k$ for every $k \in \mathbb{N}$, where $f(x)$ is the series given by (19) and

$$d(x) = s_0(x) = \sum_{n\geq 0} a_{2n+1,n}x^n$$

is the diagonal of the double series

$$d(x,t) = \sum_{n,k\geq 0} a_{2n+1,k} x^k t^n = \frac{a(x,-\sqrt{t}) - a(x,-\sqrt{t})}{2\sqrt{t}} = \frac{1+2x-x^2t}{1-(1+4x+x^2)t+x^2t^2}.$$

By Cauchy's integral theorem, we have

$$d(x) = \frac{1}{2\pi \mathrm{i}} \oint d(z; x/z) \frac{\mathrm{d}z}{z} = \frac{1}{2\pi \mathrm{i}} \oint \frac{1+2z-xz}{-xz^2+(1-4x+x^2)z-x} \mathrm{d}z$$

and consequently by the residue theorem we have

$$d(x) = \lim_{z\to z^-} \frac{1+(2-x)z}{-z(z-z^+)} = \frac{1+(2-x)z^-}{x(z^+-z^-)},$$

that is,

$$d(x) = \frac{2-5x+x^2-(2-x)\sqrt{1-6x+x^2}}{2x\sqrt{1-6x+x^2}}. \tag{21}$$

The coefficients d_n of this series appears in [25] as sequence A125190, and they can be expressed in terms of central Delannoy numbers: $d_n = (2D_{n+1} - 5D_n + D_{n-1})/2$.

In conclusion, S is a Riordan matrix. Specifically, we have $S = (d(x), f(x))$, where $d(x)$ is series (21) and $f(x)$ is series (19).

Finally, from identity (12) we obtain the explicit formula

$$s_{n,k} = \sum_{i=0}^{k} \sum_{j=0}^{\min(i,k-i)} \binom{n+k+2}{j} \binom{2n-i-j+1}{n+k+1} \binom{n+k+i-j+1}{n+k+1} (-1)^j.$$

6.3. Third Matrix

The last matrix we consider is

$$H = [h_{n,k}]_{n,k\geq 0} \begin{bmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 3 & 3 & 3 & 1 \\ 7 & 6 & 6 & 4 & 1 \\ 15 & 14 & 12 & 10 & 5 & 1 \\ 33 & 32 & 27 & 22 & 15 & 6 & 1 \\ 75 & 72 & 63 & 50 & 37 & 21 & 7 & 1 \\ 171 & 164 & 146 & 118 & 88 & 58 & 28 & 8 & 1 \\ \cdots \end{bmatrix}$$

where $h_{n,k} = \sum_{i=k}^{n} a_{n,i,i-k}$. Using recurrence (6) we have

$$\begin{aligned}
h_{n+3,k+1} &= \sum_{i=k+1}^{n+3} a_{n+3,i,i-k-1} \\
&= \sum_{i=k+1}^{n+3} (a_{n+2,i,i-k-1} + a_{n+2,i-1,i-k-1} + a_{n+2,i,i-k-2} \\
&\qquad\qquad - a_{n+1,i-1,i-k-2} - a_{n,i-1,i-k-2}) \\
&= h_{n+2,k+1} + \sum_{i=k}^{n+2} a_{n+2,i,i-k} + h_{n+2,k+2} \\
&\qquad\qquad - \sum_{i=k+1}^{n+1} a_{n+1,i,i-k-1} - \sum_{i=k+1}^{n} a_{n,i,i-k-1};
\end{aligned}$$

that is,

$$h_{n+3,k+1} = h_{n+2,k} + h_{n+2,k+1} + h_{n+2,k+2} - h_{n+1,k+1} - h_{n,k+1}. \tag{22}$$

Let $h_k(x) = \sum_{n \geq k} h_{n,k} x^n$ be the generating series for the columns of H. From recurrence (22) and the first few values of $h_{n,k}$ we obtain the equation

$$xh_{k+2}(x) - (1 - x + x^2 + x^3)h_{k+1}(x) + xh_k(x) = 0. \tag{23}$$

Now we suppose that $h_k(x) = e(x)h(x)^k$ for every $k \in \mathbb{N}$. Clearly for $k = 0$ we obtain the generating series $e(x)$ for the equidistributed antichains. Moreover, from (23) we have the equation

$$xh(x)^2 - (1 - x + x^2 + x^3)h(x) + x = 0$$

whose solution is

$$h(x) = \frac{1 - x + x^2 + x^3 - \sqrt{1 - 2x - x^2 - x^4 + 2x^5 + x^6}}{2x}. \tag{24}$$

The coefficients of this series form essentially sequence A004149 in [25]. In conclusion, H is a Riordan matrix and more precisely $H(x) = (e(x), h(x))$, where $e(x)$ is series (15) and $h(x)$ is series (24).

Finally, using identity (12) we can obtain the explicit formula

$$h_{n,k} = \sum_{i=k}^{n} \sum_{j=0}^{m} \binom{n-2i+k+1}{j} \binom{n-i-j}{i-j-k} \binom{n-i-j+k}{i-j} (-1)^j$$

where $m = \min(i, i-k, n-i, n-i+k)$.

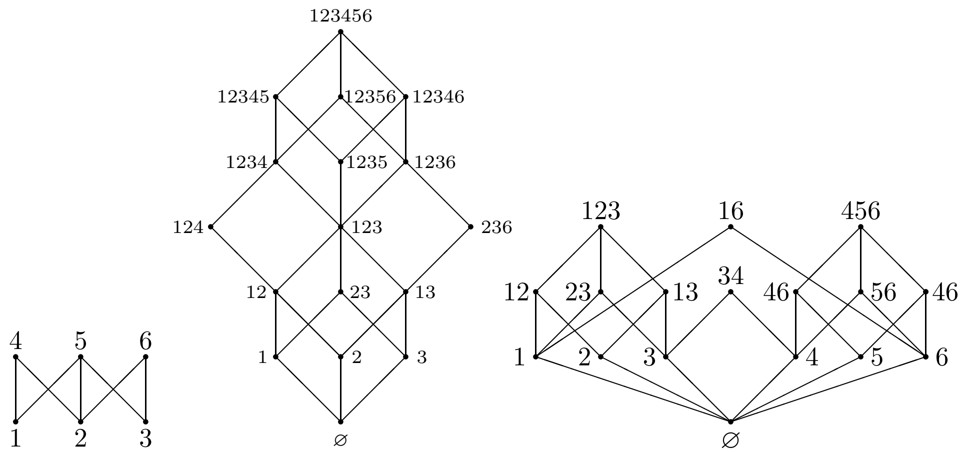

Figure 4: The garland $\mathcal{G}_3$, the lattice $\mathcal{J}(\mathcal{G}_3)$ of ideals and the semilattice $\mathcal{A}(\mathcal{G}_3)$ of antichains.

7. The Lattice of Ideals and the Semilattice of Antichains

As we recalled in the introduction, for any poset P there is a standard bijection between $\mathcal{J}(P)$ and $\mathcal{A}(P)$ defined by associating to each ideal the antichain of its maximal elements. In the particular case of garlands we can define another useful bijection $\varphi : \mathcal{J}(\mathcal{G}_n) \to \mathcal{A}(\mathcal{G}_n)$. First notice that each ideal I of $\mathcal{G}_n$ can be uniquely decomposed into the disjoint union of the set I_0 of all its element of rank 0, and the set I_1 of all its elements of rank 1. Let I_0' be the complementary set of I_0 in the set of all elements of rank 0 of $\mathcal{G}_n$. If $I = I_0 \sqcup I_1$ then we set $\varphi(I) = I_0' \sqcup I_1$. Clearly φ is well defined, since $I_0' \sqcup I_1$ is always an antichain, and it is easy to verify that it is a bijection. Equivalently, the map φ can be defined in terms of the encodings $\psi_1 : \mathcal{J}(\mathcal{G}_n) \to \mathcal{W}$ and $\psi_2 : \mathcal{A}(\mathcal{G}_n) \to \mathcal{W}$ described in Section 2. Indeed, it is easy to see that $\varphi = \psi_1\psi_2^{-1} = \psi_2^{-1} \circ \psi_1$.

Let $\mathcal{C}(\mathcal{G}_n)$ be the set of all central ideals of $\mathcal{G}_n$ and let $\mathcal{E}(\mathcal{G}_n)$ be the set of all equidistributed antichains of $\mathcal{G}_n$.

Theorem 2 *The map $\varphi : \mathcal{C}(\mathcal{G}_n) \to \mathcal{E}(\mathcal{G}_n)$ is a bijection.*

Proof. Let $I = I_0 \sqcup I_1$ be an ideal of a garland $\mathcal{G}_n$. Then

$$\varphi(I) \in \mathcal{E}(\mathcal{G}_n) \quad \Longleftrightarrow \quad |I_0'| = |I_1| \quad \Longleftrightarrow \quad |I_0| + |I_1| = n \quad \Longleftrightarrow \quad I \in \mathcal{C}(\mathcal{G}_n).$$

This means that the map $\varphi : \mathcal{J}(\mathcal{G}_n) \to \mathcal{A}(\mathcal{G}_n)$ can be restricted to the map $\varphi : \mathcal{C}(\mathcal{G}_n) \to \mathcal{E}(\mathcal{G}_n)$. Since φ is a bijection, its restriction is also a bijection. □

If we order the antichains by inclusion then $\mathcal{A}(\mathcal{G}_n)$ becomes a meet-semilattice. It has the same size of $\mathcal{J}(\mathcal{G}_n)$ but it is a different poset (see Figure 4 for an example).

However, we have

Theorem 3 *The posets $\mathcal{A}(\mathcal{G}_n)$ and $\mathcal{J}(\mathcal{G}_n)$ have the same simple graph $\mathcal{H}(\mathcal{G}_n)$ underlying their Hasse diagram.*

Proof. Let $\mathcal{H}_1(\mathcal{G}_n)$ be the simple graph underlying the Hasse diagram of $\mathcal{J}(\mathcal{G}_n)$ and similarly let $\mathcal{H}_2(\mathcal{G}_n)$ be the simple graph underlying the Hasse diagram of $\mathcal{A}(\mathcal{G}_n)$. In both cases the adjacency is defined by the cover relation: two vertices are adjacent if and only if one of them is covered by the other in the corresponding poset.

The map φ can be extended to a graph morphism $\varphi : \mathcal{H}_1(\mathcal{G}_n) \to \mathcal{H}_2(\mathcal{G}_n)$. To prove that φ indeed preserves adjacency, let I and J be two ideals adjacent in $\mathcal{H}_1(\mathcal{G}_n)$. If I is covered by J in $\mathcal{J}(\mathcal{G}_n)$, then J can be obtained by I by adding exactly one new element z. Hence

(i) if z has rank 1, then $J_0 = I_0$, $J_1 = I_1 \sqcup \{z\}$ and $\varphi(J) = J_0' \sqcup J_1 = I_0' \sqcup I_1 \sqcup \{z\} = \varphi(I) \sqcup \{z\}$; that is, $\varphi(J)$ covers $\varphi(I)$ in $\mathcal{A}(\mathcal{G}_n)$;

(ii) if z has rank 0, then $J_0 = I_0 \sqcup \{z\}$, $J_1 = I_1$ and $\varphi(I) = I_0' \sqcup I_1 = J_0' \sqcup \{z\} \sqcup J_1 = \varphi(J) \sqcup \{z\}$; that is, $\varphi(I)$ covers $\varphi(J)$ in $\mathcal{A}(\mathcal{G}_n)$.

In both cases $\varphi(I)$ and $\varphi(J)$ are adjacent in $\mathcal{H}_2(\mathcal{G}_n)$. So, in conclusion, φ is a graph isomorphism and $\mathcal{H}_1(\mathcal{G}_n)$ and $\mathcal{H}_2(\mathcal{G}_n)$ are isomorphic graphs. □

The *ladder* L_n is the simple graph on the $2n$ vertices (i, h) with $1 \le i \le n$ and $1 \le h \le 2$, where (i, h) is adjacent to (j, k) if and only if $i = j$ and $h \ne k$, or $|i - j| = 1$ and $h = k$. An *independent subset* of L_n is a set of vertices in which no pair of vertices is adjacent [16]. It is easy to see that the simple graph underlying the Hasse diagram of $\mathcal{G}_n$ is the ladder L_n and that the antichains of $\mathcal{G}_n$ correspond exactly to the independent subsets of L_n (see Figure 5).

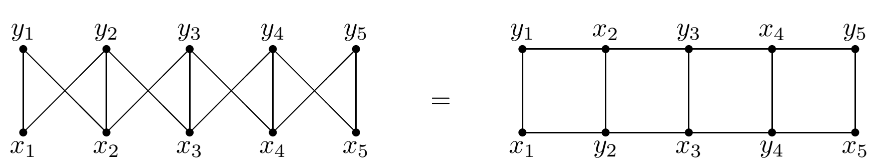

Figure 5: A garland and the associated ladder

Hence we have

Theorem 4 *The semilattice $\mathcal{A}(\mathcal{G}_n)$ of the antichains of a garland is isomorphic to the semilattice $\mathcal{I}(L_n)$ of the independent subsets of a ladder.*

Clearly the antichain polynomial $a_n(x)$ is the rank polynomial of the semilattice $\mathcal{A}(\mathcal{G}_n)$ and the independence polynomial [16] of the ladder L_n.

Let $p(x)$ and $q(x)$ be two real polynomials with degree n and $n+1$, with real distinct roots. Let $r_1 < \cdots < r_n$ and $s_1 < \cdots < s_{n+1}$ be their roots. The polynomials $p(x)$ and $q(x)$ *strictly interlace* if $s_1 < r_1 < s_2 < r_2 < \cdots < s_n < r_n < s_{n+1}$. A sequence $\{p_n(x)\}_n$ of real polynomials is a *Sturm sequence* [17] when every polynomial $p_n(x)$ has degree n, has n real distinct roots and strictly interlaces $p_{n+1}(x)$.

Theorem 5 *The antichain polynomials $a_n(x)$ form a Sturm sequence.*

Proof. The claim follows immediately from recurrence (13) [17, Corollary 2.4]. □

A sequence $\{a_0, a_1, \ldots, a_n\}$ of (positive) real numbers is *unimodal* when there exists an index k such that $a_0 \le a_1 \le \cdots \le a_k \ge a_{k+1} \ge \cdots \ge a_n$ while it is *log-concave* when $a_{k+1}a_{k-1} \le a_k^2$ for every $k = 1, 2, \ldots, n-1$. A polynomial is unimodal (resp. log-concave) when the sequence of its coefficients is unimodal (resp. log-concave).

Theorem 6 *The antichain polynomials $a_n(x)$ are log-concave and unimodal.*

Proof. Since every real polynomial having only real negative roots is log-concave [28], it follows immediately that $a_n(x)$ is log-concave. Moreover, since all its coefficients are positive, it also follows that $a_n(x)$ is unimodal. □

Since a ranked poset is rank-unimodal when its rank polynomial is unimodal, Theorem 6 implies

Theorem 7 *The semilattice $\mathcal{A}(\mathcal{G}_n)$ is rank-unimodal.*

8. Antichains and Lattice Paths

The antichains of a garland can be easily interpreted in terms of lattice paths, and more precisely in terms of *trinomial paths*; that is, lattice paths starting from $H = (0,0)$ and made of up steps $U = (1,1)$, down steps $D = (1,-1)$ and horizontal steps $(1,0)$. Every antichain A of $\mathcal{G}_n$ can be represented by means of a trinomial path γ_A as follows. Reading the garland from left to right, column by column, we write an up step U when we encounter an element of A at level 1, a down step D when we encounter an element of A at level 0, a horizontal step H when we encounter no elements. See Figure 6 for an example. Since A is an antichain, the path γ_A has neither peaks UD nor valleys DU. Moreover A is an equidistributed antichain if and only if γ_A is a *central path*; that is, if and only if γ_A ends on the x-axis.

Now we can reobtain the generating Series (15) for the equidistributed antichains with a combinatorial argument in terms of central trinomial paths with no peaks and no valleys. First of all, we consider the class $\mathcal{S}$ of all *smooth Motzkin paths*, i.e.,

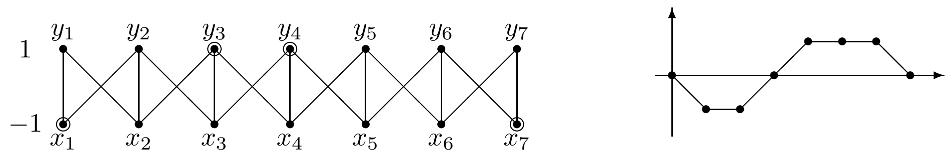

Figure 6: The antichain $A = \{x_1, y_3, y_4, x_7\}$ of $\mathcal{G}_7$ and the corresponding path $\gamma_A = DHUUHHD$.

Motzkin paths with no peaks and no valleys. Every non-empty smooth Motzkin path γ can be uniquely decomposed in one of the following ways (see Figure 7):

$$\gamma = H\gamma' \quad (\gamma' \in \mathcal{S}), \qquad \gamma = U\gamma' D \quad (\gamma' \in \mathcal{S}, \gamma' \neq \bullet),$$
$$\gamma = U\gamma' DH\gamma'' \quad (\gamma', \gamma'' \in \mathcal{S}, \gamma' \neq \bullet).$$

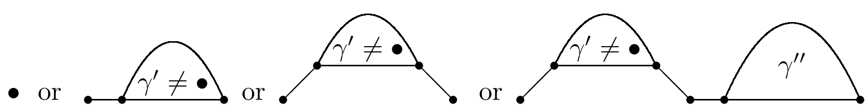

Figure 7: Main decomposition of Motzkin paths with no peaks and no valleys.

Hence, if $s(x)$ is the generating series of smooth Motzkin paths then we have the identity

$$s(x) = 1 + xs(x) + x^2(s(x) - 1) + x^3(s(x) - 1)s(x),$$

or $x^3 s(x)^2 - (1 - x - x^2 + x^3)s(x) + 1 - x^2 = 0$, and hence

$$s(x) = \frac{1 - x - x^2 + x^3 - \sqrt{1 - 2x - x^2 - x^4 + 2x^5 + x^6}}{2x^3}.$$

Now let $\mathcal{T}$ be the class of smooth central trinomial paths. Moreover let $\mathcal{T}_X$ be the class of all smooth central trinomial paths beginning with a step $X \in \{U, D, H\}$. Clearly $\mathcal{T} = 1 + \mathcal{T}_H + \mathcal{T}_U + \mathcal{T}_D$, where $\mathcal{T}_H = H\mathcal{T}$ and $\mathcal{T}_U \simeq \mathcal{T}_D$ (where the bijection is given by reflection across the x-axis). Every non-empty path $\gamma \in \mathcal{T}_U$ decomposes uniquely as $\gamma = U\gamma' D\gamma''$, where $\gamma' \in \mathcal{S}$, $\gamma' \neq \bullet$, and $\gamma'' \in \mathcal{T} \setminus \mathcal{T}_U$. If $t(x)$ and $u(x)$ denote the generating series for $\mathcal{T}$ and $\mathcal{T}_U$, then we have the identity

$$u(x) = x^2(s(x) - 1))(1 + xt(x) + u(x)).$$

Since $t(x) = 1 + xt(x) + 2u(x)$, it follows that

$$u(x) = \frac{x^2(s(x) - 1)}{1 - x + x^2 + x^3 - (x^2 + x^3)s(x)}$$

$$t(x) = \frac{1 - x^2 + x^2 s(x)}{1 - x + x^2 + x^3 - (x^2 + x^3)s(x)} = e(x). \tag{25}$$

We can also proceed in a different way as follows. We say that a path is positive (negative) when it never goes below (above) the x-axis. Negative paths correspond bijectively to positive paths by means of the reflection around the x-axis. We say that a positive (negative) Motzkin path γ is elevated if $\gamma = U\gamma' D$ ($\gamma = D\gamma' U$) where γ' is any positive (negative) Motzkin path. Clearly a (positive or negative) Motzkin path γ has no peaks and no valleys if and only if this is also true for γ'.

Every non-empty path $\gamma \in \mathcal{T}$ can be decomposed in a unique way as $\gamma = \gamma'\gamma''$ where γ' is an alternating product of positive and negative elevated Motzkin paths (where the first path can be either positive or negative) and $\gamma'' = \bullet$ or $\gamma'' = H\gamma'''$ with $\gamma''' \in \mathcal{T}$ (see Figure 8). This immediately implies the identity

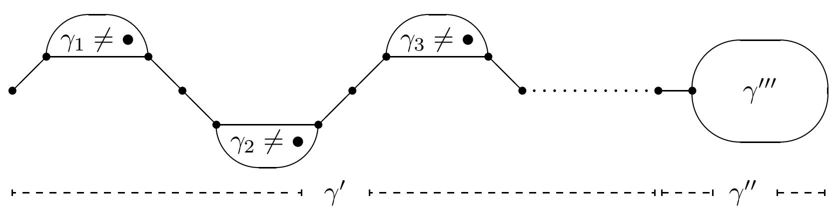

Figure 8: Decomposition of trinomial paths with no peaks and no valleys.

$$t(x) = \left(\frac{2}{1 - x^2 s(x)} - 1 \right) (1 + xt(x))$$

from which we reobtain the expression of $t(x)$ in (25).

Differentiating $s(x)$ and $t(x)$ it is possible to obtain the identities

$$\begin{aligned} x(1 - 2x - x^2 - x^4 + 2x^5 + x^6)s'(x) + (3 - 5x - 2x^2 - x^4 + x^5)s(x) \\ -3 + x + 4x^2 - x^4 - x^5 = 0 \end{aligned}$$

$$(1 - 2x - x^2 - x^4 + 2x^5 + x^6)t'(x) - (1 - x + 4x^2 + 2x^3 - x^4 - x^5)t(x) = 0$$

and hence the recurrences

$$\begin{aligned} (n + 9)s_{n+6} - (2n + 15)s_{n+5} - (n + 6)s_{n+4} - (n + 3)s_{n+2} \\ +(2n + 3)s_{n+1} + ns_n = 0, \end{aligned}$$

$$\begin{aligned} (n + 6)t_{n+6} - (2n + 11)t_{n+5} - (n + 3)t_{n+4} - 4t_{n+3} - (n + 4)t_{n+2} \\ +(2n + 3)t_{n+1} + (n + 1)t_n = 0. \end{aligned}$$

From the expression for $t(x)$ appearing in (25) we have the identity

$$(1 - x + x^2 + x^3)t(x) - (x^2 + x^3)s(x)t(x) = 1 - x^2 + x^2 s(x)$$

from which follows the relation

$$\sum_{k=0}^{n+1} s_k t_{n-k+1} + \sum_{k=0}^{n} s_k t_{n-k} = t_{n+3} - t_{n+2} + t_{n+1} - s_{n+1} + t_n.$$

Asymptotically we have

$$g_n \sim \frac{1}{2}(1+\sqrt{2})^{n+1}, \quad s_n \sim \frac{1}{n}\sqrt{\frac{\sqrt{2}-1}{n\pi}}\left(1+\sqrt{2}\right)^{n+2},$$

$$t_n \sim \frac{1}{2}\sqrt{\frac{\sqrt{2}+1}{n\pi}}\left(1+\sqrt{2}\right)^{n}.$$

The first relation is an immediate consequence of (8) while the last relation has been obtained in [19] using Darboux's theorem. Also the second relation can be obtained using Darboux's theorem, exactly as in [19]. Hence $g_{n+1} \sim (1+\sqrt{2})g_n$, $s_{n+1} \sim (1+\sqrt{2})s_n$, $t_{n+1} \sim (1+\sqrt{2})t_n$ and

$$\frac{s_n}{g_n} \sim \frac{2(1+\sqrt{2})}{n}\sqrt{\frac{\sqrt{2}-1}{n\pi}}, \qquad \frac{t_n}{g_n} \sim \sqrt{\frac{\sqrt{2}-1}{n\pi}}, \qquad \frac{s_n}{t_n} \sim \frac{2(1+\sqrt{2})}{n}.$$

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