



**BELL NUMBERS AND VARIANT SEQUENCES DERIVED FROM
A GENERAL FUNCTIONAL DIFFERENTIAL EQUATION**

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Abstract

It is well-known that the Bell numbers $\{B(n)\}_{n=0}^{\infty}$ have exponential generating function $\sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} = \exp(e^x - 1)$, which satisfies the differential equation $\frac{d}{dx}g(x) = e^x g(x)$. In this paper, we investigate certain sequences $\{G(n)\}_{n=0}^{\infty}$ whose exponential generating functions satisfy a modified form of the above differential equation, namely, the functional differential equation $\frac{d}{dx}g(x) = e^{bx}g(ax)$. For the main result of this paper, we show that when $a = -1$ and $b \in \mathbb{R}$, the sequence $\{G(n)\}_{n=0}^{\infty}$ obeys the simple second-order linear recurrence $G(n+2) = bG(n+1) - G(n)$. The proof is based on a well-known binomial series inversion formula.

1. Introduction

The Bell numbers may be defined by the linear binomial recurrence

$$B(n+1) = \sum_{k=0}^n \binom{n}{k} B(k), \quad (1)$$

with initial condition $B(0) = 1$. These numbers 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, ... have been extensively studied (see [1]). It is Sequence No. A000110 in Neil J. A. Sloane's Online Encyclopedia of Integer Sequences (OEIS). The numbers may also be defined by the exponential generating function

$$\sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} = \exp(e^x - 1). \quad (2)$$

The exponential generating function satisfies the differential equation

$$\frac{d}{dx}f(x) = e^x f(x). \quad (3)$$

Similarly, the Rao Uppuluri–Carpenter numbers obey the following recurrence,

$$D(n+1) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} D(k), \quad (4)$$

with the initial condition that $D(0) = 1$. The numbers generated by (4) are 1, 1, 0, -1, 1, 2, -9, 9, 50, -267, 413, 2180, -17731, 50533, 110176, . . . , and are listed as Sequence No. A000587 in the OEIS. They satisfy the exponential generating function

$$\sum_{n=0}^{\infty} D(n) \frac{x^n}{n!} = \exp(1 - e^{-x}). \tag{5}$$

Notice that the exponential generating function satisfies the differential equation

$$\frac{d}{dx} f(x) = e^{-x} f(x). \tag{6}$$

Motivated by the similarities between (3) and (6), we observe that they are special cases of the general functional differential equation

$$\frac{d}{dx} g(x) = e^{bx} g(ax). \tag{7}$$

Setting

$$g(x) = \sum_{n=0}^{\infty} G(n) \frac{x^n}{n!}, \tag{8}$$

we find that $G(n)$ satisfies the recurrence relation

$$G(n + 1) = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} G(k). \tag{9}$$

This allows us to think of the previously mentioned sequences as all belonging to a single general class of sequences. This is like the situation studied in [4]. We call any sequence that satisfies Equation (9) a *variant sequence*. In particular, when $a = 1$ and $b = 1$ in Equation (9), we obtained the Bell numbers $B(n)$. When $a = 1$ and $b = -1$, we obtained the Uppuluri-Carpenter numbers $D(n)$.

As it stands, any variant sequence satisfies the n^{th} order difference equation provided by Equation (9). The main result of this paper states that certain variant sequences obey a lower order recurrence. In particular, if $a = -1$ and b is *any* nonzero real number, the variant sequence obeys a second-order linear recurrence. This main result is Theorem 2.1. We then use the remaining two sections to discuss the properties of two specific variant sequences, namely the sequence given by $a = -1$ and $b = 1$, and the sequence given by $a = -1$ and $b = -1$.

2. Variant Sequences and Second-Order Difference Equations

We now assume that $a = -1$ and b is any nonzero real number. The goal is to prove that the associated variant sequence obeys a second-order recurrence. In order to prove this claim, we first need the following lemma.

Lemma 1 *Let n be a nonnegative integer. Let b be any nonzero real number. Define $\{G(n)\}_{n=0}^\infty$ by*

$$G(n + 1) = \sum_{k=0}^n (-1)^k \binom{n}{k} b^{n-k} G(k). \tag{10}$$

Then,

$$G(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} b^{n-k} G(k + 1). \tag{11}$$

Proof. Lemma 2.1 is an immediate consequence of the following well-known binomial inversion pair. Given any two sequences $\{f(n)\}_{n=0}^\infty$ and $\{g(n)\}_{n=0}^\infty$

$$g(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(k) \tag{12}$$

if and only if

$$f(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} g(k). \tag{13}$$

In our particular situation, we let $f(k) = b^{-k}G(k)$ and $g(n) = b^{-n}G(n)$. □

With Lemma 2.1 in place, we now prove that a variant sequence with $a = -1$ and b nonzero obeys a second-order linear recurrence.

Theorem 2 *Let n be a nonnegative integer. Let b be any nonzero real number. Define $\{G(n)\}_{n=0}^\infty$ by*

$$G(n + 1) = \sum_{k=0}^n (-1)^k \binom{n}{k} b^{n-k} G(k). \tag{14}$$

Then $G(n)$ satisfies

$$G(n + 2) = bG(n + 1) - G(n). \tag{15}$$

Proof. Using Lemma 2.1, we find that

$$\begin{aligned} & bG(n + 1) - G(n) \\ &= b \sum_{k=0}^n (-1)^k \binom{n}{k} b^{n-k} G(k) - \sum_{k=0}^n (-1)^k \binom{n}{k} b^{n-k} G(k + 1) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} b^{n+1-k} G(k) + \sum_{k=0}^{n+1} (-1)^k \binom{n}{k-1} b^{n+1-k} G(k) \\
 &= \sum_{k=0}^{n+1} (-1)^k \left(\binom{n}{k} + \binom{n}{k-1} \right) b^{n+1-k} G(k) \\
 &= \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} b^{n+1-k} G(k) \\
 &= G(n+2).
 \end{aligned}$$

□

Remark 3 If we define the general functional differential equation by

$$\frac{d^r}{dx^r} g(x) = e^{bx} g(ax), \tag{16}$$

we find the general recurrence is

$$G(n+r) = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} G(k), \tag{17}$$

which is a slightly more general relation using the parameter r . This provides a unification of Bell numbers, Fibonacci numbers, Uppuluri-Carpenter numbers, and the more general class of variant sequences.

3. The Variant Sequence with $a = -1, b = 1$

Define $\{F(n)\}_{n=0}^\infty$ to be the variant sequence associated with $a = -1$ and $b = 1$. Then, Equation (9) becomes

$$F(n+1) = \sum_{k=0}^n (-1)^k \binom{n}{k} F(k), \tag{18}$$

with the assumption that $F(0) = 1$. The sequence so generated is 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, ... and is listed in the OEIS as Sequence No. A010892.

Theorem 2.1 states that $\{F(n)\}_{n=0}^\infty$ obeys the second-order linear recurrence provided by Corollary 3.1.

Corollary 4 (Recurrence Relation for $F(n)$) For $n \geq 0$,

$$F(n+1) - F(n) = F(n+2). \tag{19}$$

Corollary 3.1 shows that $\{F(n)\}_{n=0}^\infty$ is actually a Fibonacci type sequence where each term is the difference of the two immediately previous terms.

Theorem 5 (Definition of $F(n)$ by its recurrence relation) *The recurrence relation $F(n + 2) = F(n + 1) - F(n)$, together with the initial values $F(0) = 1, F(1) = 1$, determines the sequence uniquely.*

It is remarkable that the sequence $\{F(n)\}_{n=0}^\infty$, originally defined by a high-order recurrence, also satisfies a second-order recurrence. We wish next to show the converse, in the sense that Equation (18) can be derived using (19). In particular, by iterating Recurrence (19), and using induction, we can prove the following result.

Lemma 6 *For any integer $n \geq 0$,*

$$F(n + r) = \sum_{k=0}^r (-1)^k \binom{r}{k} F(n - k). \tag{20}$$

We can also extend Recurrence (19) backwards to prove

Lemma 7 (Extension of meaning of $F(n)$ to negative n) *Let n be any integer. Then $F(n)$ as defined by (19) satisfies*

$$F(n) = F(1 - n). \tag{21}$$

Combining Lemmas 3.1 and 3.2, we are able to obtain the desired converse since, by letting $n = 1$ in Equation (20), we obtain

$$F(r + 1) = \sum_{k=0}^r (-1)^k \binom{r}{k} F(1 - k). \tag{22}$$

By Lemma 3.2, the right-hand side of (22) becomes

$$F(r + 1) = \sum_{k=0}^r (-1)^k \binom{r}{k} F(k), \tag{23}$$

which establishes (18).

Recurrence (19) may be solved easily using the roots of the characteristic equation $x^2 - x + 1 = 0$ to yield

Theorem 8 (First Formula for $F(n)$) *Let n be any integer. Then*

$$F(n) = \frac{3 - i\sqrt{3}}{6} \left(\frac{1 + i\sqrt{3}}{2} \right)^n + \frac{3 + i\sqrt{3}}{6} \left(\frac{1 - i\sqrt{3}}{2} \right)^n. \tag{24}$$

However, we may avoid the use of complex numbers. It is easy to show a floor function formula, that satisfies (19), with the same initial values.

Theorem 9 (Second Formula for $F(n)$) *Let n be a nonnegative integer. Then*

$$F(n) = \frac{(-1)^{\lfloor \frac{n}{3} \rfloor} + (-1)^{\lfloor \frac{n+1}{3} \rfloor}}{2}. \tag{25}$$

Equation (25) is a very useful formula. Replacing n by $n + 6$, we have an almost immediate proof that $F(n + 6) = F(n)$ so that the pattern $1, 1, 0, -1, -1, 0$ repeats forever. It is clear also that $(-1)^n F(n)$ has period 3, being the sequence $1, -1, 0, 1, -1, 0, \dots$. The periodicity is also apparent from recurrence (19). It is not difficult to show a binomial summation that also satisfies Recurrence (19).

Theorem 10 (Third Formula for $F(n)$) *Let n be a nonnegative integer. Then*

$$F(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k}. \tag{26}$$

Remark 11 The equality of Equations (25) and (26) is tabulated as binomial identity (1.75) in Gould’s book [2].

Finally, we provide an alternative formula, involving linear combinations of the floor function that satisfies Recurrence (19).

Theorem 12 (Fourth Formula for $F(n)$) *Let n be any integer. Then*

$$F(n) = \left\lfloor \frac{n+6}{6} \right\rfloor - \left\lfloor \frac{n+4}{6} \right\rfloor + \left\lfloor \frac{n+7}{6} \right\rfloor - \left\lfloor \frac{n+9}{6} \right\rfloor.$$

Remark 13 In the OEIS, where the sequence $1, 1, 0, -1, -1, 0, 1, 1, \dots$ is tabulated, Equations (25) and (26) are mentioned, but our initial definition (18) of the sequence $F(n)$ is not mentioned. However, it is mentioned that $F(n)$ is the same as the Chebychev polynomial $U(n, \frac{1}{2})$. Also the sequence is listed under the heading “Inverse of 6th cyclotomic polynomial.”

3.1. Generating Functions Associated with $F(n)$

From Equation (19), it is easy to show that the ordinary generating function for $F(n)$ is

$$\sum_{n=0}^{\infty} F(n)x^n = \frac{1}{1-x+x^2}. \tag{27}$$

We should also recall that Equation (7) implies that $f(x)$, the exponential generating function for $F(n)$, satisfies

$$\frac{d}{dx} f(x) = e^x f(-x). \tag{28}$$

By differentiating Equation (28), we obtain

$$f''(x) = e^x f(-x) - e^x f'(-x).$$

Substituting (28) into the right-hand side of the previous line, we obtain

$$f''(x) = f'(x) - e^x f'(-x). \tag{29}$$

Clearly, Equation (28) implies $f'(-x) = e^{-x} f(x)$. Thus, the previous equation may be rewritten as

$$f''(x) = f'(x) - f(x). \tag{30}$$

Note that Equation (30) is a linear homogeneous second-order differential equation which has the following general solution:

$$f(x) = c_1 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right). \tag{31}$$

4. The Variant Sequence with $a = -1$, $b = -1$

Define $\{H(n)\}_{n=0}^\infty$ to be the variant sequence associated with $a = -1$ and $b = -1$. Then, Equation (9) becomes

$$H(n + 1) = (-1)^n \sum_{k=0}^n \binom{n}{k} H(k), \tag{32}$$

with the initial condition $H(0) = 1$. The sequence generated by (32), namely, 1, 1, -2, 1, 1, -2, . . . appears in the OEIS as the simple order three periodic sequence No. A061347.

Once again, Theorem 2.1 implies that $\{H(n)\}_{n=0}^\infty$ obeys the second-order linear recurrence given as follows.

Corollary 14 (Recurrence relation for $H(n)$) *For $n \geq 0$,*

$$H(n + 2) = -H(n + 1) - H(n). \tag{33}$$

Once again, $\{H(n)\}_{n=0}^\infty$, like $\{F(n)\}_{n=0}^\infty$, is a sequence that, although originally defined by a higher order recurrence, satisfies a simple second-order recurrence. As can be expected, a converse to Corollary 4.1 exists. In other words, Equation (33) implies (32). The proof of this converse parallels the proof of the converse to Corollary 3.1, and hence will be omitted.

Recurrence (32) may be solved easily using the roots of the characteristic equation $x^2 + x + 1 = 0$ to yield

Theorem 15 (First Formula for $H(n)$) *Let n be any integer. Then*

$$H(n) = \frac{-1 + i\sqrt{3}}{2} \left(\frac{-1 + i\sqrt{3}}{2} \right)^n + \frac{3 - i\sqrt{3}}{2} \left(\frac{-1 - i\sqrt{3}}{2} \right)^n. \quad (34)$$

By using Equation (34), it is easy to show that $H(n) = H(n + 3)$.

However, we may avoid the use of complex numbers and instead use the floor function to define $H(n)$. The following two theorems show how this can be done.

Theorem 16 (Second Formula for $H(n)$) *Let n be any nonnegative integer. Define $\{\hat{H}(n)\}_{n=0}^\infty$ by*

$$\hat{H}(n) = \frac{-(-1)^{\lfloor \frac{n}{3} \rfloor} + 3(-1)^{\lfloor \frac{n+1}{3} \rfloor}}{2}.$$

Then, for $0 \leq i \leq 2$, $\hat{H}(6n + i) = H(3n + i)$.

Theorem 17 (Third Formula for $H(n)$) *Let n be any integer. Then*

$$H(n) = 3 \left\lfloor \frac{n}{3} \right\rfloor - 3 \left\lfloor \frac{n+1}{3} \right\rfloor + 1. \quad (35)$$

4.1. Generating Functions Associated with $H(n)$

From Equation (33), it is easy to show that the ordinary generating function for $H(n)$ is

$$\sum_{n=0}^\infty H(n)x^n = \frac{1 + 2x}{1 + x + x^2}. \quad (36)$$

Next, we recall that Equation (7) implies $h(x)$, the exponential generating function for $H(n)$, satisfies

$$\frac{d}{dx}h(x) = e^{-x}h(-x). \quad (37)$$

Following, with appropriate changes, the calculations given in Section 3.1, it is easy to show that

$$h(x) = c_1 e^{\frac{-1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{\frac{-1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right). \quad (38)$$

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