COEFFICIENTS IN POWERS OF THE LOG SERIES

Donald M. Davis
Department of Mathematics, Lehigh University, Bethlehem, PA, 18015, USA
dmd1@lehigh.edu

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Abstract

We determine the p-exponent in many of the coefficients of $\ell(x)^t$, where $\ell(x)$ is the power series for $\log(1+x)/x$ and t is any integer. In our proof, we introduce a variant of multinomial coefficients. We also characterize the power series $x/\log(1+x)$ by certain zero coefficients in its powers.

1. Main Divisibility Theorem

The divisibility by primes of the coefficients in the integer powers $\ell(x)^t$ of the power series for $\log(1+x)/x$, given by

$$\ell(x) := \sum_{i=0}^{\infty}(-1)^i \frac{x^i}{i+1},$$

has been applied in several ways in algebraic topology. See, for example, [1] and [4]. Our main divisibility result, Theorem 1, says that, in an appropriate range, this divisibility is the same as that of the coefficients of $(1 \pm \frac{x^{p-1}}{p})^t$. Here p is any prime and t is any integer, positive or negative. We denote by $\nu_p(-)$ the exponent of p in an integer and by $[x^n]f(x)$ the coefficient of x^n in a power series $f(x)$.

Theorem 1. *If t is any integer and $1 \le m \le p^{\nu_p(t)}$, then*

$$\nu_p\left([x^{(p-1)m}]\ell(x)^t\right) = \nu_p(t) - \nu_p(m) - m.$$

Thus, for example, if $\nu_3(t) = 2$, then, for $m = 1, \dots, 9$, the exponent of 3 in $[x^{2m}]\ell(x)^t$ is, respectively, 1, 0, −2, −2, −3, −5, −5, −6, and −9, which is the same as in $(1 \pm \frac{x^2}{3})^t$. In Section 3, we will discuss what we can say about $\nu_p([x^n]\ell(x)^t)$ when n is not divisible by $(p-1)$ and $n < (p-1)p^{\nu_p(t)}$.

The motivation for Theorem 1 was provided by ongoing work which seeks to apply the result when $p = 2$ to make more explicit some nonimmersion results for complex projective spaces described in [4]. The coefficients studied here can be directly related to Stirling numbers and generalized Bernoulli numbers ([3, Chapter 6]), but it seems that our divisibility results are new in any of these contexts.

Proving Theorem 1 led the author to discover an interesting modification of multinomial coefficients.

Definition 2. For an ordered r-tuple of nonnegative integers $(i_1,\dots,i_r)$, not all 0, we define

$$c(i_1,\dots,i_r) := \frac{(\sum i_j j)(\sum i_j - 1)!}{i_1!\cdots i_r!}.$$

Note that $c(i_1,\dots,i_r)$ equals $(\sum i_j j)/\sum i_j$ times a multinomial coefficient. Surprisingly, these numbers satisfy the same recursive formula as multinomial coefficients.

Definition 3. For positive integers $k \le r$, let E_k denote the ordered r-tuple whose only nonzero entry is a 1 in position k.

Proposition 4. *If $I = (i_1,\dots,i_r)$ is an ordered r-tuple of nonnegative integers with $\sum i_j > 1$, then*

$$c(I) = \sum_{i_k>0} c(I - E_k). \tag{1}$$

If we think of a multinomial coefficient $\binom{\sum i_j}{i_1,\cdots,i_r} := (i_1+\cdots+i_r)!/((i_1)!\cdots(i_r)!)$ as being determined by the unordered r-tuple $(i_1,\dots,i_r)$ of nonnegative integers, then it satisfies the recursive formula analogous to that of (1). For a multinomial coefficient, entries which are 0 can be omitted, but that is not the case for $c(i_1,\dots,i_r)$.

Proof of Proposition 4. The right hand side of (1) equals

$$\begin{aligned}
&\sum_k i_k \frac{(\sum i_j - 2)!}{(i_1)!\cdots(i_r)!}\left(\sum_j i_j j - k\right)\\
=\ &\frac{(\sum i_j - 2)!}{(i_1)!\cdots(i_r)!}\left(\left(\sum i_k\right)\left(\sum i_j j\right) - \sum i_k k\right)\\
=\ &\frac{(\sum i_j - 2)!}{(i_1)!\cdots(i_r)!}\left(\sum i_j j\right)\left(\sum i_j - 1\right),
\end{aligned}$$

which equals the left hand side of (1). □

Corollary 5. *If $\sum i_j > 0$, then $c(i_1,\dots,i_r)$ is a positive integer.*

Proof. Use (1) recursively to express $c(i_1,\dots,i_r)$ as a sum of various $c(E_k) = k$. □

Corollary 6. *For any ordered r-tuple $(i_1,\ldots,i_r)$ of nonnegative integers and any prime p,*

$$\nu_p\left(\sum i_j\right) \le \nu_p\left(\sum i_j j\right) + \nu_p\binom{\sum i_j}{i_1,\cdots,i_r}. \tag{2}$$

Proof. Multiply numerator and denominator of the definition of $c(i_1,\ldots,i_r)$ by $\sum i_j$ and apply Corollary 5. □

The proof of Theorem 1 utilizes Corollary 6 and also the following lemma.

Lemma 7. *If t is any integer and $\sum i_j \le p^{\nu_p(t)}$, then*

$$\nu_p\binom{t}{t-\sum i_j, i_1,\ldots,i_r} = \nu_p(t) + \nu_p\binom{\sum i_j}{i_1,\ldots,i_r} - \nu_p\left(\sum i_j\right). \tag{3}$$

Proof. For any integer t, the multinomial coefficient on the left hand side of (3) equals $t(t-1)\cdots(t+1-\sum i_j)/\prod i_j!$, and so the left hand side of (3) equals $\nu_p(t(t-1)\cdots(t+1-\sum i_j)) - \sum \nu_p(i_j!)$. Since $\nu_p(t-s) = \nu_p(s)$ provided $0 < s < p^{\nu_p(t)}$, this becomes $\nu_p(t) + \nu_p((\sum i_j - 1)!) - \sum \nu_p(i_j!)$, and this equals the right hand side of (3). □

Proof of Theorem 1. By the multinomial theorem,

$$[x^{(p-1)m}]\ell(x)^t = (-1)^{(p-1)m}\sum_I T_I,$$

where

$$T_I = \binom{t}{t-\sum i_j, i_1,\ldots,i_r}\frac{1}{\prod (j+1)^{i_j}}, \tag{4}$$

with the sum taken over all $I = (i_1,\ldots,i_r)$ satisfying $\sum i_j j = (p-1)m$. Using Lemma 7, we have

$$\nu_p(T_I) = \nu_p(t) + \nu_p\binom{\sum i_j}{i_1,\ldots,i_r} - \nu_p\left(\sum i_j\right) - \sum i_j \nu_p(j+1).$$

If $I = mE_{p-1}$, then $\nu_p(T_I) = \nu_p(t) + 0 - \nu_p(m) - m$. The theorem will follow once we show that all other I with $\sum i_j j = (p-1)m$ satisfy $\nu_p(T_I) > \nu_p(t) - \nu_p(m) - m$. Such I must have $i_j > 0$ for some $j \ne p-1$. This is relevant because $\frac{1}{p-1}j \ge \nu_p(j+1)$ with equality if and only if $j = p-1$. For I such as we are considering, we have

$$\begin{aligned}
&\nu_p(T_I) - (\nu_p(t) - \nu_p(m) - m) \qquad (5)\\
= \;&\nu_p\binom{\sum i_j}{i_1,\ldots,i_r} - \nu_p(\sum i_j) - \sum i_j \nu_p(j+1) + \nu_p(\sum i_j j) + \tfrac{1}{p-1}\sum i_j j\\
\ge \;&\sum i_j(\tfrac{1}{p-1}j - \nu_p(j+1))\\
> \;&0.
\end{aligned}$$

We have used (2) in the middle step. □

2. Zero Coefficients

While studying coefficients related to Theorem 1, we noticed the following result about occurrences of coefficients of powers of the reciprocal log series which equal 0.

Theorem 8. *If m is odd and $m > 1$, then $[x^m]\big(\frac{x}{\log(1+x)}\big)^m = 0$, while if m is even and $m > 0$, then $[x^{m+1}]\big(\frac{x}{\log(1+x)}\big)^m = 0$.*

Moreover, this property characterizes the reciprocal log series.

Corollary 9. *A power series $f(x) = 1 + \sum\limits_{i\geq 1} c_i x^i$ with $c_1 \neq 0$ has $[x^m](f(x)^m) = 0$ for all odd $m > 1$ and $[x^{m+1}](f(x)^m) = 0$ for all even $m > 0$ if and only if $f(x) = \frac{2c_1 x}{\log(1+2c_1x)}$.*

Proof. By Theorem 8, the reciprocal log series satisfies the stated property. Now assume that f satisfies this property and let n be a positive integer and $\epsilon = 0$ or 1. Since

$$[x^{2n+1}]f(x)^{2n+\epsilon} = (2n+\epsilon)(2n+\epsilon-1)c_1c_{2n} + (2n+\epsilon)c_{2n+1} + P,$$

where P is a polynomial in $c_1, \ldots, c_{2n-1}$, we see that c_{2n} and c_{2n+1} can be determined from the c_i with $i < 2n$. □

Our proof of Theorem 8 is an extension of arguments of [1] and [2]. It benefited from ideas of Francis Clarke. The theorem can be derived from results in [3, Chapter 6], but we have not seen it explicitly stated anywhere.

Proof of Theorem 8. Let $m > 1$ and

$$\left(\frac{x}{\log(1+x)}\right)^m = \sum_{i\geq 0} a_i x^i.$$

Letting $x = e^y - 1$, we obtain

$$\left(\frac{e^y-1}{y}\right)^m = \sum_{i\geq 0} a_i (e^y-1)^i. \tag{6}$$

Let j be a positive integer, and multiply both sides of (6) by $y^m e^y/(e^y-1)^{j+1}$, obtaining

$$\begin{aligned}(e^y-1)^{m-j-1}e^y &= y^m \sum_{i\geq 0} a_i (e^y-1)^{i-j-1}e^y \qquad (7)\\ &= y^m \left(a_j \frac{e^y}{e^y-1} + \sum_{i\neq j} \tfrac{a_i}{i-j}\tfrac{d}{dy}(e^y-1)^{i-j} \right).\end{aligned}$$

Since the derivative of a Laurent series has no y^{-1}-term, we conclude that the coefficient of y^{m-1} on the right-hand side of (7) is $a_j[y^{-1}](1+\frac{1}{y}\frac{y}{e^y-1}) = a_j$.

The Bernoulli numbers B_n are defined by $\frac{y}{e^y-1} = \sum \frac{B_n}{n!}y^n$. Since $\frac{y}{e^y-1}+\frac{1}{2}y$ is an even function of y, we have the well-known result that $B_n = 0$ if n is odd and $n > 1$.

Let

$$j = \begin{cases} m & m \text{ odd} \\ m+1 & m \text{ even.} \end{cases}$$

For this j, the left-hand side of (7) equals

$$\begin{cases} 1+\sum \frac{B_i}{i!}y^{i-1} & m \text{ odd} \\ -\frac{d}{dy}(e^y-1)^{-1} = -\sum \frac{(i-1)B_i}{i!}y^{i-2} & m \text{ even,} \end{cases}$$

and comparison with the coefficient of y^{m-1} in (7) implies

$$\begin{cases} a_m = \frac{B_m}{m!} = 0 & m \text{ odd} \\ a_{m+1} = -\frac{mB_{m+1}}{(m+1)!} = 0 & m \text{ even,} \end{cases}$$

yielding the theorem. □

3. Other Coefficients

In this section, a sequel to Theorem 1, we describe what can be easily said about $\nu_p([x^{(p-1)m+\Delta}]\ell(x)^t)$ when $0 < \Delta < p-1$ and $m < p^{\nu_p(t)}$. This is not relevant in the motivating case, $p = 2$. Our first result says that these exponents are at least as large as those of $[x^{(p-1)m}]\ell(x)^t$. Here t continues to denote any integer, positive or negative.

Proposition 10. *If $0 < \Delta < p-1$ and $m < p^{\nu_p(t)}$, then*

$$\nu_p\left([x^{(p-1)m+\Delta}]\ell(x)^t\right) \geq \nu_p(t) - \nu_p(m) - m.$$

Proof. We consider terms T_I as in (4) with $\sum i_j j = (p-1)m+\Delta$. Similarly to (5), we obtain

$$\nu_p(T_I) - (\nu_p(t) - \nu_p(m) - m) \;=\; \nu_p\binom{\sum i_j}{i_1,\ldots,i_r} - \nu_p\left(\sum i_j\right) - \sum i_j \nu_p(j+1) + \nu_p(m) + m. \tag{8}$$

We wish to show that this is nonnegative.

For $I = (i_1, \ldots, i_r)$, let

$$\begin{aligned}\widetilde{\nu}_p(I) &:= \nu_p\binom{\sum i_j}{i_1,\ldots,i_r} - \nu_p\left(\sum i_j\right)\\ &= \nu_p\left(\frac{1}{i_j}\binom{\sum i_j - 1}{i_1,\ldots,i_j-1,\ldots,i_r}\right),\end{aligned}$$

for any j. Thus

$$\widetilde{\nu}_p(I) \geq -\min_j \nu_p(i_j). \tag{9}$$

Ignoring the term $\nu_p(m)$, the expression (8) is

$$\geq \widetilde{\nu}_p(I) + \sum i_j(\tfrac{1}{p-1}j - \nu_p(j+1)) - \tfrac{\Delta}{p-1}. \tag{10}$$

Note that

$$\sum i_j(\tfrac{1}{p-1}j - \nu_p(j+1)) - \tfrac{\Delta}{p-1} = m - \sum i_j \nu_p(j+1)$$

is an integer and is greater than -1, and hence is ≥ 0. Thus we are done if $\widetilde{\nu}_p(I) \geq 0$.

Now suppose $\widetilde{\nu}_p(I) = -e$ with $e > 0$. By (9), all i_j are divisible by p^e. Thus $(p-1)\sum i_j(\frac{1}{p-1}j - \nu_p(j+1))$ is a positive integer and divisible by p^e. Hence it is $\geq p^e$. Therefore, (10) is an integer which is strictly greater than $-e + \frac{p^e}{p-1} - 1 = -e + \sum_{k=1}^{e-1} p^k + \frac{1}{p-1}$. Since it is an integer, we can replace the $\frac{1}{p-1}$ by 1, and obtain the nonnegative expression $\sum_{k=0}^{e-1}(p^k - 1)$. We obtain the desired conclusion, that, for each I, (10), and hence (8), is ≥ 0. □

Finally, we address the question of when does equality occur in Proposition 10. We give a three-part result, but by the third it becomes clear that obtaining additional results is probably more trouble than it is worth.

Proposition 11. *In Proposition 10,*

(a) *the inequality is strict ($\neq$) if $m \equiv 0$ (p);*

(b) *equality holds if $\Delta = 1$ and $m \not\equiv 0, 1$ (p);*

(c) *if $\Delta = 2$ and $m \not\equiv 0, 2$ (p), then equality holds if and only if $3m \not\equiv 5$ (p).*

Proof. We begin as in the proof of Proposition 10, and note that, using (2), the value of (8) is greater than or equal to

$$\nu_p(m) - \tfrac{\Delta}{p-1} + \sum i_j(\tfrac{1}{p-1}j - \nu_p(j+1)) - \nu_p((p-1)m + \Delta). \tag{11}$$

(a) If $\nu_p(m) > 0$, then $\nu_p((p-1)m + \Delta) = 0$ and so (11) is greater than 0.

In (b) and (c), we exclude consideration of the case where $m \equiv \Delta$ (p) because then $\nu_p((p-1)m+\Delta) > 0$ causes complications.

(b) If $\Delta = 1$ and $m \not\equiv 0, 1$ (p), then for $I = E_1 + mE_{p-1}$, (8) equals

$$\nu_p(m+1) - \nu_p(m+1) - m + \nu_p(m) + m = 0,$$

while for other I, (11) is

$$0 - \tfrac{1}{p-1} + \sum i_j(\tfrac{1}{p-1}j - \nu_p(j+1)) > 0.$$

(c) Assume $\Delta = 2$ and $m \not\equiv 0, 2$ (p). Then

$$\begin{aligned} T_{2E_1+mE_{p-1}} + T_{E_2+mE_{p-1}} &= \frac{t(t-1)\cdots(t-m-1)}{2!m!}\frac{1}{4p^m} + \frac{t(t-1)\cdots(t-m)}{m!}\frac{1}{3p^m} \\ &= (-1)^m \tfrac{t}{p^m}\big(\tfrac{1}{8}(-m-1+A) + \tfrac{1}{3}(1+B)\big) \\ &= (-1)^m \tfrac{t}{24p^m}(-3m+5+(3A+8B)). \qquad (12) \end{aligned}$$

Here A and B are rational numbers which are divisible by p. This is true because $\nu_p(t) > \nu_p(i)$ for all $i \le m$. Since $p > 3$, (12) has p-exponent greater than or equal to $\nu_p(t) - m$, with equality if and only if $3m - 5 \not\equiv 0$ (p). Using (11), the other terms T_I satisfy

$$\nu_p(T_I) - (\nu_p(t) - m) \ge \sum i_j(\tfrac{1}{p-1}j - \nu_p(j+1)) - \tfrac{2}{p-1} > 0.$$

□

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