ODD CATALAN NUMBERS MODULO 2^k

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Abstract

This article proves a conjecture by S.-C. Liu and J. C.-C. Yeh about Catalan numbers, which states that odd Catalan numbers can take exactly $k-1$ distinct values modulo 2^k, namely the values $C_{2^1-1}, \ldots, C_{2^{k-1}-1}$.

0. Notation

In this article we denote by $C_n := (2n)!/[(n+1)!n!]$ the n-th Catalan number. We also define $(2n+1)!! := 1 \times 3 \times \cdots \times (2n+1)$. Finally, we denote by $o(n) := n/2^a$ the odd part of n, where a is the largest power of 2 dividing n.

1. Introduction

The main result of this article is Theorem 2, which proves a conjecture by S.-C. Liu and J. C.-C. Yeh about odd Catalan numbers [4, Theorem 7.1]. To begin with, let us recall the characterization of odd Catalan numbers [1]:

Proposition 1. *A Catalan number C_n is odd if and only if $n = 2^a - 1$ for some integer a.*

The main theorem we are going to prove is the following:

Theorem 2. *For all $k \geq 2$, the numbers $C_{2^1-1}, C_{2^2-1}, \ldots, C_{2^{k-1}-1}$ all are distinct modulo 2^k, and modulo 2^k the sequence $(C_{2^n-1})_{n\geq 1}$ is constant from $k-1$ on.*

Here are a few historical references about the values of the C_n modulo 2^k. Deutsch and Sagan [2] first computed the 2-adic valuations of the Catalan numbers. Next S.-P. Eu, S.-C. Liu and Y.-N. Yeh [3] determined the modulo 8 values of the C_n. Then S.-C. Liu et J. C.-C. Yeh determined the modulo 64 values of the C_n by extending the method of Eu, Liu and Yeh in [3], in which they also stated Theorem 2 as a conjecture.

Our proof of Theorem 2 will be divided into two parts. In Section 2 we will begin with the case $k = 2$, and prove some lemmas which will be useful. In Section 3 we will treat the general case $k \geq 3$.

2. Odd Catalan Numbers Modulo 4

In this section we prove that any odd Catalan number is congruent to 1 modulo 4, which is Theorem 2 for $k = 2$. Though this result can be found in [3], we give a more "elementary" proof, in which we will also make some computations which will be used again in the sequel.

We start with two identities:

Lemma 3. *For any $a \geq 3$, the following identities hold:*

$$(2^a - 3)!! \equiv -1 \pmod{2^{a+1}}; \tag{1}$$

$$(2^a - 1)!! \equiv 1 \pmod{2^a}. \tag{2}$$

Proof. For the first identity, we reason by induction on a, the result being trivial when $a = 3$. So, let $a \geq 4$ and suppose the result holds for $a - 1$. First we have

$$(2^a - 3)!! = \prod_{k=1}^{2^{a-2}-1} (2k+1) \cdot \prod_{k=2^{a-2}}^{2^{a-1}-2} (2k+1).$$

Reversing the order of the indexes in the first product and translating the indexes in the second one, we get

$$\begin{aligned} (2^a - 3)!! &= \prod_{k=0}^{2^{a-2}-2} (2^{a-1} - (2k+1)) \cdot \prod_{k=0}^{2^{a-2}-2} (2^{a-1} + (2k+1)) \\ &= \prod_{k=0}^{2^{a-2}-2} [2^{2(a-1)} - (2k+1)^2] \\ &\equiv \prod_{k=0}^{2^{a-2}-2} [-(2k+1)^2] = -(2^{a-1} - 3)!!^2 \pmod{2^{a+1}}. \end{aligned}$$

By the induction hypothesis, $(2^{a-1} - 3)!!$ is equal to -1 or $2^a - 1$ modulo 2^{a+1}, and in either case the result follows.

We deduce from the first equality that necessarily, $(2^a - 3)!! \equiv -1 \pmod{2^a}$, so $(2^a - 1)!! \equiv (-1) \times (2^a - 1) \equiv 1 \pmod{2^a}$, whence the second equality. □

Lemma 4. *For $n = 2^a - 1$ with $a \geq 1$, we have*

$$o[(2n)!] = (2^{a+1} - 3)!! \prod_{i=1}^{a} (2^i - 1)!!; \tag{3}$$

$$o[(n+1)!] = o(n!) = \prod_{i=1}^{a} (2^i - 1)!!. \tag{4}$$

Proof. First, we have

$$\begin{aligned} o[(2n)!] &= o[2^n (2n-1)!! n!] = (2n-1)!! o(n!) \\ &= (2n-1)!! n \cdot o[(n-1)!] = (2^{a+1} - 3)!! (2^a - 1) o[(n-1)!], \end{aligned} \tag{5}$$

the penultimate equality being true because n is odd.

Therefore, since $n - 1 = 2(2^{a-1} - 1)$, we can iterate equation (5) until we get equation (3).

Using (3), the second equality can be proved as follows:

$$o[(n+1)!] = o(n!) = n \cdot o[(n-1)!] = n \cdot o[(2(2^{a-1} - 1))!] = \prod_{i=1}^{a} (2^i - 1)!!.$$

□

Now comes the main proposition of this section:

Proposition 5. *For all $a \geq 1$, $C_{2^a - 1} \equiv 1 \pmod 4$.*

Proof. Obviously this proposition is true for $a = 1, 2$; now we consider the case $a \geq 3$, to which we can apply Lemma 3. Since C_{2^a-1} is odd, by Lemma 4 we have

$$C_{2^a - 1} = \frac{o[(2n)!]}{o[(n+1)!] o(n!)} = \frac{(2^{a+1} - 3)!!}{\prod_{i=1}^{a} (2^i - 1)!!} = \frac{(2^{a+1} - 3)!!}{3 \times \prod_{i=3}^{a} (2^i - 1)!!}. \tag{6}$$

We remark that the resulting quotient is an integer. Since the denominator is odd, it is invertible modulo 4. Moreover the denominator and the numerator are all congruent to -1 by Lemma 3, whence $C_{2^a-1} \equiv 1 \pmod 4$. □

3. Proof of the General Case

To begin with, we prove that for all $k \geq 2$, the numbers $C_{2^1-1}, \ldots, C_{2^{k-1}-1}$ are distinct modulo 2^k.

Proposition 6. *Let $l \geq 2$ be an integer. For all $j \in \{1, \ldots, l-1\}$,*

$$C_{2^j - 1} \not\equiv C_{2^l - 1} \pmod{2^{l+1}}.$$

Proof. We prove this proposition by contradiction. Suppose there exists a $j \in \{1, \dots, l-1\}$ such that $C_{2^j-1} \equiv C_{2^l-1} \pmod{2^{l+1}}$. By equation (6), we have

$$\frac{(2^{j+1}-3)!!}{\prod_{i=1}^{j}(2^i-1)!!} \equiv \frac{(2^{l+1}-3)!!}{\prod_{i=1}^{l}(2^i-1)!!} \pmod{2^{l+1}}.$$

Since these two quotients are integers and their denominators are invertible modulo 4, we have by cross-multiplying

$$(2^{l+1}-3)!! \equiv (2^{j+1}-3)!! \prod_{i=j+1}^{l}(2^i-1)!! \pmod{2^{l+1}}. \tag{7}$$

By reducing equation (7) modulo 2^{j+2} and by Lemma 3, one would have

$$\begin{aligned} -1 &\equiv (2^{j+1}-3)!!(2^{j+1}-1)!! = (2^{j+1}-3)!!^2 \cdot (2^{j+1}-1) \\ &\equiv 2^{j+1}-1 \pmod{2^{j+2}}, \end{aligned}$$

which is absurd. □

Thanks to the previous proposition, we deduce easily the first claim of Theorem 2:

Corollary 7. *For $k \geq 2$, the numbers $C_{2^1-1}, C_{2^2-1}, \dots, C_{2^{k-1}-1}$ all are distinct modulo 2^k.*

To complete the proof of Theorem 2, it remains to prove that the C_{2^n-1} all are equal modulo 2^k for $n \geq k-1$.

Proposition 8. *Let $k \geq 2$, then for all $n \geq k-1$, $C_{2^n-1} \equiv C_{2^{k-1}-1} \pmod{2^k}$.*

Proof. By proposition 5, this proposition is true for $k = 2$; now we suppose $k \geq 3$. By equation (6), it suffices to show that for all $n \geq k-1$,

$$\frac{(2^{n+1}-3)!!}{\prod_{i=1}^{n}(2^i-1)!!} \equiv \frac{(2^k-3)!!}{\prod_{i=1}^{k-1}(2^i-1)!!} \pmod{2^k}.$$

Since these two quotients are all integers and their denominators are invertible modulo 4, it suffices to show that both

$$(2^k-3)!! \equiv (2^{n+1}-3)!! \pmod{2^k}$$

and

$$\prod_{i=1}^{k-1}(2^i-1)!! \equiv \prod_{i=1}^{n}(2^i-1)!! \pmod{2^k}.$$

Since $n+1 \geq k \geq 3$, we get these two equalities by Lemma 3. □

4. Going Further

Given the nice behavior of the odd Catalan numbers modulo 2^k, it is natural to wonder whether the even ones have similar properties. One approach might be to study the C_n having a given 2-adic valuation. More generally, one could consider residues modulo a prime power for other primes. See the article of Alter and Kubota [1] for results in that direction.

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