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MEAN-VALUE THEOREMS FOR MULTIPLICATIVE ARITHMETIC FUNCTIONS OF SEVERAL VARIABLES

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Abstract

Let $f : \mathbb{N}^n \to \mathbb{C}$ be an arithmetic function of n variables, where $n \geq 2$. We study the mean-value $M(f)$ of f that is defined to be

$$\lim_{x_1,\dots,x_n\to\infty} \frac{1}{x_1\cdots x_n} \sum_{m_1\leq x_1,\ \dots\ ,\ m_n\leq x_n} f(m_1,\ \dots\ ,\ m_n),$$

if this limit exists. We first generalize the Wintner theorem and then consider the multiplicative case by expressing the mean-value as an infinite product over all prime numbers. In addition, we study the mean-value of a function of the form $(m_1,\ m_2,\ \dots\ ,\ m_n) \mapsto g(\gcd(m_1,\ m_2,\ \dots\ ,\ m_n))$, where g is a multiplicative function of one variable, and express the mean-value by the Riemann zeta function.

1. Introduction

Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. The mean-value $M(f)$ of f is defined as $\lim_{x\to\infty} x^{-1}\sum_{m\leq x} f(m)$, if this limit exists. It is well-known that if $\sum_{m=1}^{\infty} m^{-1}|\sum_{d|m}\mu(d)f(m/d)| < \infty$, where μ is the Möbius function, then $M(f)$ exists and equals $\sum_{m=1}^{\infty} m^{-1}\sum_{d|m}\mu(d)f(m/d)$. This is Wintner's theorem. See, e.g., Schwarz and Spilker [4, Cor. 2.2]. Moreover, it is also well-known that if f is a multiplicative function satisfying $\sum_{p\in\mathcal{P}} p^{-1}|f(p)-1| < \infty$ and $\sum_{p\in\mathcal{P}}\sum_{k\geq 2} p^{-k}|f(p^k)| < \infty$, where $\mathcal{P}$ is the set of prime numbers, then $M(f)$ exists, and $M(f) = \prod_{p\in\mathcal{P}}(1 + \sum_{k\geq 1} p^{-k}(f(p^k) - f(p^{k-1})))$ holds (cf. Schwarz and Spilker [4, Cor. 2.3]).

We extended these theorems in [7] to the case in which $f : \mathbb{N}^2 \to \mathbb{C}$ is an arithmetic function of two variables. In this paper, we extend the aforementioned theorems to the case of an arithmetic function of n variables, where $n \geq 2$.

Toth [5] proved that the natural density of the set of n-tuples such that all pairs are coprime equals $\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{n-1}\left(1+\frac{n-1}{p}\right)$. We show in Corollary 6 that the natural density of the set of squarefree $(n-1)$-tuples such that all pairs are coprime has the same expression.

Ushiroya [7] also proved the following mean-value theorem. If g is a multiplicative function of one variable, then f defined by $f(m_1, m_2) = g(\gcd(m_1, m_2))$ is a multiplicative function of two variables. Assuming $\sum_{p\in\mathcal{P}}\sum_{k\geq 1}\frac{1}{p^{2k}}|g(p^k)-g(p^{k-1})| < \infty$, the mean-value $M(f) = \prod_{p\in\mathcal{P}}(1+\sum_{k\geq 1}\frac{1}{p^{2k}}(g(p^k)-g(p^{k-1})))$ exists. In this study, we extend this theorem to the case in which f is an arithmetic function of n variables of the form $f(m_1,\ m_2,\ \ldots\ ,\ m_n) = g(\gcd(m_1,\ m_2,\ \ldots\ ,\ m_n))$, where $n \geq 2$, and express the mean-value in terms of the Riemann zeta function.

Let S be an arbitrary set in $\mathbb{N}$ and $N_n(x, S) := \#\{(m_1,\ \ldots\ ,\ m_n) \in (\mathbb{N} \cap [1, x])^n; \gcd(m_1,\ \ldots\ ,\ m_n) \in S\}$. Cohen [2] proved that $N_n(x, S) = \dfrac{\zeta_S(n)}{\zeta(n)}x^n + T_n(x)$ holds, where $\zeta_S(n) = \sum_{m=1,\ m\in S}^{\infty}\frac{1}{m^n}$, $T_n(x) = O(x^{n-1})$ for $n > 2$, and $T_2(x) = O(x\log^2 x)$ for $n = 2$. See also [6]. From this result, it follows that the natural density of the set of n-tuples $(m_1,\ \ldots\ ,\ m_n)$ for which $\gcd(m_1,\ \ldots\ ,\ m_n)$ belongs to S equals $\lim_{x\to\infty}\dfrac{N_n(x,S)}{x^n} = \dfrac{\zeta_S(n)}{\zeta(n)}$. We note that when g is the characteristic function 1_S of S, we can obtain Cohen's result under the condition that 1_S is multiplicative by using a different method. Moreover, we present some examples, which were not treated in Cohen [2], in which g is not a characteristic function of a set in $\mathbb{N}$.

2. Notation and Some Facts

Let $n \geq 2$ be a fixed integer and $f, g : \mathbb{N}^n \to \mathbb{C}$ be arithmetic functions of n variables. The mean-value $M(f)$ of the function f is defined as

$$\lim_{x_1,\ \ldots\ ,\ x_n\to\infty}\frac{1}{x_1\cdots x_n}\sum_{m_1\leq x_1,\ \ldots\ ,\ m_n\leq x_n} f(m_1,\ \ldots\ ,\ m_n),$$

if this limit exists. Few results are known regarding the mean-values of general multiplicative functions of several variables. In this study, we investigate those mean-values by using elementary methods.

The Dirichlet convolution of f and g is defined as follows:

$$(f * g)(m_1, \ldots, m_n) = \sum_{\ell_1|m_1,\ldots,\ \ell_n|m_n} f(\ell_1, \ldots, \ell_n)g(\frac{m_1}{\ell_1}, \ldots, \frac{m_n}{\ell_n}).$$

We use the same notation μ for the function $\mu(m_1, \ldots, m_n) = \mu(m_1)\cdots\mu(m_n)$, which is the inverse of the constant 1 function under the Dirichlet convolution, i.e., $(\mu * 1)(m_1, \ldots, m_n) = \delta(m_1, \ldots, m_n)$, where $\delta(m_1, \ldots, m_n) = 1$ or 0 according to whether $m_1 = \ldots = m_n = 1$ or not.

We recall that a multiple series $\sum_{m_1,\ldots,m_n=1}^{\infty} a_{m_1,\ldots,m_n}$ with terms $a_{m_1,\ldots,m_n} \in \mathbb{C}$ is said to be convergent and to have as sum the number $A \in \mathbb{C}$ if

$$\lim_{M_1,\ldots,M_n\to\infty} \sum_{m_1\le M_1,\ldots,m_n\le M_n} a_{m_1,\ldots,m_n} = A,$$

i.e., for every $\varepsilon > 0$ there is a positive integer $M = M(\varepsilon)$ such that for every $M_1, \ldots, M_n \ge M$,

$$|\sum_{m_1\le M_1,\ldots,m_n\le M_n} a_{m_1,\ldots,m_n} - A| < \varepsilon.$$

In case of double series see, e.g., Section 4.7 in [1].

The next theorem is an extension of Wintner's theorem to the case in which f is an arithmetic function of n variables.

Theorem 1. *Let $f : \mathbb{N}^n \to \mathbb{C}$ be an arithmetic function of n variables. Suppose*

$$\sum_{m_1,\ \ldots\ ,\ m_n=1}^{\infty} \frac{1}{m_1\cdots m_n} |(f*\mu)(m_1,\ \ldots\ ,\ m_n)| < \infty. \tag{1}$$

Then, the mean-value $M(f)$ exists and

$$M(f) = \sum_{m_1,\ \ldots\ ,\ m_n=1}^{\infty} \frac{1}{m_1\cdots m_n} (f*\mu)(m_1,\ \ldots\ ,\ m_n). \tag{2}$$

Proof. Since $f = f*\delta = f*\mu*1$, we have

$$\begin{aligned}
&\sum_{m_1\le x_1,\ldots,m_n\le x_n} f(m_1,\ldots,m_n) = \sum_{m_1\le x_1,\ldots,m_n\le x_n} (f*\mu*1)(m_1,\ldots,m_n)\\
= &\sum_{m_1\le x_1,\ldots,m_n\le x_n} (f*\mu)(m_1,\ldots,m_n)[\frac{x_1}{m_1}]\cdots[\frac{x_n}{m_n}]\\
= &\sum_{m_1\le x_1,\ldots,m_n\le x_n} (f*\mu)(m_1,\ldots,m_n)\Big(\frac{x_1}{m_1}+O(1)\Big)\cdots\Big(\frac{x_n}{m_n}+O(1)\Big), \qquad (3)
\end{aligned}$$

where $[x]$ is the integer part of x. Then we have

$$\begin{aligned}
&\frac{1}{x_1\cdots x_n} \sum_{m_1\le x_1,\ldots,m_n\le x_n} f(m_1,\ldots,m_n)\\
&= \sum_{m_1\le x_1,\ldots,m_n\le x_n} \frac{(f*\mu)(m_1,\ldots,m_n)}{m_1\cdots m_n} + R_f(x_1,\ldots,x_n),
\end{aligned}$$

where

$$R_f(x_1,\ldots,x_n) \ll \sum_{u_1,\ldots,u_n} \sum_{m_1\le x_1,\ldots,m_n\le x_n} \frac{|(f*\mu)(m_1,\ldots,m_n)|}{m_1\cdots m_n} \Big(\frac{m_1}{x_1}\Big)^{u_1}\cdots\Big(\frac{m_n}{x_n}\Big)^{u_n},$$

and where the first sum is over $u_1, \ldots, u_n \in \{0,1\}$ such that at least one u_i is 1.

To complete the proof it is sufficient to show that $\lim_{x_1,\ldots,x_n\to\infty} R_f(x_1,\ldots,x_n) = 0$. To do this, fix some $u_1,\ldots,u_n \in \{0,1\}$, not all 0, and let $I = \{i; 1 \le i \le n, u_i = 1\} \neq \emptyset$. For every $\varepsilon_i > 0$ with $i \in I$,

$$\sum_{m_1\le x_1,\ldots,m_n\le x_n} \frac{|(f*\mu)(m_1,\ldots,m_n)|}{m_1\cdots m_n}\Big(\frac{m_1}{x_1}\Big)^{u_1}\cdots\Big(\frac{m_n}{x_n}\Big)^{u_n}$$

$$\le \prod_{i\in I}\varepsilon_i \sum_{\substack{m_i\le\varepsilon_i x_i \text{ for } i\in I\\ m_j\le x_j \text{ for } j\notin I}} \frac{|(f*\mu)(m_1,\ldots,m_n)|}{m_1\cdots m_n} + \sum_{\substack{m_1\le x_1,\ldots,m_n\le x_n\\ m_k>\varepsilon_k x_k \text{ for at least one } k\in I}} \frac{|(f*\mu)(m_1,\ldots,m_n)|}{m_1\cdots m_n}$$

$$\le \prod_{i\in I}\varepsilon_i \sum_{m_1,\ldots,m_n=1}^{\infty} \frac{|(f*\mu)(m_1,\ldots,m_n)|}{m_1\cdots m_n} + \sum_{\substack{m_1\le x_1,\ldots,m_n\le x_n\\ m_k>\varepsilon_k x_k \text{ for at least one } k\in I}} \frac{|(f*\mu)(m_1,\ldots,m_n)|}{m_1\cdots m_n}.$$

Here the first term is arbitrarily small (if the ε_i's are small) and the second term is also arbitrarily small if x_k is sufficiently large (using the definition of the convergence of multiple series). □

Next, we define the concept of the multiplicative function of n variables, which was given in Vaidyanathaswamy [8].

Definition 2. *Let $f : \mathbb{N}^n \to \mathbb{C}$ be an arithmetic function of n variables. We say that f is a multiplicative function of n variables if f satisfies*

$$f(\ell_1 m_1, \ \ldots \ , \ \ell_n m_n) = f(\ell_1, \ \ldots \ , \ \ell_n)\ f(m_1, \ \ldots \ , \ m_n)$$

for any $\ell_1,\ldots,\ell_n, m_1,\ldots,m_n \in \mathbb{N}$ satisfying $\gcd(\ell_1\cdots\ell_n,\ m_1\cdots m_n) = 1$.

It is known that if f and g are multiplicative functions of n variables, $f*g$ is also a multiplicative function of n variables.

Lemma 3. *Let $f : \mathbb{N}^n \to \mathbb{C}$ be a multiplicative function of n variables and $m_i = \prod_j p_j^{\ell_{ij}}$ for $1 \le i \le n$, where $p_j \in \mathcal{P}$ and $\ell_{ij} \ge 0$. Then,*

$$f(m_1, \ \ldots \ , \ m_n) = \prod_j f(p_j^{\ell_{1j}}, \ \ldots \ , \ p_j^{\ell_{nj}}).$$

Proof. Since $\gcd(p_1^{\ell_{11}}\cdots p_1^{\ell_{n1}},\ \prod_{j\ge2} p_j^{\ell_{1j}}\cdots\prod_{j\ge2} p_j^{\ell_{nj}}) = 1$, we have, by the multiplicativeness of f, that

$$f(m_1, \ \ldots \ , \ m_n) = f(\prod_j p_j^{\ell_{1j}}, \ \ldots \ , \ \prod_j p_j^{\ell_{nj}})$$

$$= f(p_1^{\ell_{11}}, \ \ldots \ , \ p_1^{\ell_{n1}})\ \ f(\prod_{j\ge2} p_j^{\ell_{1j}}, \ \ldots \ , \ \prod_{j\ge2} p_j^{\ell_{nj}}).$$

Now, Lemma 3 follows by induction on n. □

For $p_{j_1},\ p_{j_2},\ \dots,\ p_{j_n} \in \mathcal{P}$ and $\ell_1,\ \ell_2,\ \dots,\ \ell_n \in \mathbb{N} \cup \{0\}$ we set

$$\Delta_{\ell_1\ell_2\cdots\ell_n} f(p_{j_1},\, p_{j_2},\, \dots,\, p_{j_n}) := \sum_{e_1,\dots,e_n \in \{0,1\}} (-1)^{e_1+e_2+\cdots+e_n} f(p_{j_1}^{\ell_1-e_1},\, p_{j_2}^{\ell_2-e_2},\, \dots,\, p_{j_n}^{\ell_n-e_n}),$$

where we substitute $f(p_{j_1}^{\ell_1-e_1},\ p_{j_2}^{\ell_2-e_2},\ \dots,\ p_{j_n}^{\ell_n-e_n}) = 0$ if $\ell_i - e_i < 0$ for some $1 \le i \le n$. Clearly, $(f * \mu)(p_{j_1}^{\ell_1},\ p_{j_2}^{\ell_2},\ \dots,\ p_{j_n}^{\ell_n}) = \Delta_{\ell_1\ell_2\cdots\ell_n} f(p_{j_1},\ p_{j_2},\ \dots,\ p_{j_n})$ holds for any $\ell_1, \ell_2,\ \dots,\ \ell_n \ge 0$.

Theorem 4. *Let $f : \mathbb{N}^n \to \mathbb{C}$ be a multiplicative function of n variables satisfying*

$$\sum_{p \in \mathcal{P}} \sum_{\substack{\ell_1,\ \ell_2,\ \dots,\ \ell_n \ge 0 \\ \ell_1+\ell_2+\cdots+\ell_n \ge 1}} \frac{1}{p^{\ell_1+\ell_2+\cdots+\ell_n}} |\Delta_{\ell_1\ell_2\cdots\ell_n} f(p,\ p,\ \dots,\ p)| < \infty. \tag{4}$$

Then the mean-value $M(f)$ exists and

$$M(f) = \prod_{p \in \mathcal{P}} \Big(\sum_{\ell_1,\ \ell_2,\ \dots,\ \ell_n \ge 0} \frac{1}{p^{\ell_1+\ell_2+\cdots+\ell_n}} \Delta_{\ell_1\ell_2\cdots\ell_n} f(p,\ p,\ \dots,\ p) \Big). \tag{5}$$

Proof. Since the function : $(m_1,\ \dots,\ m_n) \mapsto \frac{1}{m_1\ \dots\ m_n} |f * \mu(m_1,\ \dots,\ m_n)|$ is a multiplicative function of n variables, according to Lemma 3, we have

$$\begin{aligned}
&\sum_{m_1 \le x_1,\ \dots,\ m_n \le x_n} \frac{1}{m_1 \cdots m_n} |(f * \mu)(m_1,\ \dots,\ m_n)| \\
&\le \sum_{\ell_1,\dots,\ell_n \ge 0} \Big(\prod_{p \in \mathcal{P}} \frac{1}{p^{\ell_1+\cdots+\ell_n}} |(f * \mu)(p^{\ell_1},\ \dots,\ p^{\ell_n})| \Big) \\
&= \sum_{\ell_1,\dots,\ell_n \ge 0} \Big(\prod_{p \in \mathcal{P}} \frac{1}{p^{\ell_1+\cdots+\ell_n}} |\Delta_{\ell_1\cdots\ell_n}(p,\ \dots,\ p)| \Big) \\
&\le \prod_{p \in \mathcal{P}} \Big(\sum_{\ell_1,\dots,\ell_n \ge 0} \frac{1}{p^{\ell_1+\cdots+\ell_n}} |\Delta_{\ell_1\cdots\ell_n} f(p,\ \dots,\ p)| \Big) \\
&= \prod_{p \in \mathcal{P}} \Big(1 + \sum_{\substack{\ell_1,\dots,\ell_n \ge 0 \\ \ell_1+\cdots+\ell_n \ge 1}} \frac{1}{p^{\ell_1+\cdots+\ell_n}} |\Delta_{\ell_1\cdots\ell_n} f(p,\ \dots,\ p)| \Big) \\
&\le \exp\Big(\sum_{p \in \mathcal{P}} (\sum_{\substack{\ell_1,\dots,\ell_n \ge 0 \\ \ell_1+\cdots+\ell_n \ge 1}} \frac{1}{p^{\ell_1+\cdots+\ell_n}} |\Delta_{\ell_1\cdots\ell_n} f(p,\ \dots,\ p)|) \Big) < \infty,
\end{aligned}$$

where we have used the inequality $1 + x \le exp(x)$ for $x > 0$. Therefore, according to Theorem 1, the mean-value $M(f)$ exists and clearly (5) holds. □

Example 5. *If $f(m_1,\ m_2,\ \dots\ ,\ m_n) = \mu^2(m_1 m_2 \cdots m_n)$, then f is a multiplicative function of n variables, and the mean-value $M(f)$ exists and*

$$M(f) \quad = \prod_{p\in\mathcal{P}} \Big(1-\frac{1}{p}\Big)^n \Big(1+\frac{n}{p}\Big). \tag{6}$$

Proof. Since $\mu^2(p^\ell)=0$ holds for any $\ell \ge 2$, we note that $\Delta_{\ell_1\cdots\ell_n} f(p,\dots,\ p) = \sum_{e_1,\dots,\ e_n\in\{0,1\}} (-1)^{e_1+\cdots+e_n}\mu^2(p^{(\ell_1-e_1)+\cdots+(\ell_n-e_n)})$ is 0 when $(\ell_1,\ \dots,\ \ell_n)$ is not a permutation of $(\underbrace{1,\dots,1}_{k},0,\dots,0)$ or $(2,\underbrace{1,\dots,1}_{k},0,\dots,0)$ for some $k\ge 0$. Observing that

$$\begin{aligned}
\Delta_{\underbrace{11\cdots 1}_{k}0\cdots 0} f(p,\dots,p) &= \sum_{e_1,\dots,\ e_k\in\{0,1\}} (-1)^{e_1+\cdots+e_k}\mu^2(p^{(1-e_1)+\cdots+(1-e_k)})\\
&= \sum_{e_1,\dots,\ e_k\in\{0,1\}} (-1)^{e_1+\cdots+e_k}\mu^2(p^{k-e_1-\cdots-e_k})\\
&= \mu^2(p^k) - \binom{k}{1}\mu^2(p^{k-1}) + \cdots\\
&\qquad\qquad + (-1)^{k-1}\binom{k}{k-1}\mu^2(p) + (-1)^k\\
&= (-1)^{k-1}k + (-1)^k = (-1)^k(-k+1),
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{2\underbrace{1\cdots 1}_{k}0\cdots 0} f(p,\dots,p) &= \sum_{e_1,\dots,\ e_{k+1}\in\{0,1\}} (-1)^{e_1+e_2+\cdots+e_{k+1}}\mu^2(p^{(2-e_1)+(1-e_2)+\cdots+(1-e_{k+1})})\\
&= (-1)^{k+1}\mu^2(p) = (-1)^{k+1},
\end{aligned}$$

as well as noting that the number of permutations of $(\underbrace{1,1,\dots,1}_{k},0,\ \dots,\ 0)$ is $\binom{n}{k}$ and the number of permutations of $(2,\underbrace{1,\dots,1}_{k},0,\dots,0)$ is $n\binom{n-1}{k}$ we have

$$\begin{aligned}
M(f) &= \prod_{p\in\mathcal{P}} \Big(\sum_{\ell_1,\ell_2,\dots,\ell_n\ge 0} \frac{1}{p^{\ell_1+\ell_2+\cdots+\ell_n}} \Delta_{\ell_1\ell_2\cdots\ell_n} f(p,\ p,\ \dots\ ,\ p)\Big)\\
&= \prod_{p\in\mathcal{P}} \Big(\sum_{k=0}^{n} \binom{n}{k}(-1)^k(-k+1)\frac{1}{p^k} + n\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{k+1}\frac{1}{p^{k+2}}\Big)\\
&= \prod_{p\in\mathcal{P}} \Big(-\sum_{k=0}^{n}\binom{n}{k}k\Big(-\frac{1}{p}\Big)^k + \sum_{k=0}^{n}\binom{n}{k}\Big(-\frac{1}{p}\Big)^k - \frac{n}{p^2}\sum_{k=0}^{n-1}\binom{n-1}{k}\Big(-\frac{1}{p}\Big)^k\Big).
\end{aligned}$$

Using the binomial theorem and the formula $\sum_{k=0}^{n}\binom{n}{k}kx^k = nx(1+x)^{n-1}$, we have

$$\begin{aligned} M(f) &= \prod_{p\in\mathcal{P}}\Big(-n\Big(-\frac{1}{p}\Big)\Big(1-\frac{1}{p}\Big)^{n-1}+\Big(1-\frac{1}{p}\Big)^{n}-\frac{n}{p^2}\Big(1-\frac{1}{p}\Big)^{n-1}\Big) \\ &= \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p}\Big)^{n-1}\Big(\frac{n}{p}+1-\frac{1}{p}-\frac{n}{p^2}\Big) = \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p}\Big)^{n}\Big(1+\frac{n}{p}\Big). \end{aligned}$$

□

Delange [3] proved that the set of all pairs of coprime positive integers that are squarefree posseses the natural density $\left(\frac{6}{\pi^2}\right)^2\prod_{p\in\mathcal{P}}\Big(1-\frac{1}{(p+1)^2}\Big)$, which can also be written as $\prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p}\Big)^2\Big(1+\frac{2}{p}\Big)$. Ushiroya [7] proved that if we set $f(m_1,m_2) = \mu^2(m_1m_2)$, f is a multiplicative function of two variables, and the mean-value $M(f)$ exists and equals $\prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p}\Big)^2\Big(1+\frac{2}{p}\Big)$. This is another proof of Delange's result. Since $\mu^2(m_1m_2\dots m_n)$ is the characteristic function of the set $\{(m_1,\ \dots,\ m_n)\in \mathbb{N}^n:\ m_i$ is squarefree and $\gcd(m_i,m_j)=1$ for any $1\le i\ne j\le n\}$, Example 5 is an extension of Delange's result to the case $n\ge 2$. On the other hand, Toth [5] proved that the natural density of the set where n positive integers are pairwise relatively prime equals $\prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p}\Big)^{n-1}\Big(1+\frac{n-1}{p}\Big)$. On the basis of Toth's result and Example 5, we have the following corollary.

Corollary 6. *The set* $\{(m_1,\ \dots,\ m_{n-1})\in\mathbb{N}^{n-1};\ m_i$ is squarefree and $\gcd(m_i,m_j)$ $=1$ for any $1\le i\ne j\le n-1\}$, *and the set* $\{(m_1,\ \dots,\ m_n)\in\mathbb{N}^n:\ \gcd(m_i,m_j)=$ 1 for any $1\le i\ne j\le n\}$ *have the same natural density of*

$$\prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p}\Big)^{n-1}\Big(1+\frac{n-1}{p}\Big).$$

Next, we treat the case in which a multiplicative function of n variables is a composite function of the gcd function and a multiplicative function of one variable.

Theorem 7. *Let* $g:\mathbb{N}\to\mathbb{C}$ *be a multiplicative function of one variable satisfying*

$$\sum_{p\in\mathcal{P}}\frac{|g(p)-1|}{p^n}<\infty \tag{7}$$

and

$$\sum_{p\in\mathcal{P}}\sum_{\ell\ge 2}\frac{|g(p^\ell)|}{p^{n\ell}}<\infty. \tag{8}$$

If we set $f(m_1,\ m_2,\ \dots\ ,\ m_n)=g(\gcd(m_1,\ m_2,\ \dots\ ,\ m_n))$, *then* f *is a multiplicative function of* n *variables, and* $M(f)$ *exists and*

$$M(f) = \frac{G(n)}{\zeta(n)}, \tag{9}$$

where $\zeta(n)$ *is the Riemann zeta function and* $G(n) = \sum_{m=1}^{\infty} \frac{g(m)}{m^n}$.

Proof. Clearly, $f(m_1, m_2, \ldots, m_n) = g(\gcd(m_1, m_2, \ldots, m_n))$ is a multiplicative function of n variables. Since $\Delta_{\ell_1 \ell_2 \cdots \ell_n} f(p, p, \ldots, p) \neq 0$ if and only $\ell_1 = \ell_2 = \cdots = \ell_n$, we need only consider $\Delta_{\ell\ell\cdots\ell} f(p, p, \ldots, p)$. Since

$$\begin{aligned}
&\Delta_{\ell\ell\cdots\ell} f(p, p, \ldots, p) \\
&\quad = \sum_{e_1, \ldots, e_n \in \{0,1\}} (-1)^{e_1+e_2+\cdots+e_n} f(p^{\ell-e_1}, p^{\ell-e_2}, \ldots, p^{\ell-e_n}) \\
&\quad = \sum_{e_1, \ldots, e_n \in \{0,1\}} (-1)^{e_1+e_2+\cdots+e_n} g(\gcd(p^{\ell-e_1}, p^{\ell-e_2}, \ldots, p^{\ell-e_n})) \\
&\quad = g(p^\ell) - \binom{n}{1} g(p^{\ell-1}) + \binom{n}{2} g(p^{\ell-1}) - \cdots + \binom{n}{n}(-1)^n g(p^{\ell-1}) \\
&\quad = g(p^\ell) - g(p^{\ell-1}) + g(p^{\ell-1}) \sum_{k=0}^{n} \binom{n}{k}(-1)^k = g(p^\ell) - g(p^{\ell-1}),
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{p \in \mathcal{P}} \sum_{\substack{\ell_1, \ldots, \ell_n \geq 0 \\ \ell_1 + \cdots + \ell_n \geq 1}} \frac{1}{p^{\ell_1+\ell_2+\cdots+\ell_n}} |\Delta_{\ell_1\ell_2\cdots\ell_n} f(p, p, \ldots, p)| \\
&\qquad = \sum_{p \in \mathcal{P}} \sum_{\ell \geq 1} \frac{1}{p^{n\ell}} |g(p^\ell) - g(p^{\ell-1})| \\
&\qquad = \sum_{p \in \mathcal{P}} \Big(\frac{|g(p) - 1|}{p^n} + \sum_{\ell \geq 2} \frac{|g(p^\ell) - g(p^{\ell-1})|}{p^{n\ell}} \Big).
\end{aligned}$$

The convergence of the series $\sum_{p \in \mathcal{P}} |g(p) - 1|/p^n$ follows from (7) and that of the series $\sum_{p \in \mathcal{P}} \sum_{\ell \geq 2} |g(p^\ell) - g(p^{\ell-1})|/p^{n\ell}$ follows from (7) and (8). Therefore, according to Theorem 4, $M(f)$ exists and equals

$$\begin{aligned}
\prod_{p\in\mathcal{P}}\Big(\sum_{\ell_1,\ \ell_2,\ \dots\ ,\ \ell_n\geq 0} & \frac{1}{p^{\ell_1+\ell_2+\cdots+\ell_n}}\Delta_{\ell_1\ell_2\cdots\ell_n}f(p,\ p,\ \dots\ ,\ p)\Big) \\
&= \prod_{p\in\mathcal{P}}\Big(1+\sum_{\ell\geq 1}\frac{1}{p^{n\ell}}(g(p^\ell)-g(p^{\ell-1}))\Big) \\
&= \prod_{p\in\mathcal{P}}\Big(1+\frac{g(p)-1}{p^n}+\frac{g(p^2)-g(p)}{p^{2n}}+\cdots\Big) \\
&= \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p^n}\Big)\Big(1+\frac{g(p)}{p^n}+\frac{g(p^2)}{p^{2n}}+\cdots\Big)=\frac{G(n)}{\zeta(n)}.
\end{aligned}$$

□

If g in Theorem 7 is a bounded function, (7) and (8) are obviously satisfied. Therefore, we have the following corollary.

Corollary 8. *If g in Theorem 7 satisfies $|g| \leq C$ for some $C > 0$, the mean-value $M(f)$ exists and (9) holds.*

The following corollary is a special case in [2].

Corollary 9 (Cohen [2]). *Let S be an arbitrary set in $\mathbb{N}$, where the characteristic function 1_S is multiplicative. Then, the natural density of the set of n-tuples $(m_1, \dots, m_n) \in \mathbb{N}$ such that $\gcd(m_1, \dots, m_n)$ is in S equals*

$$\frac{\zeta_S(n)}{\zeta(n)} = \frac{1}{\zeta(n)}\sum_{m=1,\ m\in S}^{\infty}\frac{1}{m^n}.$$

Although Cohen treated a more general case in which 1_S is not necessarily multiplicative, wherein Theorem 7 is not applicable, we can prove Corollary 9 by a method different from that of Cohen. Moreover, Theorem 7 is applicable to the case in which g is not a characteristic function.

When g is a multiplicative function such that G is well-known, we have a very simple expression for the mean-value. Several examples are shown below.

Example 10. *If $f(m_1,\ m_2,\ \dots\ ,\ m_n) = \mu(\gcd(m_1,\ m_2,\ \dots\ ,\ m_n))$, then f is a multiplicative function of n variables, and the mean-value $M(f)$ exists and*

$$M(f) \quad = \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p^n}\Big)^2 = \frac{1}{\zeta^2(n)}.$$

Proof. Since $g = \mu$ is a bounded function, the mean-value exists and (9) holds according to Corollary 8. It is easy to see that

$$M(f) = \frac{G(n)}{\zeta(n)} = \frac{1}{\zeta(n)}\sum_{m=1}^{\infty}\frac{\mu(m)}{m^n} = \frac{1}{\zeta^2(n)} \quad \text{with} \quad n \geq 2.$$

□

The proof of the following example is similar to that of the aforementioned example.

Example 11. *If* $f(m_1,\ m_2,\ \ldots\ ,\ m_n) = \mu^2(\gcd(m_1,\ m_2,\ \ldots\ ,\ m_n))$, *then* f *is a multiplicative function of* n *variables, and* $M(f)$ *exists and*

$$M(f) \;=\; \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p^{2n}}\Big) = \frac{1}{\zeta(2n)}.$$

We can also prove the following examples according to Theorem 7.

Example 12. *If* $f(m_1,\ m_2,\ \ldots\ ,\ m_n) = \sigma_\alpha(\gcd(m_1,\ m_2,\ \ldots\ ,\ m_n))$, *where* $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, *then* f *is a multiplicative function of* n *variables, and* $M(f)$ *exists if* $n > \alpha + 1$ *and*

$$M(f) \;=\; \prod_{p\in\mathcal{P}}\frac{1}{1-p^{\alpha-n}} = \zeta(n-\alpha).$$

Example 13. *If* $f(m_1,\ m_2,\ \ldots\ ,\ m_n) = \varphi(\gcd(m_1,\ m_2,\ \ldots\ ,\ m_n))$, *where* φ *is Euler's totient function, then* f *is a multiplicative function of* n *variables, and* $M(f)$ *exists if* $n \geq 3$ *and*

$$M(f) \;=\; \prod_{p\in\mathcal{P}}\frac{(1-\frac{1}{p^n})^2}{(1-\frac{1}{p^{n-1}})} = \frac{\zeta(n-1)}{\zeta(n)^2}.$$

Example 14. *If* $f(m_1,\ m_2,\ \ldots\ ,\ m_n) = K(\gcd(m_1,\ m_2,\ \ldots\ ,\ m_n))$, *where* $K(n) = \prod_{p|n} p$ *is the squarefree kernel of an integer* n, *then* f *is a multiplicative function of* n *variables, and* $M(f)$ *exists if* $n \geq 3$ *and*

$$M(f) \;=\; \prod_{p\in\mathcal{P}}(1+\frac{p-1}{p^n}) = \frac{1}{\zeta(n)}\prod_{p\in\mathcal{P}}\Big(1+\frac{p}{p^n-1}\Big).$$

Example 15. *If* $f(m_1,\ m_2,\ \ldots\ ,\ m_n) = (\gcd(m_1,\ m_2,\ \ldots\ ,\ m_n))^\alpha$, *then* f *is a multiplicative function of* n *variables, and* $M(f)$ *exists if* $n > \alpha + 1$ *and*

$$M(f) \;=\; \prod_{p\in\mathcal{P}}\frac{1-\frac{1}{p^n}}{1-\frac{1}{p^{n-\alpha}}} = \frac{\zeta(n-\alpha)}{\zeta(n)}.$$

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