DISTRIBUTION LAWS OF PAIRS OF DIVISORS

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Abstract

In this paper we study the distribution of pairs (d_1, d_2) of positive integers such that the product d_1d_2 divides a given integer n from a probabilistic point of view. The number of these pairs, denoted by $\tau_3(n)$, is equal to the number of ways to write n as a product of three positive integers. To these pairs we associate a random vector taking the values $((\log d_1)/(\log n), (\log d_2)/(\log n))$ with uniform probability $1/\tau_3(n)$ and its distribution function F_n. We show that the mean of F_n uniformly converges to the distribution function of the Beta two-dimensional law (Dirichlet law). Our study generalizes a work done by Deshouillers, Dress and Tenenbaum in the case of the divisors of an integer where they showed that the average distribution of divisors of a given integer follows the arcsine law.

1. Introduction

In order to study the distribution of divisors of a given integer n, Deshouillers, Dress and Tenenbaum [1], introduce the random variable D_n which takes the values $(\log d)/(\log n)$ as d runs through all divisors of n with uniform probability $1/\tau_2(n)$, where $\tau_2(n)$ is the number of divisors of n, and its distribution function $G_n(u) := \ Prob(D_n \leqslant u)$, $u \in [0, 1]$. The sequence $(G_n)_n$ does not converge pointwise in $[0, 1]$, they studied its mean value and showed that

$$\frac{1}{x}\sum_{n\leqslant x} G_n(u) = \frac{1}{x}\sum_{n\leqslant x} Prob(D_n \leqslant u) = \frac{2}{\pi}\arcsin(\sqrt{u}) + O\left(\frac{1}{\sqrt{\log x}}\right),$$

uniformly for $x \geqslant 2$ and $u \in [0, 1]$. Moreover, the order of the remainder term's magnitude is optimal if the uniformity in $[0, 1]$ is required. The method is based on the sums estimation $\Sigma_{n\leqslant x}1/\tau_2(kn)$; see Théorème T of [1] and also II.5 of [2]. In the present work we are interested in the distribution of pairs (d_1, d_2) of positive integers such that the product d_1d_2 divides n. The number of these pairs is equal to the number of ways to write n as a product of three positive integers, which will be denoted as $\tau_3(n)$. We consider the random vector

$$(X_n, Y_n) : \big\{(d_1, d_2) : d_1d_2|n\big\} \longrightarrow [0,1]\times[0,1],$$

which takes the values $(\,(\log d_1)/(\log n), (\log d_2)/(\log n)\,)$ with uniform probability equal to $1/\tau_3(n)$ and its distribution function, given by

$$F_n(u,v) := Prob(X_n \leqslant u, Y_n \leqslant v) = \frac{1}{\tau_3(n)} \sum_{qm|n,\, q\leqslant n^u,\, m\leqslant n^v} 1\,.$$

The sequence $(F_n)_n$ does not converge pointwise on $[0,1]\times[0,1]$, as can be easily seen by observing that for a fixed (u_0, v_0), $\frac{1}{3} < u_0 < \frac{2}{3}$ and $\frac{1}{3} < v_0 < \frac{2}{3}$, the subsequences $(F_p(u_0, v_0))_p$ and $(F_{p^3}(u_0, v_0))_{p^3}$ with p as a prime number, do not converge to the same limit. We will study the convegence of the mean of $(F_n)_n$:

$$\frac{1}{x}\sum_{n\leqslant x} F_n(u,v) = \frac{1}{x}\sum_{n\leqslant x} Prob(X_n \leqslant u, Y_n \leqslant v) = \frac{1}{x}\sum_{n\leqslant x}\frac{1}{\tau_3(n)} \sum_{qm|n,\, q\leqslant n^u,\, m\leqslant n^v} 1,$$

which gives the average distribution of solutions of the equation $xyz = n$ in integers $x \geqslant 1, y \geqslant 1, z \geqslant 1$. In the sequel, we will use the notation:

$$S(x; u, v) := \sum_{n\leqslant x}\frac{1}{\tau_3(n)} \sum_{qm|n,\, q\leqslant n^u,\, m\leqslant n^v} 1\,. \tag{1}$$

2. Statement of the Theorem

Denote by Γ the Euler gamma function and for $a, b \in]0, +\infty[$ let

$$B(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}dy = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\cdot$$

Set

$$T_1 = \{(u,v) \in [0,1]\times[0,1] : u+v < 1\}; T_2 = \{(u,v) \in [0,1]\times[0,1] : u+v \geqslant 1\}\,.$$

The following theorem shows that the mean of the distribution function defined above uniformly converges in T_1 to the distribution function of the Beta two-dimensional law which has parameters $1/3, 1/3, 1/3$ and uniformly converges in T_2 to a sum of distribution functions of the Beta-dimensional laws which has parameters $2/3, 1/3$.

Theorem 2.1. *1.Uniformly for $x > 1$ and $(u,v) \in T_1$, we have*

$$\frac{1}{x}\sum_{n\leqslant x} Prob(X_n \leqslant u, Y_n \leqslant v) = \frac{1}{\Gamma^3(\frac{1}{3})}\int_0^u\int_0^v y^{-\frac{2}{3}}z^{-\frac{2}{3}}(1-y-z)^{-\frac{2}{3}}dydz + O\left(\frac{1}{\sqrt[3]{\log x}}\right)\cdot$$

2.Uniformly for $x > 1$ and $(u,v) \in T_2$, we have

$$\frac{1}{x}\sum_{n\leqslant x} Prob(X_n \leqslant u, Y_n \leqslant v) = -1 + \frac{1}{B(\frac{2}{3},\frac{1}{3})}\int_0^u y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}\,dy + \frac{1}{B(\frac{2}{3},\frac{1}{3})}\int_0^v y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}\,dy + O\left(\frac{1}{\sqrt[3]{\log x}}\right)\cdot$$

Remark 2.2. The remainder term $O\left(\frac{1}{\sqrt[3]{\log x}}\right)$ in Theorem 2.1 is optimal if uniformity in (u,v) is required. Indeed, by using partial summation and lemma 3.1 below, we can show that for $0 \leqslant v < (\log 2)/(\log x)$

$$\frac{1}{x}\sum_{n\leqslant x} Prob(X_n \leqslant \frac{1}{2}, Y_n \leqslant v) \sim \frac{c_2\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\frac{1}{\sqrt[3]{\log x}}, \quad (x \to +\infty),$$

where c_2 is a constant defined in (4) below.
We also note that the transition from the first formula to the second in Theorem 2.1 is regular. Indeed, we can show that for (u,v) such $u+v=1$,

$$\begin{aligned} -1 & + \frac{1}{B(\frac{2}{3},\frac{1}{3})}\int_0^u y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}\,dy + \frac{1}{B(\frac{2}{3},\frac{1}{3})}\int_0^{1-u} y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}\,dy \\ = & \frac{1}{\Gamma^3(\frac{1}{3})}\int_0^u\int_0^{1-u} y^{-\frac{2}{3}}z^{-\frac{2}{3}}(1-y-z)^{-\frac{2}{3}}dydz \end{aligned}$$

For $x \geqslant 2$, we set

$$\epsilon_x := \frac{\log 2}{\log x}, \quad \epsilon''_x := (\frac{\log 2}{\log x})^\eta, \tag{2}$$

where $0 < \eta < 1/3$ is an arbitrary fixed number.

$$\begin{aligned} T_x &:= \{(u,v) \in [0,1]\times[0,1] : u+v \leqslant 1-\epsilon''_x\}, \\ \overline{T}_x &:= \{(u,v) \in [0,1]\times[0,1] : 1-\epsilon''_x < u+v < 1\}. \end{aligned} \tag{3}$$

For technical reasons, we divide the proof of Theorem 2.1 into two parts. In the first part we prove the first formula for (u,v) in T_x and in the second one we prove the same formula for (u,v) in $\overline{T}_x$ and we also prove the second formula for (u,v) in T_2. However, the two parts use the same ideas. In Section 3, we will give some necessary lemmas. In Section 4, we will give full proof of Theorem 2.1 for (u,v) in T_x. In Section 5, to avoid repetitions, we will just describe the proof of Theorem 2.1 for (u,v) in $\overline{T}_x$ and for (u,v) in T_2 without details. All notations introduced here will be retained throughout the rest of the article.

3. Lemmas

We introduce two multiplicative functions h_1 et h_2 that will be used in the sequel. For a prime power they are defined by

$$h_1(p^r) = \left(\sum_{j\geqslant 0} \frac{1}{p^j \tau_3(p^{j+r})}\right)\left(\sum_{j\geqslant 0} \frac{1}{p^j \tau_3(p^j)}\right)^{-1}$$

and

$$h_2(p^r) = \left(\sum_{j\geqslant 0} \frac{h_1(p^{j+r})}{p^j}\right)\left(\sum_{j\geqslant 0} \frac{h_1(p^j)}{p^j}\right)^{-1}.$$

We also set

$$c_1 := \prod_p (1-\frac{1}{p})^{\frac{1}{3}} \sum_{j\geqslant 0} \frac{p^{-j}}{C_{j+2}^2};\; c_2 := \prod_p (1-\frac{1}{p})^{\frac{1}{3}} \sum_{j\geqslant 0} \frac{h_1(p^j)}{p^j};$$
$$c_3 := \prod_p (1-\frac{1}{p})^{\frac{1}{3}} \sum_{j\geqslant 0} \frac{h_2(p^j)}{p^j}\cdot \tag{4}$$

Lemma 3.1. *The following hold:*

1. *For every $\theta \in]0,+\infty[$ and every integer $d \geqslant 1$, there is a positive constant $M_{\theta,d} := (1/3+\theta)^{\omega(d)}$, where $\omega(d)$ is the number of prime divisors of d, such that uniformly for any real number $x \geqslant 2$ and any integer $d \geqslant 1$ we have*

$$\sum_{n\leqslant x} \frac{1}{\tau_3(dn)} = \frac{c_1 h_1(d)}{\Gamma(\frac{1}{3})} \frac{x}{\log^{\frac{2}{3}}(x)} + O\left(\frac{M_{\theta,d}\, x}{\log^{\frac{5}{3}}(x)}\right);$$
$$\sum_{n\leqslant x} h_1(dn) = \frac{c_2 h_2(d)}{\Gamma(\frac{1}{3})} \frac{x}{\log^{\frac{2}{3}}(x)} + O\left(\frac{M_{\theta,d}\, x}{\log^{\frac{5}{3}}(x)}\right)\cdot$$

2. *For every $\theta \in]0,+\infty[$, and $i = 1,2$, we uniformly have for $x \geqslant 2$,*

$$\sum_{n\leqslant x} h_2(n) = \frac{c_3}{\Gamma(\frac{1}{3})} \frac{x}{\log^{\frac{2}{3}}(x)} + O\left(\frac{x}{\log^{\frac{5}{3}}(x)}\right); \sum_{n\leqslant x} \frac{h_i(n)}{n} = O\left(\log^{\frac{1}{3}}(x)\right);$$
$$\sum_{n\leqslant x} \frac{M_{\theta,n}}{n} = O\left(\log^{\frac{1}{3}+\theta}(x)\right)\cdot$$

Proof. The lemma is an immediate consequence of Théorème T of [1]. □

Lemma 3.2. *The following two equalities hold:*
1. for any $x \in]0,1[$, $\sum_{r\geqslant 0}\sum_{j\geqslant 0}\sum_{\ell\geqslant 0} \frac{x^{r+j+\ell}}{C_{r+j+\ell+2}^{r+j+\ell}} = \frac{1}{1-x}$;
2. $c_1c_2c_3 = 1$.

Proof. 1. We clearly have

$$\frac{x^{r+j+\ell}}{C_{r+j+\ell+2}^{r+j+\ell}} = \frac{2x^{r+j+\ell}}{r+j+\ell+1} - \frac{2x^{r+j+\ell}}{r+j+\ell+2} = \frac{2}{x}\int_0^x t^{r+j+\ell}dt - \frac{2}{x^2}\int_0^x t^{r+j+\ell+1}dt\,.$$

Then

$$\sum_{r\geqslant 0}\sum_{j\geqslant 0}\sum_{\ell\geqslant 0}\frac{x^{r+j+\ell}}{C_{r+j+\ell+2}^{r+j+\ell}} = \frac{2}{x}\int_0^x \frac{1}{(1-t)^3}dt - \frac{2}{x^2}\int_0^x \frac{t}{(1-t)^3}dt = \frac{1}{1-x}\cdot$$

2. From definitions, we have

$$c_1c_2c_3 = \prod_p \left(1-\tfrac{1}{p}\right)\left(\sum_{r\geqslant 0} p^{-r}\sum_{j\geqslant 0}\tfrac{p^{-j}}{C_{j+r+2}^{j+r}}\right)\left(\sum_{j\geqslant 0} p^{-j}\sum_{\ell\geqslant 0}\tfrac{p^{-\ell}}{C_{j+\ell+2}^{j+\ell}}\right)^{-1}\times$$

$$\left(\sum_{r\geqslant 0}\sum_{j\geqslant 0}\sum_{\ell\geqslant 0}\tfrac{p^{-r-j-\ell}}{C_{r+j+\ell+2}^{r+j+\ell}}\right) = \prod_p\left(1-\tfrac{1}{p}\right)\left(\sum_{r\geqslant 0}\sum_{j\geqslant 0}\sum_{\ell\geqslant 0}\tfrac{(p^{-r-j-\ell}}{C_{r+j+\ell+2}^{r+j+\ell}}\right),$$

which immediately yields $c_1c_2c_3 = 1$ by the formula proved in 1. □

Lemma 3.3. *1. For $x \geqslant 2$, let $\epsilon_x := (\log 2)/(\log x)$ and $(u,v) \in [0,1]^2$ be such that $\epsilon_x \leqslant u+v \leqslant 1-\epsilon_x$. For $2 \leqslant t \leqslant x^u$, set*

$$I(t,x,v) := \int_{\epsilon_x}^v z^{-\frac{2}{3}}(1-\frac{\log t}{\log x}-z)^{-\frac{2}{3}}dz,\ J(x,u,v) := \int_2^{x^u}(\log t)^{-\frac{2}{3}}\frac{\partial}{\partial t}I(t,x,v)\,dt\,.$$

Then, we uniformly have $J(x,u,v) = O(1)$.

2. Let

$$I(t,x,v) := \int_v^{1-\epsilon_x-\frac{\log t}{\log x}} z^{-\frac{2}{3}}(1-\frac{\log t}{\log x}-z)^{-\frac{2}{3}}dz.$$

Uniformly for $\epsilon_x \leqslant v \leqslant 1$ and $x \geqslant 2$, we have

$$J_1(x,v) := \int_2^{\frac{x}{2}} \log^{-5/3}(t)\,I(t,x,v)\frac{dt}{t} = O(1),$$

and

$$J(x,v) := \int_2^{\frac{x}{2}} \log^{-2/3}(t)\,\frac{\partial}{\partial t}I(x,t,v)\,dt = O(1).$$

Proof. 1. We have $\frac{\partial}{\partial t}I(t,x,v) = \frac{2}{3t\log x}\int_{\epsilon_x}^v z^{-\frac{2}{3}}(1-\frac{\log t}{\log x}-z)^{-\frac{5}{3}}dz$. By a change of variable $y = \log t/\log x$, we obtain $J(x,u,v) = \frac{2}{3(\log x)^{2/3}}\int_{\epsilon_x}^u\int_{\epsilon_x}^v y^{-\frac{2}{3}}z^{-2/3}(1-y-z)^{-\frac{5}{3}}dzdy$. When $y\to 0$ (resp. $y \to 1$), we have $z \to 0$ or $z \to 1$ (resp. $z \to 0$), as $\epsilon_x \leqslant u+v \leqslant 1-\epsilon_x$. The integrand is therefore equivalent to $y^{-\frac{2}{3}}z^{-\frac{2}{3}}$ or to $y^{-\frac{2}{3}}(1-z)^{-5/3}$ (resp. to $z^{-\frac{2}{3}}(1-y)^{-5/3}$). An easy calculation yields $J(x,u,v) = O(1)$.
2. The proof is similar to 1. □

Lemma 3.4. *1. For $x \geqslant 2$, let $\epsilon''_x := (\log 2)^\eta/(\log x)^\eta$, $0 < \eta < 1/3$. For $u \in [\epsilon_x, 1-\epsilon''_x]$, $v \in [\sqrt{\epsilon_x}, 1-\epsilon''_x]$, $u+v \leqslant 1-\epsilon''_x$, and $2 \leqslant q \leqslant x^u$, we uniformly have*

$$\widehat{I}(x,v) := \int_{x^{\sqrt{\epsilon_x}}}^{x^v} \log^{-\frac{5}{3}}(t)\,\log^{-\frac{2}{3}}(\frac{x/q}{t})\,\frac{dt}{t} = O\left((\log x)^{-1+\frac{4}{3}\eta}\right),$$

and

$$\widehat{\widehat{I}}(x,v) := \int_{x^{\sqrt{\epsilon_x}}}^{x^v} \log^{-\frac{2}{3}}(t)\,\log^{-\frac{5}{3}}(\frac{x/q}{t})\,\frac{dt}{t} = O\left((\log x)^{-1+\frac{4}{3}\eta}\right).$$

2. For $\epsilon_x \leqslant u \leqslant 1-\epsilon''_x$, $2 \leqslant m \leqslant x^{\sqrt{\epsilon_x}}$ and $u+v \leqslant 1-\epsilon''_x$, we uniformly have

$$\overline{I}(x,u) := \int_2^{x^u} \log^{-\frac{5}{3}}(t)\,\log^{-\frac{2}{3}}(\frac{x/m}{t})\,\frac{dt}{t} = O\left((\log x)^{-\frac{2}{3}+\frac{2}{3}\eta}\right),$$

and

$$\overline{\overline{I}}(x,u) := \int_2^{x^u} \log^{-\frac{2}{5}}(t)\,\log^{-\frac{5}{3}}(\frac{x/m}{t})\,\frac{dt}{t} = O\left((\log x)^{-\frac{2}{3}+\frac{2}{3}\eta}\right).$$

3. For $\epsilon_x \leqslant v \leqslant 1-\epsilon''_x$ and $2 \leqslant q \leqslant \sqrt{x}$, we uniformly have

$$\int_{x^v}^{\frac{x}{2q}} \log^{-\frac{2}{3}+\theta}(t)\log^{-\frac{5}{3}}(x/qt)\frac{dt}{t} = O\left((\log x)^{-\frac{2}{3}+\theta}\right).$$

4. For $\epsilon_x \leqslant v \leqslant 1-\epsilon_x$ and $2 \leqslant q \leqslant x^{1-v}$, we uniformly have

$$\int_{x^v}^{x/2q} \log^{-5/3}(t)\,\log^{-2/3}(\frac{x/q}{t})\,\frac{dt}{t} = O\left((\log x)^{-\frac{4}{3}+\frac{4}{3}\eta}\right).$$

Proof. The proofs of the four statements are similar. Let us prove the first one. We write

$$\widehat{I}(x,v) = \log^{-2/3}(x/q)\int_{x^{\sqrt{\epsilon_x}}}^{x^v} \log^{-5/3}(t)\,(1-\frac{\log t}{\log(x/q)})^{-2/3}\,\frac{dt}{t},$$

and by the change of variable $y = 1-\frac{\log t}{\log(x/q)}$, we get

$$\begin{aligned}
\widehat{I}(x,v) &= \log^{-4/3}(x/q)\int_{1-\frac{v\log x}{\log(x/q)}}^{1-\frac{\sqrt{\epsilon_x}\log x}{\log(x/q)}} y^{-2/3}(1-y)^{-5/3}\,dy \\
&\leqslant \log^{-4/3}(x/q)(1-\frac{v\log x}{\log(x/q)})^{-2/3}\int_{1-\frac{v\log x}{\log(x/q)}}^{1-\frac{\sqrt{\epsilon_x}\log x}{\log(x/q)}} (1-y)^{-5/3}\,dy \\
&\leqslant \tfrac{3}{2}\log^{-4/3}(x/q)(\tfrac{1-u-v}{1-u})^{-2/3}(\tfrac{\sqrt{\epsilon_x}\log x}{\log(x/2)})^{-2/3} \ll \log^{-1+(4/3)\eta}(x).
\end{aligned}$$

□

4. Proof of Theorem 2.1 for (u,v) in T_x

Recall that the notation $\overline{T}_x$ has been introduced in (3), ϵ_x and ϵ''_x in (2) and $S(x;u,v)$ in (1). First, we note that Theorem 2.1 is obvious for x bounded. From now on we suppose that x is sufficently large. We divide T_x into two zones: $[0,\epsilon_x]\times[0,1]\cup[0,1]\times[0,\epsilon_x]$, and $\mathcal{D}_0 := \left\{(u,v)\in[\epsilon_x, 1-\epsilon''_x]^2, u+v\leqslant 1-\epsilon''_x\right\}$. In the first zone, we show that $S(x,u,v)$ has the same order of magnitude as the remainder term (see Lemma 4.1 below). In order to study the sum $S(x;u,v)$ in the second zone, we decompose it as follows: $S(x,u,v) = S_1(x,u,v) - S_2(x,u,v) - S_3(x,u,v) - S_4(x,u,v)$, with

$$S_1(x,u,v) := \sum_{\substack{q\leqslant x^u, m\leqslant x^v\\ d\leqslant \frac{x}{qm}}} \frac{1}{\tau_3(qmd)}; \quad S_2(x,u,v) := \sum_{\substack{n^u\leqslant q\leqslant x^u, n^v<m\leqslant x^v\\ n=qmd\leqslant x}} \frac{1}{\tau_3(qmd)};$$

$$S_3(x,u,v) := \sum_{\substack{q\leqslant n^u, n^v<m\leqslant x^v\\ n=dmq\leqslant x}} \frac{1}{\tau_3(qmd)}; \quad S_4(x,u,v) := \sum_{\substack{n^u\leqslant q\leqslant x^u, m\leqslant n^v\\ n=qmd\leqslant x}} \frac{1}{\tau_3(qmd)}\cdot$$

We then show that $S_2(x,u,v)$, $S_3(x,u,v)$ and $S_4(x,u,v)$ have the same order of magnitude as the remainder term (see Lemma 4.2 below) and that $S_1(x,u,v)$ provides the main term (see Lemma 4.3 below).

Lemma 4.1. *Uniformly for $x\geqslant 2$ and $(u,v)\in[0,1]\times[0,\epsilon_x[\cup[0,\epsilon_x[\times[0,1]$, we have*

$$S(x,u,v) = O\left(\frac{x}{\sqrt[3]{\log x}}\right)\cdot$$

Proof. By symmetry, it suffices to prove the lemma for $(u,v)\in[0,1]\times[0,\epsilon_x[$. We have

$$S(x,u,v)\leqslant \sum_{q\leqslant x^u}\sum_{d\leqslant x/q}\frac{1}{\tau_3(dq)}\leqslant \sum_{q\leqslant x}\sum_{d\leqslant x/q}\frac{1}{\tau_3(dq)}\cdot$$

The condition $dq\leqslant x$ implies that $d\leqslant\sqrt{x}$ or $q\leqslant\sqrt{x}$, we clearly have

$$\sum_{q\leqslant x}\sum_{d\leqslant x/q}\frac{1}{\tau_3(dq)} \ll \sum_{q\leqslant\sqrt{x}}\sum_{d\leqslant x/q}\frac{1}{\tau_3(dq)} =: L\,.$$

Applying Lemma 3.1 and observing that $q\leqslant\sqrt{x}$, we get

$$L\ll x\log^{-2/3}(x)\sum_{q\leqslant\sqrt{x}}\frac{h_1(q)}{q}\ll\frac{x}{\sqrt[3]{\log x}}\cdot$$

□

Lemma 4.2. *Let $\epsilon_x = \log 2/\log x$. Uniformly for $x\geqslant 2$ and $(u,v)\in[\epsilon_x,1]^2$, we have*

$$S_i := S_i(x,u,v) = O\left(x/\sqrt[3]{\log x}\right), \quad (i=2,3,4)\cdot$$

Proof. Let $M_2 := \min(M_3, M_4)$, with $M_3 = \frac{x^{1-v}}{q}$, $M_4 = \frac{x^{1-u}}{m}$ and

$$\widetilde{S_3} = \widetilde{S_3}(x,u,v) := \sum_{\substack{q\leqslant x^u, m\leqslant x^v\\ d\leqslant M_3}} \frac{1}{\tau_3(qmd)}, \quad \widetilde{S_4} = \widetilde{S_4}(x,u,v) := \sum_{\substack{q\leqslant x^u, m\leqslant x^v\\ d\leqslant M_4}} \frac{1}{\tau_3(qmd)}\cdot$$

We clearly have

$$S_2 \leqslant \sum_{\substack{q\leqslant x^u, m\leqslant x^v\\ d\leqslant M_2}} \frac{1}{\tau_3(qmd)} \leqslant \min(\widetilde{S_3}, \widetilde{S_4}),\ S_3 \leqslant \widetilde{S_3},\ S_4 \leqslant \widetilde{S_4},\ \widetilde{S_3}(x,u,v) = \widetilde{S_4}(x,v,u)\cdot$$

Then to prove the lemma, it suffices to prove that $\widetilde{S_4} = O\left(x/\sqrt[3]{\log x}\right)$ uniformly in $[\epsilon_x, 1]^2$. Set $\epsilon'_x := \sqrt{\log 2/\log x}$. We will first give the estimate for $(u,v) \in [\epsilon_x, \epsilon'_x]^2$. By Lemma 3.1, we get

$$\begin{aligned}\widetilde{S_4} &\ll x^{1-u} \textstyle\sum_{q\leqslant x^u, m\leqslant x^v} \frac{h_1(qm)}{m} \log^{-2/3}(\frac{x^{1-u}}{m})\\ &\ll x(u(1-u-v))^{-2/3}\log^{-4/3}(x) \textstyle\sum_{m\leqslant x^v} \frac{h_2(m)}{m} \ll \frac{x}{\sqrt[3]{\log x}}\cdot\end{aligned}$$

Let us now estimate $\widetilde{S_4}$ for (u,v) in $[\epsilon'_x, 1] \times [\epsilon_x, 1]$. We distinguish two cases:
1st case, suppose that $u+v \leqslant 1-\epsilon'_x$. Lemma 3.1 gives

$$\widetilde{S_4} \ll x(u(1-u-v))^{-2/3}\log^{-4/3}(x) \sum_{m\leqslant x^v} \frac{h_2(m)}{m} \ll \frac{x}{\sqrt[3]{\log x}},$$

as $u^{-2/3} \leqslant (\epsilon'_x)^{-2/3} \ll \log^{1/3}(x)$ and $(1-u-v)^{-2/3} \ll (\epsilon'_x)^{-2/3} \ll \log^{1/3}(x)$.
2nd case, suppose that $u+v > 1-\epsilon'_x$. Then, we can write $\widetilde{S_4} = T_1 + T_2$, with

$$T_1 := \sum_{m\leqslant x^{1-u-\epsilon'_x}, q\leqslant x^u, d\leqslant M_4} \frac{1}{\tau_3(qmd)},\ T_2 := \sum_{x^{1-u-\epsilon'_x} < m\leqslant x^v, q\leqslant x^u, d\leqslant M_4} \frac{1}{\tau_3(qmd)}\cdot$$

Consider T_1. By Lemma 3.1 we obtain, as before,

$$\begin{aligned}T_1 &\ll \textstyle\sum_{m\leqslant x^{1-u-\epsilon'_x}, q\leqslant x^u} x^{1-u}\frac{h_1(qm)}{m}\log^{-2/3}(x^{1-u}/m)\\ &\ll xu^{-2/3}\log^{-2/3}(x)(\epsilon'_x)^{-2/3}\log^{-2/3}(x)\log^{1/3}(x) \ll \frac{x}{\sqrt[3]{\log x}}\cdot\end{aligned}$$

Consider now T_2. The condition $d \leqslant M_4$ that is $dm \leqslant x^{1-u}$ implies that $d \leqslant \sqrt{x^{1-u}}$ or $m \leqslant \sqrt{x^{1-u}}$. We therefore have, by symmetry, $T_2 \leqslant T_3 + 2\,T_4$ with

$$T_3 := \sum_{q\leqslant x^u, m\leqslant\sqrt{x^{1-u}}, d\leqslant\sqrt{x^{1-u}}} \frac{1}{\tau_3(qmd)},\ T_4 := \sum_{q\leqslant x^u, m\leqslant\sqrt{x^{1-u}}, \sqrt{x^{1-u}}<d\leqslant\frac{x^{1-u}}{m}} \frac{1}{\tau_3(qmd)}\cdot$$

In order to evaluate T_3 and T_4 we consider three cases:
1st case. We suppose that $\epsilon'_x \leqslant u \leqslant 1-\epsilon'_x$. By Lemma 3.1, we obtain

$$T_3 \ll x^{\frac{1-u}{2}} x^u x^{\frac{1-u}{2}} \log^{-2/3}(x^{\frac{1-u}{2}})\log^{-2/3}(x^u)\log^{-2/3}(x^{\frac{1-u}{2}}) \ll \frac{x}{\log x},$$

and

$$T_4 \ll x^{1-u} \log^{-2/3}(x^{\frac{1-u}{2}}) x^u \log^{-2/3}(x^u) \sum_{m \leqslant \sqrt{x^{1-u}}} \frac{h_2(m)}{m} \ll \frac{x}{\sqrt[3]{\log x}}.$$

2nd case. We suppose that $1-\epsilon'_x < u \leqslant 1-\epsilon_x$. We obtain

$$T_3 \ll x^{\frac{1-u}{2}} x^u x^{\frac{1-u}{2}} \log^{-2/3}(x^{\frac{1-u}{2}}) \log^{-2/3}(x^u) \log^{-2/3}(x^{\frac{1-u}{2}}) \ll \frac{x}{(\log x)^{2/3}},$$

and

$$T_4 \ll x^{1-u} \log^{-2/3}(x^{\frac{1-u}{2}}) x^u \log^{-2/3}(x^u) \sum_{m \leqslant \sqrt{x^{1-u}}} \frac{h_2(m)}{m} \ll \frac{x}{\sqrt[3]{\log x}}.$$

3rd case. We suppose that $1-\epsilon_x < u \leqslant 1$. We have, $1 \leqslant d \leqslant x^{\frac{1-u}{2}} < x^{\frac{\epsilon_x}{2}} = \sqrt{2}$ then $d=1$. Similarly $m=1$. Hence

$$T_3 = T_4 = \sum_{q \leqslant x} \frac{1}{\tau_3(q)} \ll x(\log x)^{-2/3}.$$

To complete the proof, it remains to estimate $\widetilde{S_4}$ for (u,v) in $[\epsilon_x, \epsilon'_x] \times [\epsilon'_x, 1]$. Using the notations T_3 and T_4 above, we have $\widetilde{S_4} \leqslant T_3 + 2\,T_4$ and by Lemma 3.1, we easily see that $T_3 \ll x/\log^{4/3} x$ and $T_4 \ll x/\sqrt[3]{\log x}$ for (u,v) in $[\epsilon_x, \epsilon'_x] \times [\epsilon'_x, 1]$. □

Lemma 4.3. *Uniformly for $(u,v) \in \mathcal{D}_0$ and $x \geqslant 2$, we have*

$$S_1 = \frac{x}{\Gamma^3(\frac{1}{3})} \int_0^u y^{-2/3}(1-y)^{-1/3} \int_0^{\frac{v}{1-y}} t^{-2/3}(1-t)^{-2/3}\, dt\, dy + O\left(\frac{x}{\sqrt[3]{\log x}}\right).$$

Proof. Write the decomposition $S_1 = S_{1,1} + S_{1,2}$ with

$$S_{1,1} := \sum_{q \leqslant x^u} \sum_{x^{\sqrt{\epsilon_x}} < m \leqslant x^v} \sum_{d \leqslant \frac{x}{qm}} \frac{1}{\tau_3(qmd)}, \quad S_{1,2} := \sum_{q \leqslant x^u} \sum_{m \leqslant x^{\sqrt{\epsilon_x}}} \sum_{d \leqslant \frac{x}{qm}} \frac{1}{\tau_3(qmd)}.$$

We will estimate these two quantities.

`1. Estimation of` $S_{1,1}$. The application of Lemma 3.1(1) gives

$$\sum_{d \leqslant \frac{x}{qm}} \frac{1}{\tau_3(qmd)} = x \frac{c_1}{\Gamma(\frac{1}{3})} \frac{h_1(qm)}{qm} \log^{-\frac{2}{3}}(\frac{x}{qm}) + O\left(M_{qm,\theta} \frac{x}{qm} \log^{-\frac{5}{3}}(\frac{x}{qm})\right). \qquad (5)$$

Consider the remainder term. Recall that $M_{qm,\theta} = (\frac{1}{3}+\theta)^{\omega(qm)}$, $\theta > 0$. Hence

$$\sum_{q \leqslant x^u} \sum_{x^{\sqrt{\epsilon_x}} < m \leqslant x^v} \frac{M_{qm,\theta}}{qm} \leqslant \sum_{q \leqslant x^u} \frac{(\frac{1}{3}+\theta)^{\omega(q)}}{q} \sum_{m \leqslant x^v} \frac{(\frac{1}{3}+\theta)^{\omega(m)}}{m} \ll \log^{2(\theta+\frac{1}{3})}(x),$$

by Lemma 3.1(2) and $\log^{-5/3}(x/qm) \leqslant \log^{-5/3}(x^{1-u-v}) \leqslant \log^{-5/3}(x^{\epsilon''_x}) \ll \log^{-5/3+(5/3)\eta}(x)$. From the choices $0 < \theta < (1-(5/2)\eta)/3$ and $\eta < 2/5$, it follows that

$$O\left(x \sum_{q \leqslant x^u} \sum_{x^{\sqrt{\epsilon_x}} < m \leqslant x^v} \frac{M_{qm,\theta}}{qm} \log^{-\frac{5}{3}}(\frac{x}{qm})\right) = O\left(\frac{x}{\sqrt[3]{\log x}}\right). \tag{6}$$

Now, we consider the main term in (5). By partial summation and Lemma 3.1(1) we have

$$\begin{aligned} \sum_{x^{\sqrt{\epsilon_x}} < m \leqslant x^v} \frac{h_1(qm)}{m} \log^{-\frac{2}{3}}(\frac{x}{qm}) &= -\int_{x^{\sqrt{\epsilon_x}}}^{x^v} \left(\sum_{m \leqslant t} h_1(qm)\right) d(\frac{\log^{-\frac{2}{3}}(\frac{x/q}{t})}{t}) \\ &+ O\left(\log^{-1+\frac{2}{3}\eta}(x)\, h_2(q)\right). \end{aligned} \tag{7}$$

Taking the sum over q, and using Lemma 3.1(2), we see that the remainder term is

$$O\left(\log^{-1+\frac{2}{3}\eta}(x) \sum_{q \leqslant x^u} \frac{h_2(q)}{q}\right) = O\left(\frac{1}{\sqrt[3]{\log x}}\right). \tag{8}$$

Now, consider the first term in the second member of (7). By Lemma 3.1(1), we get

$$\frac{c_2 h_2(q)}{\Gamma(\frac{1}{3})} \int_{x^{\sqrt{\epsilon_x}}}^{x^v} \log^{-2/3}(t) \log^{-2/3}(\frac{x/q}{t}) \frac{dt}{t} + O(h_2(q)\widehat{\widehat{I}}(x,v)) + O(M_{q,\theta'}\widehat{I}(x,v)), \tag{9}$$

where $\widehat{I}(x,v)$ and $\widehat{\widehat{I}}(x,v)$ are defined in Lemma 3.4 (1). Summing over q and using Lemma 3.4 (1) and Lemma 3.1 (2) in each of the two remainder terms, we get

$$\begin{aligned} &O\left((\log x)^{-1+\frac{4}{3}\eta} \sum_{q \leqslant x^u} \frac{M_{q,\theta'}}{q}\right) = O\left(\frac{1}{\sqrt[3]{\log x}}\right); \\ &O\left(\widehat{I}(x,v) \sum_{q \leqslant x^u} \frac{h_2(q)}{q}\right) = O\left(\frac{1}{\sqrt[3]{\log x}}\right). \end{aligned} \tag{10}$$

In the first term of (9), by the changes of de variables $y = \frac{\log t}{\log x/q}$ and $z = y(1-\frac{\log q}{\log x})$, and summation over q, we get

$$\frac{c_2}{\Gamma(\frac{1}{3})} \log^{-\frac{1}{3}}(x) \sum_{q \leqslant x^u} \frac{h_2(q)}{q} \int_{\sqrt{\epsilon_x}}^{v} z^{-\frac{2}{3}}(1 - \frac{\log q}{\log x} - z)^{-\frac{2}{3}} dz. \tag{11}$$

It remains to study the quantity

$$\begin{aligned} K &:= \sum_{q \leqslant x^u} \frac{h_2(q)}{q} \int_{\sqrt{\epsilon_x}}^{v} z^{-\frac{2}{3}}(1 - \frac{\log q}{\log x} - z)^{-\frac{2}{3}} dz \\ &= \sum_{2 \leqslant q \leqslant x^u} \frac{h_2(q)}{q} \int_{\sqrt{\epsilon_x}}^{v} z^{-\frac{2}{3}}(1 - \frac{\log q}{\log x} - z)^{-\frac{2}{3}} dz + O(1). \end{aligned} \tag{12}$$

By partial summation, Lemma 3.1 (2) and Lemma 3.3 (1) with the notations therein, we get

$$\begin{aligned} K &= -\int_2^{x^u}\left(\sum_{q\leqslant t} h_2(q)\right) d\left(\frac{1}{t}\int_{\sqrt{\epsilon_x}}^{v} z^{-\frac{2}{3}}(1-\tfrac{\log t}{\log x}-z)^{-\frac{2}{3}}dz\right) + O(1)\\ &= \frac{c_3}{\Gamma(\frac{1}{3})}\int_2^{x^u}\log^{-\frac{2}{3}} t\left(\frac{I(t,x,v)}{t}-\frac{\partial}{\partial t}I(t,x,v)\right)dt + O(1)\\ &= \frac{c_3}{\Gamma(\frac{1}{3})}\int_2^{x^u}\log^{-\frac{2}{3}}(t)\,\frac{I(t,x,v)}{t}\,dt + O\left(J(x,u,v)\right)) + O(1)\\ &= \frac{c_3}{\Gamma(\frac{1}{3})}\int_2^{x^u}\log^{-\frac{2}{3}}(t)\,\frac{I(t,x,v)}{t}\,dt + O(1)\cdot \end{aligned}$$

Now, the changes of variables $y=\frac{\log t}{\log x}$, and $t=\frac{z}{1-y}$ yield

$$\begin{aligned} K &= \frac{c_3}{\Gamma(\frac{1}{3})}\log^{1/3}(x)\int_{\epsilon_x}^{u} y^{-2/3}(1-y)^{-1/3}\int_{\frac{\sqrt{\epsilon_x}}{1-y}}^{\frac{v}{1-y}} t^{-2/3}(1-t)^{-2/3}\,dt\,dy\\ &\quad +O(1). \end{aligned} \tag{13}$$

Finally by collecting successively (5), (6),(7), (8), (9), (10), (11), (12), (13) and using the equality $c_1c_2c_3=1$ established in Lemma 3.2 (2) we get

$$S_{1,1}=\frac{x}{\Gamma^3(\frac{1}{3})}\int_{\epsilon_x}^{u} y^{-2/3}(1-y)^{-1/3}\int_{\frac{\sqrt{\epsilon_x}}{1-y}}^{\frac{v}{1-y}} t^{-2/3}(1-t)^{-2/3}\,dt\,dy + O\left(\frac{x}{\sqrt[3]{\log x}}\right). \tag{14}$$

2. Estimation of $S_{1,2}$. We invert the first two sums that we estimate:

$$S_{1,2}=\sum_{m\leqslant x^{\sqrt{\epsilon_x}}}\sum_{q\leqslant x^u}\sum_{d\leqslant\frac{x}{qm}}\frac{1}{\tau_3(qmd)}\cdot$$

The method described above works. To study various remainder terms, we use Lemma 3.4 (2). We obtain in each case $O\left(x/\sqrt[3]{\log x}\right)$. we obtain

$$\begin{aligned} S_{1,2} &= \frac{x}{\Gamma^3(\frac{1}{3})}\int_{\epsilon_x}^{\sqrt{\epsilon_x}} y^{-2/3}(1-y)^{-1/3}\int_{\frac{\sqrt{\epsilon_x}}{1-y}}^{\frac{u}{1-y}} t^{-2/3}(1-t)^{-2/3}\,dt\,dy + O\left(\frac{x}{\sqrt[3]{\log x}}\right)\\ &= \frac{x}{\Gamma^3(\frac{1}{3})}\int_{\sqrt{\epsilon_x}}^{u} y^{-2/3}(1-y)^{-1/3}\int_{\frac{\epsilon_x}{1-y}}^{\frac{\sqrt{\epsilon_x}}{1-y}} t^{-2/3}(1-t)^{-2/3}\,dt\,dy + O\left(\frac{x}{\sqrt[3]{\log x}}\right)\cdot \end{aligned} \tag{15}$$

The last equality is obtained by a change of variables.

Finally, we obtain the required formula for $S_1=S_{1,1}+S_{1,2}$ from (14) and (15). This completes the proof of Lemma 4.3. □

5. Description of the Proof of Theorem 2.1 for (u,v) in $\overline{T}_x$ and for (u,v) in T_2

Recall that the notation $\overline{T}_x$ has been introduced in (3), ϵ_x and ϵ''_x in (2) and $S(x;u,v)$ in (1). Also $T_2=\{(u,v)\in[0,1]\times[0,1]: u+v\geqslant 1\}$. In $\overline{T}_x\cup T_2$, we will use the expression of $S(x,u,v)$ given in the following lemma which reduces the estimate of $S(x,u,v)$ to that of the three quantities Σ_i, $1 \leqslant i \leqslant 3$, with

$$\Sigma_1 = \sum_{2\leqslant q\leqslant x; x^v<m\leqslant x; dmq\leqslant x} \frac{1}{\tau_3(dmq)}; \quad \Sigma_2 = \sum_{x^u<q\leqslant x; 2\leqslant m\leqslant x; dmq\leqslant x} \frac{1}{\tau_3(dmq)};$$

$$\Sigma_3 = \sum_{x^u<q\leqslant x; x^v<m\leqslant x; dmq\leqslant x} \frac{1}{\tau_3(dmq)}\cdot$$

Lemma 5.1. *For every $(u,v) \in [\epsilon_x, 1]^2$ and $x \geqslant 2$, we uniformly have*

$$S(x,u,v) = [x] - \Sigma_1 - \Sigma_2 + \Sigma_3 + O\left(\frac{x}{\sqrt[3]{\log x}}\right)\cdot$$

By following the proof of Theorem 2.1 in T_x, we can prove the following two lemmas:

Lemma 5.2. *Uniformly for $\epsilon_x \leqslant v \leqslant 1$, $\epsilon_x \leqslant u \leqslant 1$, and $x \geqslant 2$, we have,*

$$\begin{aligned}\Sigma_1 &= \tfrac{x}{\Gamma^3(\frac{1}{3})}\int_0^{1-v} y^{-\frac{2}{3}}(1-y)^{-\frac{1}{3}}\int_{\frac{v}{1-y}}^1 s^{-\frac{2}{3}}(1-s)^{-\frac{2}{3}}\,dy\,ds + O\left(\tfrac{x}{\sqrt[3]{\log x}}\right)\\ &= x - \tfrac{x}{B(\frac{2}{3},\frac{1}{3})}\int_0^v y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}\,dy + O\left(\tfrac{x}{\sqrt[3]{\log x}}\right),\end{aligned}$$

and

$$\begin{aligned}\Sigma_2 &= \tfrac{x}{\Gamma^3(\frac{1}{3})}\int_0^{1-u} y^{-\frac{2}{3}}(1-y)^{-\frac{1}{3}}\int_{\frac{u}{1-y}}^1 s^{-\frac{2}{3}}(1-s)^{-\frac{2}{3}}\,dy\,ds + O\left(\tfrac{x}{\sqrt[3]{\log x}}\right)\\ &= x - \tfrac{x}{B(\frac{2}{3},\frac{1}{3})}\int_0^u y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}\,dy + O\left(\tfrac{x}{\sqrt[3]{\log x}}\right)\cdot\end{aligned}$$

Remark 5.3. We trivially have $\Sigma_3 = 0$ if $u + v \geqslant 1$. So, the formula of Theorem 2.1 for $(u,v) \in T_2$ results from the estimates of Σ_1 and Σ_2, given in Lemma 5.2.

Lemma 5.4. *Uniformly for $(u,v) \in \overline{T}_x$, with $u + v < 1$ and $x \geqslant 2$, we have*

$$\begin{aligned}\Sigma_3 &= x - \tfrac{x}{B(\frac{2}{3},\frac{1}{3})}\int_0^u y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}\,dy - \tfrac{x}{B(\frac{2}{3},\frac{1}{3})}\int_0^v y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}\,dy\\ &+ \tfrac{x}{\Gamma^3(\frac{1}{3})}\int_0^u y^{-2/3}(1-y)^{-1/3}\int_0^{\frac{v}{1-y}} t^{-2/3}(1-t)^{-2/3}\,dt\,dy + O\left(\tfrac{x}{\sqrt[3]{\log x}}\right)\cdot\end{aligned}$$

The Proof of Theorem 2.1 in $\overline{T}_x$ and in T_2 results from Lemmas 5.2 and 5.3 and Remark 5.1. Note that by a change of variable $z = (1-y)t$, we have

$$\begin{aligned}&\frac{1}{\Gamma^3(\frac{1}{3})}\int_0^u y^{-2/3}(1-y)^{-1/3}\int_0^{\frac{v}{1-y}} t^{-2/3}(1-t)^{-2/3}\,dt\,dy\\ &= \frac{1}{\Gamma^3(\frac{1}{3})}\int_0^u\int_0^v y^{-\frac{2}{3}}z^{-\frac{2}{3}}(1-y-z)^{-\frac{2}{3}}\,dy\,dz.\end{aligned}$$

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