SOME WEIGHTED SUMS OF PRODUCTS OF LUCAS SEQUENCES

Emrah Kılıç
Mathematics Department, TOBB University of Economics and Technology, Ankara, Turkey
ekilic@etu.edu.tr

Neşe Ömür
Mathematics Department, Kocaeli University, İzmit Kocaeli, Turkey
neseomur@kocaeli.edu.tr

Received: 8/16/12, Accepted: 4/21/13, Published: 5/10/13

Abstract

In this paper, we consider the weighted sums of products of Lucas sequences of the form

$$\sum_{k=0}^{n} \binom{n}{k} r_{mk} s_{m(tn+k)},$$

where r_n and s_n are the terms of Lucas sequences $\{U_n\}$ and $\{V_n\}$ for some positive integers t and m. By using generating function methods, we compute the weighted sums of products of Lucas sequences and show that these sums could be expressed via terms of the Lucas sequences.

1. Introduction

Define second order linear recurrences $\{U_n\}$ and $\{V_n\}$ for $n > 0$ as

$$\begin{aligned} U_n &= pU_{n-1} + U_{n-2}, \\ V_n &= pV_{n-1} + V_{n-2}, \end{aligned}$$

where $U_0 = 0,\ U_1 = 1$ and $V_0 = 2,\ V_1 = p$, respectively. If $p = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

The Binet formulas of $\{U_n\}$ and $\{V_n\}$ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = \left(p \pm \sqrt{\Delta}\right)/2$ and $\Delta = p^2 + 4$.

Let $A(x)$ and $B(x)$ be the exponential generating functions of sequences $\{a_n\}$ and $\{b_n\}$, that is,

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!} \text{ and } B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}.$$

Then the convolution of them is given by

$$A(x)B(x) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{x^n}{n!}.$$

Many authors have considered and computed many kinds of binomial sums as well as weighted binomial sums with terms of certain number sequences. As a consequence of convolution of two exponential generating functions, we have the following results from the literature (see [2]):

$$\sum_{i=0}^{n} \binom{n}{i} F_{mi} L_{mn-mi}, \quad \sum_{i=0}^{n} \binom{n}{i} F_{mi} F_{mn-mi}, \tag{1.1}$$

$$\sum_{i=0}^{n} \binom{n}{i} L_{mi} F_{mn-mi}, \quad \sum_{i=0}^{n} \binom{n}{i} L_{mi} L_{mn-mi}. \tag{1.2}$$

In this paper, motivated by (1.1) and (1.2), we consider the following new four kinds of weighted binomial sums with the product of terms of the sequences $\{U_n\}$ and $\{V_n\}$:

$$\sum_{i=0}^{n} \binom{n}{i} U_{mi} V_{m(kn+i)}, \quad \sum_{i=0}^{n} \binom{n}{i} V_{mi} U_{m(kn+i)},$$
$$\sum_{i=0}^{n} \binom{n}{i} U_{mi} U_{m(kn+i)}, \quad \sum_{i=0}^{n} \binom{n}{i} V_{mi} V_{m(kn+i)},$$

for some integers k and m. We consider the sums above and then show that the sums could nicely be expressed in terms of the terms of the sequences $\{U_n\}$ and $\{V_n\}$. Because of the indices of the terms in the sums, the convolution of exponential generating functions can not be used for computing these sums. Our approach for computing these kind of sums is mainly to use generating function methods and the Binet formula of the sequences. For computing weighted binomial sums with the product of terms of binary sequences and using generating functions in deriving combinatorial identities, we refer to [1, 3].

2. The Main Results

First we give a useful auxiliary lemma and its direct consequences. After this we give our main results. By the Binet formulas of $\{U_n\}$ and $\{V_n\}$, we have the following results without proof:

Lemma 1. *For odd m,*

$$1+\alpha^{2m}=\alpha^m U_m\sqrt{\Delta} \text{ and } \left(1+\beta^{2m}\right)=-\beta^m U_m\sqrt{\Delta},$$

and for even m,

$$1+\alpha^{2m}=\alpha^m V_m \text{ and } 1+\beta^{2m}=\beta^m V_m.$$

As straightforward consequences of Lemma 1, we have the following results.

Corollary 1. *Let m be a nonnegative odd integer. Then for $n>0$,*

$$\sum_{i=0}^{n}\binom{n}{i}U_{2mi}=U_m^n\begin{cases}\Delta^{\frac{n-1}{2}}V_{mn} & \text{if } n \text{ is odd,}\\ \Delta^{\frac{n}{2}}U_{mn} & \text{if } n \text{ is even.}\end{cases}$$

Proof. By Lemma 1, we write for odd n,

$$\left(1+\alpha^{2m}\right)^n=\alpha^{mn}U_m^n\Delta^{\frac{n}{2}} \text{ and } \left(1+\beta^{2m}\right)^n=-\beta^{mn}U_m^n\Delta^{\frac{n}{2}}$$

or

$$\sum_{i=0}^{n}\binom{n}{i}\alpha^{2mi}=\alpha^{mn}U_m^n\left(p^2+4\right)^{\frac{n}{2}} \text{ and } \sum_{i=0}^{n}\binom{n}{i}\beta^{2mi}=-\beta^{mn}U_m^n\left(p^2+4\right)^{\frac{n}{2}}$$

and so

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\left(\frac{\alpha^{2mi}-\beta^{2mi}}{\alpha-\beta}\right) &= \Delta^{\frac{n-1}{2}}U_m^n\left(\alpha^{mn}+\beta^{mn}\right)\\ \sum_{i=0}^{n}\binom{n}{i}U_{2mi} &= \Delta^{\frac{n-1}{2}}U_m^nV_{mn},\end{aligned}$$

as claimed. For even n, consider

$$\left(1+\alpha^{2m}\right)^n=\alpha^{mn}U_m^n\Delta^{\frac{n}{2}} \text{and } \left(1+\beta^{2m}\right)^n=\beta^{mn}U_m^n\Delta^{\frac{n}{2}}$$

or

$$\sum_{i=0}^{n}\binom{n}{i}\alpha^{2mi}=\alpha^{mn}U_m^n\Delta^{\frac{n}{2}} \text{ and } \sum_{i=0}^{n}\binom{n}{i}\beta^{2mi}=\beta^{mn}U_m^n\Delta^{\frac{n}{2}}$$

and so

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\left(\alpha^{2mi}-\beta^{2mi}\right) &= \Delta^{\frac{n}{2}}U_m^n\left(\alpha^{mn}-\beta^{mn}\right)\\ \sum_{i=0}^{n}\binom{n}{i}U_{2mi} &= \Delta^{\frac{n}{2}}U_m^nU_{mn},\end{aligned}$$

as claimed. $\square$

For example, for odd $m > 0$ and $n > 0$, we have

$$\sum_{i=0}^{n} \binom{n}{i} F_{2mi} = F_m^n \begin{cases} 5^{\frac{n-1}{2}} L_{mn} & \text{if } n \text{ is odd,} \\ 5^{\frac{n}{2}} F_{mn} & \text{if } n \text{ is even,} \end{cases}$$

which can be found in [4].

Corollary 2. *Let m be a nonnegative even integer. Then for $n > 0$,*

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} U_{2mi} &= V_m^n U_{mn}, \\ \sum_{i=0}^{n} \binom{n}{i} V_{2mi} &= V_m^n V_{mn}. \end{aligned}$$

Proof. By Lemma 1, we have that for even m,

$$1 + \alpha^{2m} = \alpha^m V_m \text{ and } 1 + \beta^{2m} = \beta^m V_m$$

and write

$$\left(1 + \alpha^{2m}\right)^n = \alpha^{mn} V_m^n \text{ and } \left(1 + \beta^{2m}\right)^n = \beta^{mn} V_m^n,$$

which, by the binomial theorem, gives us

$$\sum_{i=0}^{n} \binom{n}{i} \alpha^{2mi} = \alpha^{mn} V_m^n \text{ and } \sum_{i=0}^{n} \binom{n}{i} \beta^{2mi} = \beta^{mn} V_m^n.$$

By subtracting these two equalities side by side and the Binet formula of $\{U_n\}$, we obtain

$$\sum_{i=0}^{n} \binom{n}{i} U_{2mi} = V_m^n U_{mn}.$$

By adding the above two equalities and the Binet formula of $\{V_n\}$, we obtain

$$\sum_{i=0}^{n} \binom{n}{i} V_{2mi} = V_m^n V_{mn},$$

as claimed. □

For even $m > 0$ and $n > 0$, we obtain

$$\sum_{i=0}^{n} \binom{n}{i} F_{2mi} = L_m^n F_{mn} \quad \text{and} \quad \sum_{i=0}^{n} \binom{n}{i} L_{2mi} = L_m^n L_{mn}.$$

Theorem 1. *Let k be a nonnegative integer. For odd m,*

$$\sum_{i=0}^{n}\binom{n}{i}U_{mi}V_{kmn+mi}=\Delta^{\lfloor\frac{n}{2}\rfloor}U_m^n\begin{cases}U_{(k+1)mn} & \text{if } n \text{ is even,}\\ V_{(k+1)mn} & \text{if } n \text{ is odd.}\end{cases}\tag{2.1}$$

For even m,

$$\sum_{i=0}^{n}\binom{n}{i}U_{mi}V_{kmn+mi}=V_m^nU_{mn(k+1)}-2^nU_{kmn}.$$

Proof. Multiplying the left-hand side of (2.1) by z^n and summing over n and by the Binet formulas of $\{U_n\}$ and $\{V_n\}$, we derive for odd m,

$$\begin{aligned}
&\sum_{n\geq 0}z^n\sum_{i=0}^{n}\binom{n}{i}U_{mi}V_{kmn+mi}\\
=\ &\frac{1}{\alpha-\beta}\sum_{n\geq 0}z^n\sum_{i=0}^{n}\binom{n}{i}\left(\left(\alpha^{kmn+2mi}-\beta^{kmn+2mi}\right)-\left(\alpha^{kmn}-\beta^{kmn}\right)(-1)^i\right)\\
=\ &\frac{1}{\alpha-\beta}\left(\sum_{i\geq 0}\alpha^{2mi}\sum_{n\geq 0}\binom{n}{i}\left(\alpha^{km}z\right)^n-\sum_{i\geq 0}\beta^{2mi}\sum_{n\geq 0}\binom{n}{i}\left(\beta^{km}z\right)^n\right)\\
&-\frac{1}{\alpha-\beta}\sum_{n\geq 0}\sum_{i=0}^{n}\binom{n}{i}\left(\alpha^{kmn}-\beta^{kmn}\right)(-1)^{mi}z^n\\
=\ &\frac{1}{\alpha-\beta}\sum_{i\geq 0}\left(\frac{\left(\alpha^{m(k+2)}z\right)^i}{\left(1-\left(\alpha^{km}z\right)\right)^{i+1}}-\frac{\left(\beta^{m(k+2)}z\right)^i}{\left(1-\left(\beta^{km}z\right)\right)^{i+1}}\right)\\
&-\frac{1}{\alpha-\beta}\left(\sum_{i\geq 0}\frac{\left((-1)^m\alpha^{km}z\right)^i}{\left(1-\left(\alpha^{km}z\right)\right)^{i+1}}-\sum_{i\geq 0}\frac{\left((-1)^m\beta^{km}z\right)^i}{\left(1-\left(\beta^{km}z\right)\right)^{i+1}}\right)\\
=\ &\frac{1}{\alpha-\beta}\left(\frac{1}{\left(1-\alpha^{km}z\right)}\frac{1}{1-\frac{\alpha^{m(k+2)}z}{1-\alpha^{km}z}}-\frac{1}{\left(1-\beta^{km}z\right)}\frac{1}{1-\frac{\beta^{m(k+2)}z}{1-\beta^{km}z}}\right)\\
&-\frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha^{km}\left(1+(-1)^m\right)z}-\frac{1}{1-\beta^{km}\left(1+(-1)^m\right)z}\right)\\
=\ &\frac{1}{\alpha-\beta}\left(\frac{1}{1-z\alpha^{km}\left(1+\alpha^{2m}\right)}-\frac{1}{1-z\beta^{km}\left(1+\beta^{2m}\right)}\right)-\frac{1}{\alpha-\beta}(1-1)\\
=\ &\frac{1}{\alpha-\beta}\sum_{n\geq 0}\left(\alpha^{kmn}\left(1+\alpha^{2m}\right)^n-\beta^{kmn}\left(1+\beta^{2m}\right)^n\right)z^n.
\end{aligned}$$

Therefore, we get the identity

$$\sum_{i=0}^{n}\binom{n}{i}U_{mi}V_{kmn+mi}=\sum_{i=0}^{n}\binom{n}{i}U_{2mi+kmn}.$$

Using Lemma 1, we write for even n,

$$\begin{aligned}
& \frac{1}{\alpha-\beta}\left(\alpha^{kmn}\left(1+\alpha^{2m}\right)^n-\beta^{kmn}\left(1+\beta^{2m}\right)^n\right) \\
= & \frac{1}{\alpha-\beta}\left(\alpha^{kmn}\left(\alpha^m U_m\sqrt{\Delta}\right)^n-\beta^{kmn}\left(-\beta^m U_m\sqrt{\Delta}\right)^n\right) \\
= & \frac{1}{\alpha-\beta}\left(\alpha^{kmn}\alpha^{mn}U_m^n\Delta^{\frac{n}{2}}-\beta^{kmn}\beta^{mn}U_m^n\Delta^{\frac{n}{2}}\right) \\
= & U_m^n\Delta^{\frac{n}{2}}U_{(k+1)mn}.
\end{aligned}$$

On the other hand, we get for odd n

$$\begin{aligned}
& \frac{1}{\alpha-\beta}\left(\alpha^{kmn}\left(1+\alpha^{2m}\right)^n-\beta^{kmn}\left(1+\beta^{2m}\right)^n\right) \\
= & \frac{1}{\alpha-\beta}\left(\alpha^{kmn}\left(\alpha^m U_m\sqrt{\Delta}\right)^n-\beta^{kmn}\left(-\beta^m U_m\sqrt{\Delta}\right)^n\right) \\
= & \frac{1}{\sqrt{p^2+4}}\left(\alpha^{kmn}\alpha^{mn}U_m^n\Delta^{\frac{n}{2}}+\beta^{kmn}\beta^{mn}U_m^n\Delta^{\frac{n}{2}}\right) \\
= & \alpha^{kmn}\alpha^{mn}U_m^n\Delta^{\frac{n-1}{2}}+\beta^{kmn}\beta^{mn}U_m^n\Delta^{\frac{n-1}{2}} \\
= & U_m^n\Delta^{\frac{n-1}{2}}V_{(k+1)mn},
\end{aligned}$$

as claimed. By combining the above two results, the proof is complete for the case m is odd.

Now we consider the case m is even:

$$\begin{aligned}
& \frac{1}{\alpha-\beta}\left(\sum_{i\geq 0}\alpha^{2mi}\sum_{n\geq 0}\binom{n}{i}\left(\alpha^{km}z\right)^n-\sum_{i\geq 0}\beta^{2mi}\sum_{n\geq 0}\binom{n}{i}\left(\beta^{km}z\right)^n\right) \\
& -\frac{1}{\alpha-\beta}\sum_{n\geq 0}\sum_{i=0}^{n}\binom{n}{i}\left(\alpha^{kmn}-\beta^{kmn}\right)z^n \\
= & \frac{1}{\alpha-\beta}\sum_{i\geq 0}\left(\frac{\left(\alpha^{m(k+2)}z\right)^i}{\left(1-\left(\alpha^{km}z\right)\right)^{i+1}}-\frac{\left(\beta^{m(k+2)}z\right)^i}{\left(1-\left(\beta^{km}z\right)\right)^{i+1}}\right) \\
& -\frac{1}{\alpha-\beta}\left(\sum_{i\geq 0}\frac{\left(\alpha^{km}z\right)^i}{\left(1-\left(\alpha^{km}z\right)\right)^{i+1}}-\sum_{i\geq 0}\frac{\left(\beta^{km}z\right)^i}{\left(1-\left(\beta^{km}z\right)\right)^{i+1}}\right) \\
= & \frac{1}{\alpha-\beta}\left(\frac{1}{\left(1-\alpha^{km}z\right)}\frac{1}{1-\frac{\alpha^{m(k+2)}z}{1-\alpha^{km}z}}-\frac{1}{\left(1-\beta^{km}z\right)}\frac{1}{1-\frac{\beta^{m(k+2)}z}{1-\beta^{km}z}}\right) \\
& -\frac{1}{\alpha-\beta}\left(\frac{1}{1-2\alpha^{km}z}-\frac{1}{1-2\beta^{km}z}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\alpha-\beta}\left(\frac{1}{1-z\alpha^{km}\left(1+\alpha^{2m}\right)}-\frac{1}{1-z\beta^{km}\left(1+\beta^{2m}\right)}\right)\\
&\quad -\frac{1}{\alpha-\beta}\left(\frac{1}{1-2\alpha^{km}z}-\frac{1}{1-2\beta^{km}z}\right)\\
&= \frac{1}{\alpha-\beta}\sum_{n\geq 0}\left(\alpha^{kmn}\left(1+\alpha^{2m}\right)^n-\beta^{kmn}\left(1+\beta^{2m}\right)^n-2^n\left(\alpha^{kmn}-\beta^{kmn}\right)\right)z^n
\end{aligned}$$

By Lemma 1, we write

$$\begin{aligned}
&\frac{1}{\alpha-\beta}\left(\alpha^{kmn}\left(\alpha^m V_m\right)^n-\beta^{kmn}\left(\beta^m V_m\right)^n-2^n\left(\alpha^{kmn}-\beta^{kmn}\right)\right)\\
&= \frac{1}{\alpha-\beta}\left(V_m^n\left(\alpha^{mn(k+1)}-\beta^{mn(k+1)}\right)-2^n\left(\alpha^{kmn}-\beta^{kmn}\right)\right)\\
&= V_m^n U_{mn(k+1)}-2^n U_{kmn},
\end{aligned}$$

as claimed. □

Theorem 2. *Let k be a nonnegative integer. For odd m,*

$$\sum_{i=0}^{n}\binom{n}{i}V_{mi}V_{kmn+mi}=\Delta^{\left\lfloor\frac{n+1}{2}\right\rfloor}U_m^n\begin{cases}V_{(k+1)mn} & \text{if } n \text{ is even,}\\ U_{(k+1)mn} & \text{if } n \text{ is odd.}\end{cases}\tag{2.2}$$

For even m,

$$\sum_{i=0}^{n}\binom{n}{i}V_{mi}V_{kmn+mi}=V_m^n V_{(k+1)mn}+2^n V_{kmn}.$$

Proof. Multiplying the left-hand side of (2.2) by z^n and summing over n, we write

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{i=0}^{n}\binom{n}{i}V_{mi}V_{kmn+mi}z^n\\
&= \sum_{n\geq 0}\sum_{i=0}^{n}\binom{n}{i}\left(\alpha^{mi}+\beta^{mi}\right)\left(\alpha^{kmn+mi}+\beta^{kmn+mi}\right)z^n\\
&= \sum_{n\geq 0}\sum_{i=0}^{n}\binom{n}{i}\left(\alpha^{kmn+2im}+\beta^{kmn+2mi}+(-1)^{im}\left(\alpha^{kmn}+\beta^{kmn}\right)\right)z^n\\
&= \sum_{i\geq 0}\alpha^{2mi}\sum_{n\geq 0}\binom{n}{i}\left(\alpha^{km}z\right)^n+\sum_{i\geq 0}\beta^{2mi}\sum_{n\geq 0}\binom{n}{i}\left(\beta^{km}z\right)^n\\
&\quad +\sum_{i\geq 0}(-1)^{im}\sum_{n\geq 0}\binom{n}{i}\left(\alpha^{kmn}+\beta^{kmn}\right)z^n\\
&= \sum_{i\geq 0}\left(\frac{\left(\alpha^{km+2m}z\right)^i}{\left(1-\alpha^{km}z\right)^{i+1}}+\frac{\left(\beta^{km+2m}z\right)^i}{\left(1-\beta^{km}z\right)^{i+1}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i\geq 0} \left(\frac{\left((-1)^m \alpha^{km} z \right)^i}{\left(1-\alpha^{km}z\right)^{i+1}} + \frac{\left((-1)^m \beta^{km} z \right)^i}{\left(1-\beta^{km}z\right)^{i+1}} \right) \\
&= \quad \frac{1}{1 - z\alpha^{km}\left(1+\alpha^{2m}\right)} + \frac{1}{1 - z\beta^{km}\left(1+\beta^{2m}\right)} \\
&\quad + \frac{1}{1 - z\alpha^{km}\left(1+(-1)^m\right)} + \frac{1}{1 - z\beta^{km}\left(1+(-1)^m\right)}.
\end{aligned}
$$

If m is even, then by Lemma 1, we write

$$
\begin{aligned}
&\sum_{n\geq 0} \sum_{i=0}^{n} \binom{n}{i} V_{mi} V_{kmn+mi} z^n \\
&= \quad \frac{1}{1 - z\alpha^{m(k+1)}V_m} + \frac{1}{1 - z\beta^{m(k+1)}V_m} + \frac{1}{1-2z\alpha^{km}} + \frac{1}{1-2z\beta^{km}} \\
&= \quad \sum_{n\geq 0} \left(\left(\alpha^{m(k+1)n} + \beta^{m(k+1)n} \right) V_m^n + 2^n \left(\alpha^{kmn} + \beta^{kmn} \right) \right) z^n
\end{aligned}
$$

which gives us

$$
\sum_{i=0}^{n} \binom{n}{i} V_{mi} V_{kmn+mi} = V_m^n V_{(k+1)mn} + 2^n V_{kmn}.
$$

If m is odd, then

$$
\begin{aligned}
&\sum_{n\geq 0} \sum_{i=0}^{n} \binom{n}{i} V_{mi} V_{kmn+mi} z^n \\
&= \quad \frac{1}{1 - z\alpha^{km}\left(\alpha^m U_m \sqrt{p^2+4}\right)} + \frac{1}{1 - z\beta^{km}\left(-\beta^m U_m \sqrt{p^2+4}\right)} \\
&= \quad \sum_{n\geq 0} \left(\left(\alpha^{(k+1)m} U_m \sqrt{p^2+4} \right)^n + \left(-\beta^{m(k+1)} U_m \sqrt{p^2+4} \right)^n \right) z^n.
\end{aligned}
$$

Now we consider two cases: first if n is odd, then we obtain

$$
\begin{aligned}
&\sum_{n\geq 0} \sum_{i=0}^{n} \binom{n}{i} V_{mi} V_{kmn+mi} z^n \\
&= \quad \sum_{n\geq 0} \left(\alpha^{(k+1)mn} U_m^n \left(p^2+4\right)^{\frac{n}{2}} - \beta^{m(k+1)n} U_m^n \left(p^2+4\right)^{\frac{n}{2}} \right) z^n \\
&= \quad \sum_{n\geq 0} \left(U_m^n \Delta^{\frac{n}{2}} \left(\alpha^{(k+1)mn} - \beta^{m(k+1)n} \right) \right) z^n \\
&= \quad \sum_{n\geq 0} \left(U_m^n \Delta^{\frac{n-1}{2}} \left(\frac{\alpha^{(k+1)mn} - \beta^{m(k+1)n}}{\alpha - \beta} \right) \right) z^n \\
&= \quad \sum_{n\geq 0} \left(U_m^n \Delta^{\frac{n-1}{2}} U_{(k+1)mn} \right) z^n.
\end{aligned}
$$

Second, if n is even, then we obtain

$$\begin{aligned}\sum_{n\geq 0}\sum_{i=0}^{n}\binom{n}{i}V_{mi}V_{kmn+mi}z^{n} &= \sum_{n\geq 0}\left(\alpha^{(k+1)mn}U_m^n\Delta^{\frac{n}{2}}+\beta^{(k+1)mn}U_m^n\Delta^{\frac{n}{2}}\right)z^n\\ &= \sum_{n\geq 0}\left(U_m^n\left(p^2+4\right)^{\frac{n}{2}}\left(\alpha^{(k+1)mn}+\beta^{(k+1)mn}\right)\right)z^n\\ &= \sum_{n\geq 0}U_m^n\Delta^{\frac{n}{2}}V_{(k+1)mn}z^n.\end{aligned}$$

By combining the last two results, we prove the claim for odd m. Thus the proof is complete. □

Similar to the proof methods of Theorems 1 and 2, we give the following results without proof.

Theorem 3. *Let k be a nonnegative integer. For odd m,*

$$\sum_{i=0}^{n}\binom{n}{i}V_{mi}U_{kmn+mi}=\Delta^{\left\lfloor\frac{n}{2}\right\rfloor}U_m^n\begin{cases}U_{(k+1)mn} & \text{if } n \text{ is even,}\\ V_{(k+1)mn} & \text{if } n \text{ is odd.}\end{cases}$$

For even m,

$$\sum_{i=0}^{n}\binom{n}{i}V_{mi}U_{kmn+mi}=V_m^nU_{mn(k+1)}+2^nU_{kmn}$$

Theorem 4. *Let k be a nonnegative integer. For odd m,*

$$\sum_{i=0}^{n}\binom{n}{i}U_{mi}U_{kmn+mi}=\Delta^{\left\lfloor\frac{n-1}{2}\right\rfloor}U_m^n\begin{cases}V_{(k+1)mn} & \text{if } n \text{ is even,}\\ U_{(k+1)mn} & \text{if } n \text{ is odd.}\end{cases}$$

For even m,

$$\sum_{i=0}^{n}\binom{n}{i}U_{mi}U_{kmn+mi}=\frac{1}{\Delta}\left(V_m^nV_{(k+1)mn}-2^nV_{kmn}\right).$$

References

[1] E. Kılıç, Y. Ulutas and N. Omur, Formulas for weighted binomial sums with the powers of terms of binary recurrences, *Miskolc Math. Notes* **13 (1)** (2012), 53-65.

[2] T. Koshy, Fibonacci and Lucas numbers with applications. Pure and Applied Mathematics, Wiley-Interscience, New York, 2001.

[3] P. Stanica, Generating functions, weighted and non-weighted sums for powers of second-order recurrence sequences, *Fibonacci Quarterly* **41.4** (2003), 321-333.

[4] S. Vajda, Fibonacci & Lucas numbers, and the golden section. John Wiley & Sons, Inc., New York, 1989.