A COMBINATORIAL INTERPRETATION OF THE CATALAN AND BELL NUMBER DIFFERENCE TABLES

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Abstract

We study the recurrence relations and derive the generating functions of the entries along the rows and diagonals of the Catalan and Bell number difference tables.

– In Memory of Professor Herb Wilf

1. Introduction

One of the most famous sequences in all of enumerative combinatorics is the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

They possess numerous fascinating properties and appear in many problems in mathematics and computer science; see, for example, [1, 3]. A glimpse of the old classic [2], and more recently, [4, 7], reveals there are hundreds of books and articles written about the sequence $\{C_n\}_{n=0}^{\infty}$. One of the reasons it occurs so frequently in enumeration is a well-known recurrence relation that it satisfies:

$$C_0 = 1, \qquad C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}, \quad n \geq 1.$$

Interestingly, C_n also satisfies several other less famous recurrence relations. The first is Touchard's formula

$$C_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2r} 2^{n-2r} C_r.$$

The second is an n^{th} difference type recurrence formula

$$C_n = \sum_{r=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{r-1} \binom{n-r+1}{r} C_{n-r} \tag{1}$$

They can be found on, for example, page 319 and 322 respectively of [4].

The structure of Equation (1) prompts us to study

$$K_n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} C_r = \sum_{r=0}^{n} \frac{(-1)^{n-r}}{r+1} \binom{n}{r} \binom{2r}{r}. \tag{2}$$

In the OEIS [6], $\{K_n\}_{n=0}^{\infty}$ is sequence A005043. This sequence enumerates certain Motzkin and Dyck paths. We notice that $\{K_n\}_{n=0}^{\infty}$ is the leftmost diagonal of the Catalan number difference table (see Table 1). The difference table contains an infinite number of rows and diagonals. Only the first two diagonals and first four rows are found in the OEIS. Furthermore, the OEIS only provides combinatorial meaning for the two diagonals and first two rows. We wonder if there is a systematic interpretation of the combinatorial meaning for all the rows and diagonals of the Catalan difference table.

An answer can be found in the numeration of certain $n \times 1$ non-interlocking letter columns discussed in Chapter 4 of [5]. The purpose of this paper is to explain this association. During the process of exploring this connection we are able to derive recurrence formulas and generating functions for each of the rows and diagonals.

We close this section with a derivation of the generating function $K(t)$ of the sequence $\{K_n\}_{n=0}^{\infty}$. It is easy to show that

$$\frac{(-1)^k}{k+1} \binom{2k}{k} = 2 \cdot 4^k \binom{\frac{1}{2}}{k+1}.$$

Together with

$$\sum_{n=k}^{\infty} \binom{n}{k} (-t)^{n-k} = \sum_{r=0}^{\infty} \binom{k+r}{r} (-t)^r = \frac{1}{(1+t)^{k+1}},$$

we find

$$\begin{aligned}
K(t) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{k+1}\binom{n}{k}\binom{2k}{k}t^n \\
&= 2\sum_{k=0}^{\infty}(-4t)^k\binom{\frac{1}{2}}{k+1}\sum_{n=k}^{\infty}\binom{n}{k}(-t)^{n-k} \\
&= -\frac{1}{2t}\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k+1}\left(\frac{-4t}{1+t}\right)^{k+1} \\
&= -\frac{1}{2t}\left[\left(1-\frac{4t}{1+t}\right)^{\frac{1}{2}}-1\right] \\
&= \frac{1}{2t}\left(1-\sqrt{\frac{1-3t}{1+t}}\right).
\end{aligned}$$

As we shall see in Section 3, $K(t)$ plays an important role in finding the generating functions of the diagonals of the Catalan difference table.

2. Rows of the Catalan Difference Table

In the next two sections we shall combinatorially analyze the difference table of the Catalan numbers. Table 1 shows a portion of this infinite table. The top row consists of the Catalan numbers C_n, where $n \geq 0$. The main diagonal contains the numbers $1, 0, 1, 1, 3, 6, 15, 91, 252, 603, \ldots$. We shall show that this is precisely the sequence $\{K_n\}_{n=0}^{\infty}$.

1	1	2	5	14	42	132	429	1430	4862	16796	...
	0	1	3	9	28	90	297	1001	3432	11934	...
		1	2	6	19	62	207	704	2431	8502	...
			1	4	13	43	145	497	1727	6071	...
				3	9	30	102	352	1230	4344	...
					6	21	72	250	878	3114	...
						15	51	178	628	2236	...
							36	127	450	1608	...
								91	323	1158	...
									232	835	...
										603	...
											⋱

Table 1: A portion of the Catalan number difference table.

We use $c_{i,n}$, where $i \geq 0$ and $n \geq i$, to denote the entry in the ith row and nth column in this table. Obviously, $c_{0,n} = c_n$ are the Catalan numbers. To maintain consistency we will write c_n instead of C_n in the rest of the paper.

2.1. The First Row of the Difference Table

The top row of the difference table contains the Catalan numbers c_n, where $n \geq 0$. They enumerate many combinatorial objects. Standard interpretations include the following:

- The number of ways to triangulate the interior of a convex $(n+2)$-gon.
- The number of rooted binary trees of height n.
- The number of ways to parenthesize n pairs of left and right parentheses.

For our purpose we use an interpretation different from these three.

Consider an $n \times 1$ column of letters selected from an alphabet $\mathcal{A} = \{\alpha_i\}_{i \geq 1}$. For brevity we shall call it an ***n-column***. Denote entries in an n-column by $a_1 a_2 \ldots a_n$. An n-column is said to be ***non-skipping*** if, for any i, all predecessors of a_i within $\mathcal{A}$ must have already appeared among $a_1 a_2 \ldots a_{i-1}$. In other words, after $a_1 a_2 \ldots a_i$ is generated, a_{i+1} is either a letter that has already appeared, or the next letter in $\mathcal{A}$ that has not appeared yet. Note that the definition implies that a non-skipping n-column always starts with $a_1 = \alpha_1$.

In addition, call a letter column ***interlocking*** if there exist $1 \leq i < j < k < \ell \leq n$ such that $a_i \neq a_j$, $a_i = a_k$, and $a_j = a_\ell$. In a way, we can say that the pattern $a_i a_j$, where $a_i \neq a_j$, repeats again later within the same column as $a_k a_\ell$, with the understanding that neither a_i and a_j nor a_k and a_ℓ need to be adjacent.

For demonstrative purposes we shall use the English letters. In Figure 1 both columns are non-skipping, but the one on the left is non-interlocking, while the one on the right is interlocking due to the repetition of the AB pattern.

A	A
B	B
A	C
C	A
A	C
A	B

Figure 1: Examples of non-interlocking and interlocking letter columns

We are interested in the enumeration of non-interlocking and non-skipping (NINS) n-columns. Notice that in an NINS n-column, if a_k is the last occurrence of α_1, then, because of the non-skipping property, we may assume that for some $t \geq 1$, all the letters $\alpha_1, \alpha_2, \ldots, \alpha_t$ appear within $a_1 a_2 \ldots a_k$. Now each of the letters $\alpha_2, \alpha_3, \ldots, \alpha_t$ cannot appear in the segment $a_{k+1}a_{k+2}\ldots a_n$, for otherwise, along with $a_k = \alpha_1$, one of the patterns $\alpha_1\alpha_i$, where $1 \leq i \leq t$, would have reappeared. This means only *new* letters $\alpha_{t+1}, \alpha_{t+2}, \ldots$ can appear in $a_{k+1}a_{k+2}\ldots a_n$. The non-skipping property also requires $a_{k+1} = \alpha_{t+1}$. In addition, the segment $a_{k+1}a_{k+2}\ldots a_n$ still obeys the NINS condition. Therefore, over all NINS n-columns with a_k as the last occurrence of α_1, the segments $a_{k+1}a_{k+2}\ldots a_n$ are in one-to-one correspondence with NINS $(n-k)$-columns. This important observation plays a key role in our analysis.

Theorem 2.1 *Let u_n denote the number of NINS n-columns. Define $u_0 = 1$. Then $u_n = c_n$.*

Proof. It is easy to verify that $u_1 = 1$, $u_2 = 2$, and $u_3 = 5$. Their respective letter columns are depicted in Figure 2.

A	A	A	A	A	A	A	A
	B	A	B	B	A	B	A
			C	B	B	A	A

Figure 2: NINS letter columns of size at most three.

Let a_k be the last occurrence of the letter $\alpha_1 = \text{A}$ within an NINS n-column. Obviously, $1 \leq k \leq n$. Notice that $a_1 a_2 \ldots a_{k-1}$ is a NINS $(k-1)$-column. In this regard, defining $u_0 = 1$ counts the number of the null column. Our earlier remark asserts that all the letters from a_{k+1} to a_n must be new, and the segment $a_{k+1}a_{k+2}\ldots a_n$ can be reduced to an NINS $(n-k)$-column. Figure 3 lists the NINS 4-columns according to this classification.

A	A	A	A	A	A	A	A	A	A	A	A	A	A
B	B	B	B	B	A	A	B	A	B	B	A	B	A
C	C	B	C	B	B	B	A	A	C	B	B	A	A
D	C	C	B	B	C	B	C	B	A	A	A	A	A
$k=1$					$k=2$		$k=3$		$k=4$				

Figure 3: Classification of NINS 4-columns.

It follows that

$$u_0 = 1, \qquad u_n = \sum_{k=1}^{n} u_{k-1}u_{n-k}, \quad n \geq 1.$$

Since u_n satisfies the same initial condition and recurrence relation as the Catalan numbers, we find $u_n = c_n$. □

A closing remark: an alternate proof of Theorem 2.1 using parentheses can be found in Chapter 4 of [5].

2.2. The General Row of the Difference Table

The second row of the Catalan difference table forms the sequence $\{c_{1,n}\}_{n=1}^{\infty}$. By construction, we have $c_{1,n} = c_n - c_{n-1}$, which leads to our first result regarding $c_{1.n}$.

Theorem 2.2 *For $n \geq 2$, the number of NINS n-columns that starts with AB is $c_{1,n}$. Equivalently, $c_{1,n}$ is the number of NINS n-columns that do not start with AA.*

Proof. Without any restriction, there are c_n NINS columns of size $n \times 1$. Because of non-skipping property, any such column has to start with AA or AB. Hence we need to remove those that start with AA.

If an NINS n-column starts with AA, by removing the first A, we obtain an NINS $(n-1)$-column. Conversely, starting with any NINS $(n-1)$-column, by appending an A on top, we form an NINS n-column that starts with AA. Therefore there are c_{n-1} such columns, and $c_{1,n} = c_n - c_{n-1}$ counts the number of n-columns that do not start with AA (consequently they must start with AB). □

We say that a letter column has a ***repetition of type i***, or it has a ***repetition at position i***, if $a_i = a_{i+1}$. If $i = n$, an n-column has a repetition of type n if it ends with an A. Equivalently, we may define repetition of type i as having the property $a_i = a_{(i+1) \bmod n}$. Hence $c_{1,n}$ counts the number of NINS n-columns that do not have repetition at position 1. Notice that, in particular, $c_{1,1} = 0$ because the 1×1 letter column A has a type 1 repetition.

We can restate Theorem 2.2 as: for $n \geq 1$, the number $c_{1,n}$ enumerates NINS n-columns that do not have repetition at the first position. It turns out that this result also holds for other rows in the Catalan difference table.

Theorem 2.3 *For $j \geq 1$, the sequence $\{c_{j,n}\}_{n=j}^{\infty}$ counts the number of NINS n-columns that do not have any repetition at the first j positions.*

Proof. Induct on j. The case of $j = 1$ was proved in Theorem 2.2. Assume, for some $j \geq 2$, the sequence $\{c_{j-1,n}\}_{n=j-1}^{\infty}$ counts the number of NINS n-columns that do not have type i repetitions for $1 \leq i \leq j-1$. We now analyze the meaning of $\{c_{j,n}\}_{n=j}^{\infty}$, where $c_{j,n} = c_{j-1,n} - c_{j-1,n-1}$.

To account for all NINS n-columns that do not have repetition at the first j positions, we start with the $c_{j-1,n}$ columns that do not have repetition at the first $j-1$ positions, and remove from them those that do have a type j repetition.

The case of $n = j$ requires a separate treatment. Among the $c_{j-1,j}$ columns that do not have repetition at the first $j-1$ positions, some also have type j repetition, hence must be discarded from our consideration. By definition, these columns have $a_j = A$. Notice that no repetition at position $j-1$ implies $a_{j-1} \neq A$. Hence $a_1a_2\dots a_{j-1}$ is an NINS $(j-1)$-column that does not have repetition at the first $j-1$ positions.

Conversely, by adding an A at the bottom of an NINS $(j-1)$-column that does not have repetition at its first $j-1$ positions will create a repetition of type j. Therefore, we need to throw out $c_{j-1,j-1}$ of the original $c_{j-1,j}$ columns. Hence the number of NINS n-columns that do not have repetition at the first j positions is $c_{j-1,j} - c_{j-1,j-1}$, which is precisely $c_{j,j}$.

Now we look at $n \geq j+1$. Any column of this type must have $a_{j-1} \neq a_j$ and $a_j = a_{j+1}$. By deleting a_j, we form a new NINS $(n-1)$-column $a_1a_2\dots a_{j-1}a_{j+1}\dots a_n$, which does not have repetition at its first $j-1$ positions. Conversely, starting with NINS $(n-1)$-column $b_1b_2\dots b_{n-1}$ that does not have repetition at its first $j-1$ positions, we can form an n-column $b_1b_2\dots b_{j-1}b_jb_jb_{j+1}\dots b_{n-1}$ that does have a type j repetition. Therefore the number of forbidden columns is $c_{j-1,n-1}$.

We have proved that, for $n \geq j+1$, the number $c_{j,n} = c_{j-1,n} - c_{j-1,n-1}$ gives the number of NINS n-columns that do not have repetition at their first j positions. This completes the induction. □

The combinatorial meaning of $c_{j,n}$ yields the following recurrence relation, in which we adopt the usual convention that a summation is empty if its upper limit is less than its lower limit.

Theorem 2.4 *For $n-1 \geq j \geq 1$,*

$$c_{j,n} = c_{j-1,n-1} + \sum_{k=3}^{j} c_{k-1,k-1}c_{j-k,n-k} + \sum_{k=\max(3,j+1)}^{n} c_{n-k}c_{j,k-1}.$$

Proof. Consider any NINS n-column with no repetition at its first j positions. Let a_k be the last occurrence of A. Obviously $k \neq 2$. If $k = 1$, then $a_2a_3\dots a_n$ is an NINS $(n-1)$-column over the alphabet $\mathcal{A} - \{A\} = \mathcal{A} - \{\alpha_1\}$ that does not have any repetition at its first $j-1$ positions. Hence there are $c_{j-1,n-1}$ such NINS columns.

For $3 \leq k \leq j$, the upper portion $a_1a_2\dots a_{k-1}$ (recall that $a_{k-1} \neq A$) is an NINS $(k-1)$-column without repetition at its first $k-1$ positions. This upper portion can be formed in $c_{k-1,k-1}$ ways. The lower portion $a_{k+1}a_{k+2}\dots a_n$, as we had remarked before, is essentially an NINS $(n-k)$-column. Since this is part of a letter column that originally does not have repetition at positions 1 through j, we know that the lower portion does not have any repetition at its first $j-k$ positions. Hence it can be formed in $c_{j-k,n-k}$ ways. We have found that $\sum_{k=3}^{j} c_{k-1,k-1}c_{j-k,n-k}$ NINS columns of size $n \times 1$ meet our condition.

For each k that satisfies $\max(3, j+1) \le k \le n$, the same old familiar argument yields $c_{j,k-1}c_{n-k}$ columns. By combining the counts over all possible values of k, we obtain the desired recurrence relation. □

Here are a few examples:

$$\begin{aligned}
c_{1,n} &= c_{n-1} + \sum_{k=3}^{n} c_{n-k}c_{1,k-1}, \quad n \ge 2, \\
c_{2,n} &= c_{1,n-1} + \sum_{k=3}^{n} c_{n-k}c_{2,k-1}, \quad n \ge 3, \\
c_{3,n} &= c_{2,n-1} + c_{n-3} + \sum_{k=4}^{n} c_{n-k}c_{3,k-1}, \quad n \ge 4, \\
c_{4,n} &= c_{3,n-1} + c_{1,n-3} + c_{n-4} + \sum_{k=5}^{n} c_{n-k}c_{4,k-1}, \quad n \ge 5, \\
c_{5,n} &= c_{4,n-1} + c_{2,n-3} + c_{1,n-4} + 3c_{n-5} + \sum_{k=6}^{n} c_{n-k}c_{5,k-1}, \quad n \ge 6.
\end{aligned}$$

The generating function for the Catalan numbers is well-known [1, 2, 3, 7] to be

$$C(t) = \sum_{t=0}^{\infty} c_n t^n = \frac{1-\sqrt{1-4t}}{2t}.$$

The generating function for $\{c_{j,n}\}_{n=j}^{\infty}$ can be derived recursively.

Theorem 2.5 *Let* $C_j(t) = \sum_{n=j}^{\infty} c_{j,n}t^n$. *Then* $C_0(t) = C(t) = \frac{1-\sqrt{1-4t}}{2t}$, *and, for* $j \ge 1$,

$$C_j(t) = (1-t)C_{j-1}(t) - c_{j-1,j-1}t^{j-1}.$$

Proof. Since $c_{j,n} = c_{j-1,n} - c_{j-1,n-1}$, we obtain

$$\begin{aligned}
\sum_{n=j}^{\infty} c_{j,n}t^n &= \sum_{n=j}^{\infty} c_{j-1,n}t^n - t\sum_{n=j}^{\infty} c_{j-1,n-1}t^{n-1} \\
&= [C_{j-1}(t) - c_{j-1,j-1}t^{j-1}] - tC_{j-1}(t),
\end{aligned}$$

which simplifies to the stated result. □

For examples,

$$\begin{aligned}
C_1(t) &= (1-t)C_0(t) - 1 &&= (1-t)C(t) - 1, \\
C_2(t) &= (1-t)C_1(t) &&= (1-t)^2C(t) - (1-t), \\
C_3(t) &= (1-t)C_2(t) - t^2 &&= (1-t)^3C(t) - (1-2t+2t^2).
\end{aligned}$$

These examples suggests that it is possible to express $C_j(t)$ in terms of $C(t)$ explicitly.

Theorem 2.6 *For $j \geq 0$,*

$$C_j(t) = (1-t)^j C(t) - \sum_{k=0}^{j-1} \left[\sum_{i=0}^{k} (-1)^i \binom{j}{i} c_{k-i} \right] t^k.$$

Proof. For $n \geq j$, we know, from the construction of the difference table,

$$c_{j,n} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} c_{n-i}. \tag{3}$$

Thus

$$\begin{aligned} \sum_{n=j}^{\infty} c_{j,n} t^n &= \sum_{i=0}^{j} (-1)^i \binom{j}{i} t^i \sum_{n=j}^{\infty} c_{n-i} t^{n-i} \\ &= \sum_{i=0}^{j} (-1)^i \binom{j}{i} t^i \left[C(t) - \sum_{\ell=0}^{j-i-1} c_\ell t^\ell \right] \\ &= (1-t)^j C(t) - \sum_{i=0}^{j-1} \sum_{\ell=0}^{j-i-1} (-1)^i \binom{j}{i} c_\ell t^{i+\ell} \\ &= (1-t)^j C(t) - \sum_{k=0}^{j-1} \left[\sum_{i=0}^{k} (-1)^i \binom{j}{i} c_{k-i} \right] t^k, \end{aligned}$$

which is what we want to prove. □

It is clear from the proof that a similar result also holds in *any* difference table.

3. Diagonals of the Catalan Difference Table

We now turn our attention to the sequences $\{\tilde{c}_{j,n}\}_{n=j}^{\infty}$, where $j = 0, 1, 2, \ldots$, formed by the diagonal entries in the Catalan difference table. Table 2 illustrates the meaning of the notation we use to identify the entries. It is clear that $\tilde{c}_{j,n} = c_{n-j,n}$. Hence $\tilde{c}_{j,n}$ counts the number of NINS n-columns that do not have repetition at their first $n-j$ positions.

3.1. The First Diagonal of the Catalan Difference Table

The entry $\tilde{c}_n = \tilde{c}_{0,n}$ on the first diagonal counts the number of $n \times 1$ NINS columns that do not have a repetition at their first n positions. Recall that this means

$\tilde{c}_{0,0}$	$\tilde{c}_{1,1}$	$\tilde{c}_{2,2}$	$\tilde{c}_{3,3}$	$\tilde{c}_{4,4}$	$\cdots$
	$\tilde{c}_{0,1}$	$\tilde{c}_{1,2}$	$\tilde{c}_{2,3}$	$\tilde{c}_{3,4}$	$\cdots$
		$\tilde{c}_{0,2}$	$\tilde{c}_{1,3}$	$\tilde{c}_{2,4}$	$\cdots$
			$\tilde{c}_{0,3}$	$\tilde{c}_{1,4}$	$\cdots$
				$\tilde{c}_{0,4}$	$\cdots$
					$\ddots$

Table 2: Renaming of the entries in the Catalan difference table.

these columns do not end with an A. Naturally we set $\tilde{c}_0 = 1$. The combinatorial interpretation also implies that $\tilde{c}_1 = 0$. For $n \geq 2$, we obtain the following recurrence relation.

Theorem 3.1 *For $n \geq 2$,*

$$\tilde{c}_n = \sum_{k=1}^{n-1} \tilde{c}_{k-1} \left(\tilde{c}_{n-k} + \tilde{c}_{n-k-1}\right).$$

Proof. Let a_k be the last occurrence of A. Since the column cannot end with an A, we find $1 \leq k \leq n-1$. In addition, $a_{k-1} \neq A$, hence $a_1 a_2 \ldots a_{k-1}$ is an NINS $(k-1)$-column that does not have a repetition at its first $k-1$ positions. Thus it can be formed in $\tilde{c}_{k-1}$ ways. The remaining question is: in how many ways can $a_{k+1} a_{k+2} \ldots a_n$ be formed?

Notice that all these letters are different from those used in the upper portion $a_1 a_2 \ldots a_k$, and a_{k+1} is the next unused letter. The lower portion is essentially an NINS $(n-k)$-column. However, a_n may or may not be equal to a_{k+1}. If $a_n \neq a_{k+1}$, what we have is an NINS $(n-k)$-column that does not have a repetition at its first $n-k$ positions, hence it can be formed in $\tilde{c}_{n-k}$ ways. If $a_n = a_{k+1}$, then $a_{n-1} \neq a_{k+1}$. The removal of a_n will produce an NINS $(n-k-1)$-column that does not have a repetition at its first $n-k-1$ positions, hence the lower portion can be formed in $\tilde{c}_{n-k-1}$ in this case. Combining what we have found yields the given recurrence. □

This recurrence relation, along with the standard convolution argument, yields the generating function of the sequence $\{\tilde{c}_n\}_{n=0}^{\infty}$.

Theorem 3.2 *Let $\widetilde{C}(t) = \sum_{n=0}^{\infty} \tilde{c}_n t^n$, then*

$$\widetilde{C}(t) = \frac{(1+t) - \sqrt{1-2t-3t^2}}{2t(1+t)}.$$

Proof. Using the recurrence from Theorem 3.1, we find

$$\sum_{n=2}^{\infty} \tilde{c}_n t^n = t \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \tilde{c}_{k-1} \tilde{c}_{n-k} \right) t^{n-1} + t^2 \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \tilde{c}_{k-1} \tilde{c}_{n-k-1} \right) t^{n-2}.$$

In terms of $\widetilde{C}(t)$, we can rewrite it as

$$\widetilde{C}(t) - \tilde{c}_0 - \tilde{c}_1 t = t\widetilde{C}(t)[\widetilde{C}(t) - \tilde{c}_0] + t^2\widetilde{C}(t) \cdot \widetilde{C}(t).$$

Since $\tilde{c}_0 = 1$ and $\tilde{c}_1 = 0$, this reduces to the equation

$$(t + t^2)\widetilde{C}^2(t) - (1 + t)\widetilde{C}(t) + 1 = 0.$$

Solving for $\widetilde{C}(t)$ finishes the proof. □

Note that we can write

$$\widetilde{C}(t) = \frac{1}{2t}\left(1 - \frac{\sqrt{(1+t)(1-3t)}}{1+t}\right) = \frac{1}{2t}\left(1 - \sqrt{\frac{1-3t}{1+t}}\right),$$

which is precisely $K(t)$ in Section 1. This proves that $K_n = \tilde{c}_n$ for all $n \geq 0$.

3.2. Other Diagonals of the Catalan Difference Table

For $j \geq 1$, we find a recurrence for the sequence $\{\tilde{c}_{j,n}\}_{n=j}^{\infty}$.

Theorem 3.3 *For $n - 1 \geq j \geq 1$,*

$$\tilde{c}_{j,n} = \sum_{\ell=1}^{j} \tilde{c}_{j-\ell,j-\ell}\tilde{c}_{\ell-1,n-j+\ell-1} + \sum_{k=1}^{n-j} \tilde{c}_{k-1}\tilde{c}_{j,n-k}.$$

Proof. Let a_k be the last occurrence of A in an NINS n-column that does not have a repetition at its first $n - j$ positions. Then $1 \leq k \leq n$. Since $n - j \geq 1$, we know that $a_2 \neq A$. Hence, technically, $k \neq 2$. Nevertheless, we shall see that it will be taken care of, numerically, in the summation.

First consider $n - j + 1 \leq k \leq n$. In this case, the upper portion $a_1a_2 \ldots a_{k-1}$ is an NINS $(k-1)$-column that does not have a repetition at the first $n - j$ positions, so it can be formed in $\tilde{c}_{k-1-(n-j),k-1}$ ways. The lower portion $a_{k+1}a_{k_2} \ldots a_n$ is essentially an NINS $(n - k)$-column with no restriction in regard to repetitions, hence it can formed in $c_{n-k} = \tilde{c}_{n-k,n-k}$ ways. This case produces

$$\sum_{k=n-j+1}^{n} \tilde{c}_{n-k,n-k}\tilde{c}_{k-1-(n-j),k-1} = \sum_{\ell=1}^{j} \tilde{c}_{j-\ell,j-\ell}\tilde{c}_{\ell-1,n-j+\ell-1}$$

columns.

For $1 \leq k \leq n - j$, the upper portion $a_1a_2 \ldots a_{k-1}$ does not have a repetition in the first $k - 1$ positions, hence it can be formed in $\tilde{c}_{k-1}$ ways. Note that $\tilde{c}_1 = 0$, which explains why $k \neq 2$. The lower portion $a_{k+1}a_{k+2} \ldots a_n$ is essentially an NINS

$(n-k)$-column that does not have repetition in its first $n-j-k$ positions, it can be formed in $\tilde{c}_{j,n-k}$ ways. There are altogether

$$\sum_{k=1}^{n-j} \tilde{c}_{k-1}\tilde{c}_{j,n-k}$$

columns in this case. Combining this with the first case yields the stated recurrence relation. □

For $1 \le j \le 4$, we obtain these examples:

$$\begin{aligned}
\tilde{c}_{1,n} &= \tilde{c}_{n-1} + \sum_{k=1}^{n-1} \tilde{c}_{k-1}\tilde{c}_{1,n-k}, \quad n \ge 2, \\
\tilde{c}_{2,n} &= \tilde{c}_{n-2} + \tilde{c}_{1,n-1} + \sum_{k=1}^{n-2} \tilde{c}_{k-1}\tilde{c}_{2,n-k}, \quad n \ge 3, \\
\tilde{c}_{3,n} &= 2\tilde{c}_{n-3} + \tilde{c}_{1,n-2} + \tilde{c}_{2,n-1} + \sum_{k=1}^{n-3} \tilde{c}_{k-1}\tilde{c}_{3,n-k}, \quad n \ge 4, \\
\tilde{c}_{4,n} &= 5\tilde{c}_{n-4} + 2\tilde{c}_{1,n-3} + \tilde{c}_{2,n-2} + \tilde{c}_{3,n-1} + \sum_{k=1}^{n-4} \tilde{c}_{k-1}\tilde{c}_{4,n-k}, \quad n \ge 5.
\end{aligned}$$

The generating function $\widetilde{C}_j(t) = \sum_{n=j}^{\infty} \tilde{c}_{j,n} t^n$ can be expressed in terms of $\widetilde{C}(t)$, as follows.

Theorem 3.4 *For $j \ge 0$,*

$$\widetilde{C}_j(t) = (1+t)^j \widetilde{C}(t) - \sum_{k=0}^{j-1} \left[\sum_{i=0}^{k} \binom{j}{i} \tilde{c}_{k-i} \right] t^k.$$

Proof. The construction of the difference table ensures that for all $n \ge j \ge 1$,

$$\tilde{c}_{j,n} = \tilde{c}_{j-1,n} + \tilde{c}_{j-1,n-1}.$$

By applying it repeatedly, we obtain

$$\tilde{c}_{j,n} = \sum_{i=0}^{j} \binom{j}{i} \tilde{c}_{n-i}.$$

Note its similarity to Equation (3). The rest of the proof is almost identical to that of Theorem 2.6, and hence is omitted. □

To illustrate Theorem 3.4, we list below the results for $1 \leq j \leq 4$.

$$\begin{aligned}
\widetilde{C}_1(t) &= (1+t)\widetilde{C}(t) - 1, \\
\widetilde{C}_2(t) &= (1+t)^2\widetilde{C}(t) - (1+2t), \\
\widetilde{C}_3(t) &= (1+t)^3\widetilde{C}(t) - (1+3t+4t^2), \\
\widetilde{C}_4(t) &= (1+t)^4\widetilde{C}(t) - (1+4t+7t^2+9t^3).
\end{aligned}$$

Once again, we note that the same argument could be applied to the diagonal entries of *any* difference table to produce similar results.

4. The Bell Number Difference Table

Thus far, we have discussed the correlation between NINS n-columns and the Catalan number difference table. It is the non-interlocking property that gives rise to the convolution-type recurrence relation that is often found in Catalan numbers. What if we drop the non-interlocking property? The answer can be found in the Bell number B_n, which counts the number of partitions of an n-set.

Theorem 4.1 *The number of non-skipping n-columns, denoted NS n-columns, is B_n.*

Proof. Given any $n \times 1$ non-skipping letter column, the subsets containing subscripts whose respective positions hold the same letter form a partition of $[n] = \{1, 2, \ldots, n\}$. Conversely, given any partition of $[n]$, we can name the subsets $S_A, S_B, \ldots$, such that their smallest elements are in ascending order. Next, let $a_i = k$ if $i \in S_k$. The result is an $n \times 1$ non-skipping letter column. This proves that the $n \times 1$ non-skipping columns are in one-to-one correspondence with the partitions of $[n]$, hence they are counted by the Bell number B_n. □

The Bell number B_n can be defined recursively as

$$B_0 = 1, \qquad B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k, \quad n \geq 1.$$

This enables us to obtain another proof.

Proof. (Alternate Proof) Let v_n denote the number of non-skipping n-columns. We adopt the convention that $v_0 = 1$, which can be considered as the number of null columns. For $n \geq 1$, assume that there are k occurrences of A other than the first one. Hence $0 \leq k \leq n-1$. The locations of these k occurrences can be selected in $\binom{n-1}{k}$ ways. Once the A's are placed, the remaining $n-k$ spaces essentially form an NS $(n-k)$-column that uses B, C, ... Therefore

$$v_n = \sum_{k=0}^{n-1} \binom{n-1}{k} v_{n-k}.$$

Since v_n and B_n obey the same recurrence and share the same initial condition, we conclude that $v_n = B_n$ for all $n \geq 0$. □

Since no interlocking occurs when $n \leq 3$, we find $B_1 = C_1 = 1$, $B_2 = C_2 = 3$, and $B_3 = C_3 = 5$. When $n = 4$ only one interlocking NS 4-column can be found, namely, ABAB. It follows that $B_4 = 15$, and $C_4 = 14$.

Naturally one may ask if there is any combinatorial meaning attached to the Bell number difference table (see Table 3). As in the case of the Catalan difference table, we denote the numbers in this table by $b_{j,n}$, where $n \geq j$, and let $b_{0,n} = b_n = B_n$.

1	1	2	5	15	52	203	877	4140	21147	115975	...
	0	1	3	10	37	151	674	3263	17007	94828	...
		1	2	7	27	114	523	2589	13744	77821	...
			1	5	20	87	409	2066	11155	64077	...
				4	15	67	322	1657	9089	52922	...
					11	52	255	1335	7432	43833	...
						41	203	1080	6097	36401	...
							162	877	5017	30304	...
								715	4140	25287	...
									3425	21147	...
										17722	...
											⋱

Table 3: A portion of the Bell number difference table.

4.1. Rows of the Bell Difference Table

The proofs of Theorems 2.2 and 2.3 do not rely on the non-interlocking property, hence we can apply them to non-skipping columns.

Theorem 4.2 *For $j \geq 1$ the sequence $\{b_{j,n}\}_{n=j}^{\infty}$ counts the number of NS n-columns that do not have any repetition at the first j positions.*

The argument for the recurrence relation that the $c_{j,n}$ satisfy does depend on the non-interlocking property. It appears that $b_{j,n}$ does not obey any simple recurrences.

Nonetheless, it is clear from the structure of the difference table that

$$b_{j,n} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} b_{n-i}.$$

There is another way to connect $b_{j,n}$ to the Bell numbers.

Theorem 4.3 *For $n \geq 1$,*

$$b_{1,n+1} = \sum_{k=0}^{n-1} \binom{n-1}{k} b_{n-k} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} b_{\ell+1}.$$

Proof. Once again let k count the number of A's that occur beyond the initial A in an NS $(n+1)$-column that begins with AB. Hence $0 \leq k \leq n-1$. The placement of these A's is enumerated by $\binom{n-1}{k}$. Once these A's are placed, the remaining $n-k$ spaces are completed with an arbitrary NS $(n-k)$-column over $\mathcal{A}-\{A\}$ which starts with B. The number of such letter columns is b_{n-k}. This completes the proof of the first equality. The second is obtained by a change in the index of summation. □

This result can be generalized.

Theorem 4.4 *For $n \geq j \geq 1$,*

$$b_{j,n+1} = \sum_{k=0}^{n-j} \binom{n-j}{k} b_{n-k} = \sum_{\ell=0}^{n-j} \binom{n-j}{\ell} b_{\ell+j}.$$

Proof. It suffices to prove the first equality. Induct on j. We have just proved the case of $j = 1$. So we may assume the result holds for $j-1$ for some $j \geq 2$. Then, using Pascal's identity, we find

$$\begin{aligned} b_{j,n+1} &= b_{j-1,n+1} - b_{j-1,n} \\ &= \sum_{k=0}^{n-j+1} \binom{n-j+1}{k} b_{n-k} - \sum_{k=0}^{n-j} \binom{n-j}{k} b_{n-k-1} \\ &= \sum_{k=0}^{n-j+1} \binom{n-j+1}{k} b_{n-k} - \sum_{k=1}^{n-j+1} \binom{n-j}{k-1} b_{n-k} \\ &= b_n + \sum_{k=1}^{n-j+1} \left[\binom{n-j+1}{k} - \binom{n-j}{k-1} \right] b_{n-k} \\ &= \sum_{k=0}^{n-j} \binom{n-j}{k} b_{n-k}, \end{aligned}$$

which completes the induction. □

Corollary 4.5 *For $j \geq 1$, we have*

$$b_{j,j+1} = b_j, \qquad \text{and} \qquad b_{j,j+2} = b_j + b_{j+1}.$$

For $j \geq 2$, we find

$$b_{j,j} = \sum_{k=0}^{j-2} (-1)^k b_{j-1-k}.$$

Proof. The first two results are direct consequences of Theorem 4.4. Apply

$$b_{j,j} = b_{j-1,j} - b_{j-1,j-1} = b_{j-1} - b_{j-1,j-1}$$

repeatedly to derive the last identity. □

The simplicity of Theorem 4.4 and Corollary 4.5 suggest that there may exist simple combinatorial proofs. We invite the readers to find them.

Bell numbers can be stated without using any recurrence. For instance, Dubinski's formula asserts that

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!},$$

from which we obtain the following *ordinary* generating function

$$B(t) = \sum_{n=0}^{\infty} B_n t^n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{(1-kt)k!}.$$

Because of the remark following Theorem 2.6, we immediately obtain the next result.

Theorem 4.6 *Let $B_j(t) = \sum_{n=j}^{\infty} b_{j,n} t^n$. For $j \geq 0$.*

$$B_j(t) = (1-t)^j B(t) - \sum_{k=0}^{j-1} \left[\sum_{i=0}^{k} (-1)^i \binom{j}{i} b_{k-i} \right] t^k.$$

We are often more interested in the *exponential* generating function

$$\mathfrak{B}(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = \exp\left(e^t - 1\right).$$

For $j \geq 1$, the result for $b_{j,n}$ becomes rather easy if we study a non-standard form of exponential generating function:

$$\mathfrak{B}_j(t) = \sum_{n=j}^{\infty} b_{j,n+1} \frac{t^n}{(n-j)!}.$$

Although it may look odd, it is precisely its peculiar set up that allows us to apply Theorem 4.4 efficiently.

Theorem 4.7 *For $j \geq 1$, the "shifted" exponential generating function for the truncated sequence $\{b_{j,n+1}\}_{n=j}^{\infty}$ is*

$$\mathfrak{B}_j(t) = t^j e^t \mathfrak{B}^{(j)}(t),$$

where $\mathfrak{B}^{(j)}(t)$ denotes $D_t^j \mathfrak{B}(t)$.

Proof. Using Theorem 4.4 we find

$$\begin{aligned} \mathfrak{B}_j(t) &= \sum_{n=j}^{\infty} \frac{t^n}{(n-j)!} \sum_{k=0}^{n-j} \binom{n-j}{k} b_{n-k} \\ &= t^j \sum_{k=0}^{\infty} \sum_{n=k+j}^{\infty} \frac{t^k}{k!} \cdot \frac{b_{n-k} t^{n-k-j}}{(n-k-j)!} \\ &= t^j \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} b_{j+\ell} \frac{t^\ell}{\ell!} \right), \end{aligned}$$

from which the result follows. □

4.2. Diagonals of the Bell Difference Table

Following the same convention we adopted in the Catalan difference table, we name the diagonal entries in the Bell difference table $\tilde{b}_{j,n}$, where $n \geq j \geq 0$, and $\tilde{b}_{j,n} = b_{n-j,n}$. Combinatorially, $\tilde{b}_{j,n}$ counts the number of NS n-columns that do not have any repetition in its first $n-j$ positions.

Theorem 4.8 *For $n \geq j \geq 0$,*

$$\tilde{b}_{j,n} = \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} b_{n-k}.$$

Proof. Analytically, since $\tilde{b}_{j,n} = b_{n-j,n}$, this is a direct consequence due to the construction of the difference table. What follows is a combinatorial proof. Let S denote the of set of all NS n-columns. For $1 \leq i \leq n-j$ define S_i to be the subset of S that contains all the NS n-columns with a repetition at position i. Obviously, $|S| = b_n$. Consider $S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}$. We may assume the subscripts $i_1, i_2, \ldots, i_k$ form ℓ clusters of consecutive integers of sizes $m_1, m_2, \ldots, m_\ell$. For each q, the q^{th} cluster of consecutive subscripts give rise to a block of $m_q + 1$ repeated letters in the n-columns. By deleting the first m_q occurrences of these repeated letters for each q, we obtain a NS column of size $n - \sum_{q=1}^{\ell} m_q = n - k$. Since the converse is also true, we see that $\left|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}\right| = n - k$ regardless of the choices of $i_1, i_2, \ldots, i_k$. The result follows from principle of inclusion-exclusion. □

Corollary 4.5 yields these special values for the first three diagonals:

$$\begin{aligned} \tilde{b}_{0,n} &= \sum_{k=0}^{n-2} (-1)^k b_{n-1-k}, \quad n \geq 2, \\ \tilde{b}_{1,n} &= b_{n-1}, \qquad n \geq 2, \\ \tilde{b}_{2,n} &= b_{n-2} + b_{n-1}, \quad n \geq 3. \end{aligned} \tag{4}$$

More generally, Theorem 4.4 becomes

Theorem 4.9 *For $n-1 \geq j \geq 0$,*

$$\tilde{b}_{j+1,n+1} = \sum_{k=0}^{j} \binom{j}{k} b_{n-k} = \sum_{\ell=0}^{j} \binom{j}{\ell} b_{n+\ell-j}.$$

We can also compute the generating functions in the usual way.

Theorem 4.10 *Let $\widetilde{B}_j(t) = \sum_{n=j}^{\infty} \tilde{b}_{j,n} t^n$. Then $\widetilde{B}(t) = \widetilde{B}_0(t) = \frac{1+tB(t)}{1+t}$, and, for $j \geq 0$,*

$$\widetilde{B}_j(t) = (1+t)^j \widetilde{B}(t) - \sum_{k=0}^{j-1} \left[\sum_{i=0}^{k} \binom{j}{i} \tilde{b}_{k-i} \right] t^k.$$

Proof. It suffices to derive the generating function $\widetilde{B}(t) = \widetilde{B}_0(t)$. Define $b_{-1} = 1$ so that (4) can be rewritten as

$$\tilde{b}_{0,n} = \sum_{k=0}^{n} (-1)^k b_{n-1-k}, \qquad n \geq 0.$$

Then

$$\widetilde{B}(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k b_{n-1-k} t^n = \left(\sum_{k=0}^{\infty} (-1)^k t^k \right) \left(\sum_{\ell=0}^{\infty} b_{\ell-1} t^\ell \right).$$

Hence $\widetilde{B}(t) = \frac{1+tB(t)}{1+t}$. □

For exponential generating functions, we study the shifted version

$$\widetilde{\mathfrak{B}}_j(t) = \sum_{n=j}^{\infty} \tilde{b}_{j,n} \frac{t^n}{(n-j)!}.$$

Theorem 4.11 *For $j \geq 0$, we find $\widetilde{\mathfrak{B}}_j(t) = t^j e^{-t} \mathfrak{B}^{(j)}(t)$. In particular, $\widetilde{\mathfrak{B}}_0(t) = e^{-t} \mathfrak{B}(t) = \exp(e^t - t - 1)$.*

Proof. Because of Theorem 4.8, we find

$$\begin{aligned}
\widetilde{\mathfrak{B}}_j(t) = \sum_{n=j}^{\infty} \frac{t^n}{(n-j)!} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} b_{n-k} &= t^j \sum_{k=0}^{\infty} \sum_{n=k+j}^{\infty} \frac{(-t)^k}{k!} \cdot \frac{b_{n-k} t^{n-k-j}}{(n-k-j)!} \\
&= t^j \left(\sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} b_{j+\ell} \frac{t^\ell}{\ell!} \right) \\
&= t^j e^{-t} \mathfrak{B}^{(j)}(t),
\end{aligned}$$

from which the special case of $j = 0$ follows easily. □

5. Closing Remarks

Interested readers may want to study the difference tables of other famous sequences, especially those with well-known combinatorial meanings.

We could also extend the definition (2) to

$$K_{n,s,t} = \sum_{r=0}^{n} \frac{(-1)^{n-r}}{r+a} \binom{n}{r} \binom{2r+b}{r}.$$

Is there any combinatorial interpretation for this number?

We showed $K_n = \tilde{c}_n$ by proving that they share the same generating function. Is it possible to find a direct combinatorial proof? The algebraic structure of (2) suggests some kind of application of the principle of inclusion-exclusion may work.

We invite the readers to investigate these and other related problems.

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