PISOT NUMBERS AND CHROMATIC ZEROS

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Abstract
In this article we show that Pisot numbers of even degree and their powers cannot be roots of chromatic polynomials. We also consider the family of smallest Pisot numbers of odd degree. We show that they cannot be roots of chromatic polynomials of connected graphs with a certain maximum number of vertices.

1. Introduction

An *algebraic integer* is a root of a monic polynomial with integer coefficients. A *Pisot number* is a real algebraic integer greater than 1 such that all its Galois conjugates are of norm smaller than 1. The *degree of the Pisot number* is the degree of its minimal polynomial. The Pisot numbers are also known as *Pisot-Vijayaraghavan numbers* or *PV numbers*. The best well-known is the golden mean, i.e. $(\sqrt{5}+1)/2$. The Pisot numbers form an infinite closed set ([6]). Due to their properties, these numbers play an important role in number theory (*cf.* [6, 7, 18]), harmonic analysis (*cf.* [6, 11, 16]), dynamical systems and ergodic theory (*cf.* [13, 14, 15, 17, 19]) and tilings (*cf.* [4, 12, 21, 22]).

One of the best known family of Pisot numbers is the so-called n-bonacci numbers, a generalization of the golden mean, i.e. the real root greater than 1, of the polynomial $x^n - x^{n-1} - \cdots - x - 1$ (*cf.* [8]).

The chromatic polynomial of a graph counts the number of its proper vertex colorings. More precisely, let G be a graph and t a positive integer. A *t-colouring* is a map from the set of vertices of G to $\{1, \ldots, t\}$ such that the images of adjacent vertices are different. Let $P_G(t)$ be the number of different t-colourings of G, it turns out that $P_G(t)$ is a polynomial in t ([5]), so we call $P_G(t)$ the *chromatic polynomial of G*. A root of the chromatic polynomial of G is called *chromatic zero* or *chromatic root*. Sokal ([20]) proved that the complex chromatic zeros are dense in the complex

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plane. An important question is what kind of algebraic integers could be chromatic zeros, see [1] and references within, for different partial answers to this question. In particular, studies have been done concerning which algebraic numbers are not chromatic zeros. Alikhani and Peng ([2]) showed the golden mean is not a chromatic zero and in [3] they showed that n-bonacci numbers are not chromatic zeros when n is even. In the case of n odd, the authors showed that the n-bonacci number cannot be a chromatic zero of a connected graph having at most $4n+2$ vertices. In the present article we generalized those results to Pisot numbers. In particular we show in Theorem 1 that a Pisot number of even degree and its natural powers are not chromatic zeros. In Theorems 3 and 4, we consider some important families of Pisot numbers of odd degree. Using similar arguments to the proof of Theorem 5 in [3], we prove that those Pisot numbers could not be chromatic zeros of connected graphs having certain maximum number of vertices.

It is well-known that the chromatic zeros are contained in the complement of $(-\infty, 0) \cup (0, 1) \cup (1, 32/27]$ ([10]). Since the smallest Pisot number (also known as the *plastic number*), the real root of $x^3 - x - 1$ (*cf.* [18]), is approximately 1.32, we ask when a Pisot number is a chromatic zero. In Theorems 3 and 4 we give a negative answer of this question for the smallest isolated Pisot numbers.

We consider the following families of polynomials:

1. $\Psi_n(x) := x^{2n+1} - x^{2n-1} - \cdots - x - 1$,
2. $\Phi_n(x) := x^{2n+1} - x^{2n} - x^{2n-2} - \cdots - x^2 - 1$,
3. $\Xi_n(x) := x^n(x^2 - x - 1) + x^2 - 1$,

where n is a positive integer. These polynomials are irreducible in $\mathbb{Z}[x]$ and have a real root greater than 1, which is a Pisot number (*cf.* [6]). Some of the Pisot numbers associated to these polynomials are well-known: $\Xi_1(x) = \Psi_1(x) = x^3 - x - 1$ is the minimal polynomial of the smallest Pisot number, ξ_1. The minimal polynomial of the second smallest Pisot number, $\xi_2 \approx 1.38$, is $\Xi_2(x) = x^4 - x^3 - 1$ (*cf.* [18]). We denote by ψ_n, ϕ_n and ξ_n the Pisot numbers associated to the polynomial $\Psi_n(x)$, $\Phi_n(x)$ and $\Xi_n(x)$, respectively. They were considered by C.L. Siegel in [18]. They are the smallest Pisot numbers and they are ordered as follows (*cf.* [6]):

$$\xi_1 = \psi_1 < \xi_2 < \xi_3 < \phi_1 < \xi_4 < \psi_2 < \xi_5 < \phi_2 < \xi_6 < \psi_3 < \cdots <$$
$$\cdots < \phi_n < \xi_{2n+2} < \psi_{n+1} < \xi_{2n+3} < \phi_{n+1} < \cdots < \frac{1+\sqrt{5}}{2}.$$

Moreover they are isolated points of the set of Pisot numbers and the golden mean is the smallest limit point of the set of Pisot numbers (*cf.* [6]).

We do not know if for all positive integers n and $l > 1$, the numbers ψ_n^l, ϕ_n^l are not chromatic zeros. And similarly for ξ_n^l, with n odd.

2. Results

Theorem 1. *A Pisot number of even degree is not a chromatic zero. Moreover the natural powers of a Pisot number of even degree are not chromatic zeros.*

Proof. Let θ be a Pisot number of even degree and $\Theta(x)$ its minimal polynomial. By definition, the degree of $\Theta(x)$ is even. So it has at least two real roots: one root is θ, and the other root θ' is inside the unit circle. So θ' is in the interval $(-1, 1)$. It cannot be zero due to the minimality of $\Theta(x)$. Therefore the root θ' is in $(-1, 0) \cup (0, 1)$ which is not possible for a chromatic polynomial. Hence $\Theta(x)$ is not a chromatic polynomial.

Let $C(x)$ be a polynomial with integer coefficients having θ as a root. So $\Theta(x)$ is a factor of $C(x)$, and hence $C(x)$ has a root in $(-1, 0) \cup (0, 1)$. Therefore $C(x)$ is not a chromatic polynomial and θ is not a chromatic zero.

Suppose that θ^n is a chromatic zero, for some positive integer n. So there exists a chromatic polynomial $P(x)$ such that θ^n is one of its roots. It follows that θ is a root of $Q(x) := P(x^n)$. Hence $\Theta(x)$ is a factor of $Q(x)$. We have seen in the first part of the proof that there exists $\theta' \in (-1, 0) \cup (0, 1)$, a root of $\Theta(x)$. Therefore θ' is a root of $Q(x)$, so θ'^n is a root of $P(x)$. Since $\theta'^n \in (-1, 0) \cup (0, 1)$, $P(x)$ could be a chromatic polynomial. We conclude that θ^n is not a chromatic zero. □

Theorem 2 ([9]). *Let G be a graph with n vertices and k connected components. Then the chromatic polynomial of G is of the form*

$$P_G(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_k t^k$$

where a_i are integers such that $a_n = 1$ and $(-1)^{n-i} a_i > 0$, for $k \leq i \leq n$. Moreover, if G has at least one edge, then 1 is a root of $P_G(t)$.

We prove the following theorem using arguments based on the proof of Theorem 5 in [3].

Theorem 3. *Let ψ_n be the Pisot number whose minimal polynomial is $\Psi_n(x)$. If $n \geq 2$ then ψ_n is not a chromatic zero of a connected graph with at most $4n + 1$ vertices.*

Proof. According to Theorem 2, the polynomial $\Psi_n(x)$ is not a chromatic polynomial. If ψ_n is a root of a chromatic polynomial $C(x) = \sum_{i=0}^{m} a_i x^i$, we have that $\Psi_n(x)$ is a proper factor of $C(x)$ and $m > 2n + 1$. So there exists a polynomial $D(x)$, such that $C(x) = \Psi_n(x) D(x)$ and

$$D(x) = b_{m-2n-1} x^{m-2n-1} + \cdots + b_1 x + b_0.$$

So when we develop the product $\Psi_n(x)D(x)$, we get the following equations when $m-2n-1 \leq 2n-1$:

$$-b_0 = a_0, \quad \cdots \quad -b_{m-2n-1} - \cdots - b_1 - b_0 = a_{m-2n-1},$$

and for $m-2n-1 = 2n$:

$$-b_0 = a_0, \quad \cdots \quad -b_{m-2n-1} - \cdots - b_1 = a_{m-2n-1}.$$

Due to Theorem 2, $b_0 = a_0 = 0$, since 0 is a root of a chromatic polynomial. And by the same theorem, the number 1 is a root of $C(x)$. Since $\Psi_n(1) \neq 0$, we have $D(1) = 0$, so $b_{m-2n-1} + \cdots + b_1 + b_0 = 0$, i.e. $a_{m-2n-1} = 0$. This fact contradicts Theorem 2. Therefore ψ_n is not a chromatic zero of a connected graph with m vertices, where $m-2n-1 \leq 2n$, i.e. $m \leq 4n+1$. □

Theorem 4. *Let ξ_n and ϕ_n the Pisot numbers whose minimal polynomials are $\Xi_n(x)$ and $\Phi_n(x)$, respectively.*

(a) The plastic number, i.e. ξ_1, is not a chromatic zero of a connected graph with at most 7 vertices.

(b) The Pisot number ξ_3 is not a chromatic zero of a connected graph with at most 10 vertices.

(c) The Pisot number ξ_n, with $n \geq 5$ and odd, is not a chromatic zero of a connected graph with at most $n+5$ vertices.

(d) The Pisot number ϕ_n, with $n \geq 2$, is not a chromatic zero of a connected graph with at most $2n+4$ vertices.

Proof. Due to Theorem 2, $\Xi_1(x)$ is not a chromatic polynomial. We suppose that ξ_1 is a root of the chromatic polynomial $C(x) = \sum_{i=0}^{m} a_i x^i$, with $m > 3$, so $\Xi_1(x)$ is a factor of $C(x)$. Let $D(x)$ be as in the proof of Theorem 3, with $C(x) = D(x)\Xi_1(x)$. If $m-3 \leq 4$, then we have the following equations:

$$-b_0 = a_0, \quad -b_1 - b_0 = a_1, \quad -b_1 - b_2 = a_2, \quad b_0 - b_2 - b_3 = a_3, \quad b_1 - b_3 - b_4 = a_4.$$

As in the proof of Theorem 3, $b_0 = a_0 = 0$, and $C(1) = D(1) = 0$, so

$$\begin{aligned} a_0 + \cdots + a_4 &= 0, \\ -b_0 + (-b_0 - b_1) + (-b_1 - b_2) + (b_0 - b_2 - b_3) + (b_1 - b_3 - b_4) &= 0, \\ -b_2 - b_3 &= 0, \\ a_3 &= 0. \end{aligned}$$

Hence $C(x)$ could not be a chromatic polynomial, by Theorem 2 . Therefore ξ_1 is not a chromatic zero of a connected graph having m vertices with $m \leq 7$. This completes the proof of statement (a).

The polynomial $\Xi_3(x)$ is not a chromatic polynomial, due to Theorem 2. Let $C(x) = \sum_{i=0}^{m} a_i x^i$, with $m > 5$, be a polynomial having ξ_3 as a root. We suppose that $C(x)$ is chromatic, so there exists $D(x) = \sum_{j=0}^{m-5} b_j x^j$ such that $C(x) = D(x)\Xi_3(x)$. If $m \leq 10$ we have the following equations:

$$\begin{array}{lll} -b_0 = a_0,\ -b_1 = a_1, & b_0 - b_2 = a_2, & -b_0 + b_1 - b_3 = a_3, \\ -b_0 - b_1 + b_2 - b_4 = a_4, & b_0 - b_1 - b_2 + b_3 - b_5 = a_5. & \end{array}$$

As in (a) $b_0 = a_0 = 0$ and $C(1) = D(1) = 0$, so

$$\begin{array}{rcl} a_0 + \cdots + a_5 & = & 0, \\ -b_1 - b_2 + (b_1 - b_3) + (-b_1 + b_2 - b_4) + (-b_1 - b_2 + b_3 - b_5) & = & 0, \\ -b_1 + b_3 & = & 0, \\ -a_3 & = & 0. \end{array}$$

Hence $C(x)$ could not be a chromatic polynomial, by Theorem 2 . Therefore ξ_3 is not a chromatic zero of a connected graph having m vertices with $m \leq 10$. This completes the proof of statement (b).

The polynomial $\Xi_n(x)$ is not a chromatic polynomial, due to Theorem 2. Let n be an odd number and $n \geq 5$. Let $C(x) = \sum_{i=0}^{m} a_i x^i$, with $m > n+2$, be a polynomial having ξ_n as a root. We suppose that $C(x)$ is chromatic, so there exists $D(x) = \sum_{j=0}^{m-n-2} b_j x^j$ such that $C(x) = D(x)\Xi_n(x)$. If $m - n - 2 \leq 3$ we have the following equations:

$$-b_0 = a_0, \quad -b_1 = a_1, \quad b_0 - b_2 = a_2, \quad b_1 - b_3 = a_3.$$

As in (a), $b_0 = a_0 = 0$ and $C(1) = D(1) = 0$, so

$$\begin{array}{rcl} a_0 + a_1 + a_2 + a_3 & = & 0, \\ -b_0 - b_1 + b_0 - b_2 + b_1 - b_3 & = & 0, \\ b_1 & = & 0, \\ -a_1 & = & 0. \end{array}$$

This contradicts the assumption that $C(x)$ is chromatic. Therefore ξ_n, with $n \geq 5$ is not a chromatic zero of a connected graph having m vertices with $m \leq n+5$. This completes the proof of statement (c).

Let $n \geq 2$. Due to Theorem 2, $\Phi_n(x)$ is not a chromatic polynomial. We suppose that ϕ_n is a root of the chromatic polynomial $C(x) = \sum_{i=0}^{m} a_i x^i$, with $m > 2n+1$, so

There exists $D(x)=\sum_{j=0}^{m-2n-1} b_j x^j$ such that $C(x)=D(x)\Phi_n(x)$. If $m-2n-1\leq 3$ we have the following equations:

$$-b_0=a_0,\quad -b_1=a_1,\quad -b_0-b_2=a_2,\quad -b_1-b_3=a_3.$$

As in the previous proofs, $b_0=a_0=0$ and $C(1)=D(1)=0$, so

$$\begin{aligned} a_0+a_1+a_2+a_3 &= 0,\\ -b_0-b_1-b_0-b_2-b_1-b_3 &= 0,\\ -b_1 &= 0,\\ a_1 &= 0. \end{aligned}$$

Therefore ϕ_n, with $n\geq 2$, is not a chromatic zero of a connected graph having m vertices with $m\leq 2n+4$. This completes the proof of statement (d). □

Open problem. We do not know if the numbers ψ_n^l, ϕ_n^l are chromatic zeros, for integers $n\geq 1$ and $l>1$; and similarly for ξ_n^l, when n is odd.

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