A NOTE ON IRREGULARITIES OF DISTRIBUTION

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Abstract

Let $d \geq 0$ be a fixed integer. Suppose $X = (x_1, x_2, x_3, \ldots)$ is a sequence in $[0, 1)$ with the property that for every $n \leq N$, each of the intervals $[\frac{k-1}{n}, \frac{k}{n}),\ 1 \leq k \leq n$, contains *at least one* x_i with $1 \leq i \leq n + d$. We show that this implies that $N = O(d^3)$. This is a generalization of a question raised by Steinhaus some 50 years ago.

1. Introduction

In [4], H. Steinhaus [4] raised the following question. Does there exist for every positive integer N, a sequence of real numbers $x_1, x_2, \ldots, x_N$ in $[0, 1)$ such that every $n \in \{1, 2, \ldots, N\}$ and every $k \in \{1, 2, \ldots, n\}$, we have

$$\frac{k-1}{n} \leq x_i < \frac{k}{n}$$

for some $i \in \{1, 2, \ldots, n\}$? If not, what is the largest possible value of N for which this can hold?

It was first shown by M. Warmus [5] (see also [4]) that the largest possible value of N is 17. In fact, he showed there are 768 essentially different solutions for $N = 17$. For example, one of these solutions is given by choosing the x_i to satisfy the following inequalities:

$$\begin{array}{llll}
\frac{4}{7} \leq x_1 < \frac{7}{12}, & \frac{2}{7} \leq x_2 < \frac{5}{17}, & \frac{16}{17} \leq x_3 < 1, & \frac{1}{14} \leq x_4 < \frac{1}{13}, \\
\frac{8}{11} \leq x_5 < \frac{11}{15}, & \frac{5}{11} \leq x_6 < \frac{6}{13}, & \frac{1}{7} \leq x_7 < \frac{2}{13}, & \frac{14}{17} \leq x_8 < \frac{5}{6}, \\
\frac{3}{8} \leq x_9 < \frac{5}{13}, & \frac{11}{17} \leq x_{10} < \frac{2}{3}, & \frac{3}{14} \leq x_{11} < \frac{3}{13}, & \frac{15}{17} \leq x_{12} < \frac{11}{12}, \\
\frac{1}{12} \leq x_{13} < \frac{9}{17}, & 0 \leq x_{14} < \frac{1}{17}, & \frac{13}{17} \leq x_{15} < \frac{4}{5}, & \frac{5}{16} \leq x_{16} < \frac{6}{17}, \\
\frac{10}{17} \leq x_{17} < \frac{11}{17}. & . & &
\end{array}$$

A generalization of Steinhaus' question was treated in 1970 in a paper of E. R. Berlekamp and the author [1]. Here, for each integer $d \geq 0$, we define $s(d)$ to be the largest integer such that there is a sequence $x_1, x_2, \ldots, x_{s(d)}$ with $x_i \in [0, 1)$ so that for all $n \leq s(d) - d$ and all $k \in \{1, 2, \ldots, n\}$, there is *at least one* $i \in \{1, 2, \ldots, n+d\}$ such that $\frac{k-1}{n} \leq x_i < \frac{k}{n}$. The result of Warmus shows that $s(0) = 17$.

In addition to giving a different proof that $s(0) = 17$, the authors of [1] also presented an argument asserting that $s(d) < 4^{(d+2)^2}$ for general d. There are several things to be said about the claim in this paper.

First, the form of this bound already follows (as pointed out in [1]) from a fundamental inequality of Roth [3]. In particular, he proves the following. Let $X = (x_1, x_2, \ldots)$ be a sequence of points in the interval $[0, 1)$, $I \subseteq [0, 1)$ a subinterval, $|I|$ its length and $X_n(I)$ the number of $x_m \in I$ with $m \leq n$. Then

$$\sup_{I \subseteq [0,1)} | X_n(I) - n|I| \,| > c\sqrt{\log n} \tag{1}$$

for some suitable absolute constant $c > 0$. This easily implies that $s(d) < \exp(c'd^2)$ for an appropriate absolute constant c'.

The second point to make is that it was recently pointed out by David and Moshe Newman [2] that the proof given in [1] is incomplete. The purpose of this note is to fill this gap by a different argument which also yields a much stronger upper bound on $s(d)$.

2. The Main Result

Theorem 1 *There is an absolute constant c such that for all $d \geq 1$, $s(d) < cd^3$.*

Proof. (sketch) Fix $d \geq 1$ and let $X = (x_1, x_2, \ldots)$ be a sequence of points in $[0, 1)$. Let $I_{k,n}$ denote the subinterval $[\frac{k-1}{n}, \frac{k}{n})$. Assume that for $1 \leq n \leq N$, and $1 \leq k \leq n$, each subinterval $I_{k,n}$ contains at least one x_i with $1 \leq i \leq n + d$. Our goal is to show that $N = N(d) = O(d^3)$. We first want to transform our problem as follows. Namely, at time n, let us expand the time line by a factor of n. Thus, the interval $I_{k,n}$ becomes the (unit) interval $I_k = [k - 1, k)$ and each point x_i gets replaced by nx_i. In this form, our condition now is that for $1 \leq k \leq n$, each of the intervals I_k contains at least one of the points nx_i for some i satisfying $1 \leq i \leq n+d$. Furthermore, this must hold for all n satisfying $1 \leq n \leq N$.

With this interpretation, we can think of each point x_i as starting from the origin at time 0, and then, as time progresses, continuously moving to the right along the positive time axis with a velocity x_i. Let us use the notation $x_i(t) = tx_i$ to denote the position of x_i at time t. Thus, $x_i(0) = 0$ for all i. If we stop the clock at some integer time n and see where the first $n + d$ points $x_i(n)$ are, the hypothesis implies that each of the intervals $I_k = [k-1, k),\ 1 \le k \le n$, has at least one of these points in it.

Our proof sketch proceeds as follows. We first choose some (large) time t_0 and identify a (large) set of disjoint pairs of points $(y_i(t_0), z_i(t_0))$, with $y_i(t_0) \in I_{4i}$ and $z_i(t_0) \in I_{4i+2}$. Thus, the differences $g_i(t_0) = z_i(t_0) - y_i(t_0)$ are bounded between 1 and 3. We will next select $d+1$ of these, say $g_{i_1}(t_0), g_{i_2}(t_0), \ldots, g_{i_{d+1}}(t_0)$, with $g_{i_j}(t_0) = z_{i_j}(t_0) - y_{i_j}(t_0)$ which all are close to each other in size, i.e., so that all $|g_{i_j}(t_0) - g_{i_k}(t_0)| < \alpha$ for some small $\alpha = \alpha(d)$. We will then let time advance from t_0 to some later time $t_1 > t_0$. This will have the effect of (linearly) expanding the $g_{i_j}(t)$ by a factor of roughly $\frac{t_1}{t_0}$ (since $y_{i_j}(t_0) = t_0 y_{i_j}$ goes to $y_{i_j}(t_1) = t_1 y_{i_j}$, etc). We will choose t_1 so that all the sizes of the (expanded) differences $g_{i_j}(t_1) = z_{i_j}(t_1) - y_{i_j}(t_1)$ now lie between 3 and $3+\delta$ for a small $\delta = \delta_i(d)$. (Strictly speaking, the differences should be bounded away from 3, but this does not affect the basic argument). Note that we can always choose $t_1 \le 3t_0$. The next (and final) step is to start letting the time variable advance, keeping track of where the terms of each pair $y_{i_j}(t)$ and $z_{i_j}(t)$ are as the time variable assumes successive integer values.

Suppose at some time t a particular pair $(y_{i_j}(t),\, z_{i_j}(t))$ has $y_{i_j}(t)$ just barely less than some integer m. That is, at time t, $y_{i_j}(t) \in I_m = [m-1, m)$ but at time $t+1$, $y_{i_j}(t+1) \in I_{m+1} = [m, m+1)$. However, at time t, we still have $z_{i_j}(t) \in I_{m+4}$ (since $g_{i_j}(t) = z_{i_j}(t) - y_{i_j}(t)$ is just barely greater than 3). By the hypothesis on the original sequence X, the intervals I_{m+1}, I_{m+2} and I_{m+3} must each also have points $x_m(t)$ in them at time t. However, when we advance the time to $t+1$ and $y_{i_j}(t+1)$ now moves out of I_m and into I_{m+1}, its mate $z_{i_j}(t+1)$ is still in I_{m+4} (since $g_{i_j}(t+1)$ is still only slightly larger than 3). Consequently, at time $t+1$, the 4 intervals $I_{m+1}, I_{m+2}, I_{m+3}$ and I_{m+4} now have **5** points in them! Furthermore, the situation persists until the time advances to the point where $z_{i_j}(t)$ leaves I_{m+4}, which is most of the time that $y_{i_j}(t)$ is still moving across I_{m+1} . We will call these times at which there are 5 $x_j's$ in 4 consecutive $I_r's$ "good" times for the pair (y_{i_j}, z_{i_j}). While we don't know exactly when these various transitions occur for different pairs, it is enough to show that for a suitable finishing time t_2, the *fraction* of the number of time steps that are "good" for each pair exceeds the fraction $\frac{d}{d+1}$ of the total number of time steps rom t_1 to t_2. From this it will follow that there must be some common time which is simultaneously good for all $d+1$ pairs. This is a contradiction since at this time, we would have $d+1$ disjoint sets of 4 consecutive intervals I_k which each have 5 points x_i in them, forcing some other interval I_l to have no point x_i in it, contradicting our initial hypothesis.

To make this all work, it is simply a matter of choosing the various parameters carefully (actually, one doesn't have to be too careful). For example, we need to make sure that that the values y_i (and z_i) are not too small (since it takes roughly $\frac{1}{y_i}$ time steps for $y_i(t)$ to go from m to $m+1$). We would like this minimum numbers of steps to cross a unit interval to be relatively large.

It turns out that with a modest amount of computation, the following choice of parameters can be shown to work. Namely, start by taking $t = 5000d^3$ (for example). We will choose our y_i and z_i from the intervals I_{4r} to I_{8r+2} where $r = 20d^2$. This will guarantee that we can find $d+1$ differences $g_{i_j}(t) = |z_{i_j}(t) - y_{i_j}(t)|$ which differ in size from each other by at most $\frac{1}{10d}$. We then expand these (by letting time increase to $t_1 \leq 3t_0 \leq 15000d^3$) so that the sizes of all the enlarged $g_{i_j}(t_0)$ satisfy $3 < g_{i_j}(t_0) < 3 + \frac{1}{9d}$. Finally, we compute the fraction of time steps which are "good" time for each difference $z_{i_j}(t) - y_{i_j}(t)$ from times t_1 to $t_2 = t_1 + 1000d^2$. This turns out to be at least the fraction $\frac{d}{d+1}$ of the total number of times steps during time period from t_1 to t_2. This in turn implies that there is some time t during this period which is good for *all* $d+1$ of the differences $z_{i_j}(t) - y_{i_j}(t)$, which is a contradiction. This implies that the initial value of N must be less than $t_1 + 1000d^2 < 16000d^3$ and consequently $s(d) < 16000d^3$ (this bound is actually rather loose). Checking the details is not difficult (although messy) and we leave this to the interested reader. This completes our proof sketch.

3. Some Remarks

It is not clear what the truth for the upper bound of $s(d)$ really is. Could it actually be *linear* in d? Or even sublinear? It would be interesting to know what the correct growth rate for s(d) is. While it is known that $s(1) \geq 23$, we have no nontrivial estimates for lower bounds for $s(d)$. It would appear that some new ideas may be needed to attack this problem.

References

[1] E. R. Berlekamp and R. L. Graham, Irregularities in the Distributions of Finite Sequences, *J. Number Th.* **2**, (1970), 152–161.

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[3] K. F. Roth, On irregularities of distribution, *Mathematika* **1** (1954), 73–79.

[4] H. Steinhaus, One Hundred Problems in Elementary Mathematics, *Basic Books*, New York, (1964).

[5] M. Warmus, A Supplementary Note on the Irregularities of Distributions, *J. Number Th.* **8**, (1976), 260–263.