A ZERO-SUM THEOREM OVER $\mathbb{Z}$

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Abstract

A *zero-sum* sequence of integers is a sequence of nonzero terms that sum to 0. Let $k > 0$ be an integer and let $[-k, k]$ denote the set of all nonzero integers between $-k$ and k. Let $\ell(k)$ be the smallest integer ℓ such that any zero-sum sequence with elements from $[-k, k]$ and length greater than ℓ contains a proper nonempty zero-sum subsequence. In this paper, we prove a more general result which implies that $\ell(k) = 2k - 1$ for any $k > 1$.

1. Introduction

For any multiset S, let $|S|$ denote the number of elements in S, let $\max(S)$ denote the maximum element in S, and let $\Sigma S = \sum_{s \in S} s$. Let A and B be nonempty multisets of positive integers. The pair $\{A, B\}$ is said to be *irreducible* if $\Sigma A = \Sigma B$, and for every nonempty proper mutisubsets $A' \subsetneq A$ and $B' \subsetneq B$, $\Sigma A' \neq \Sigma B'$ holds. If $\{A, B\}$ fails to be irreducible, we say that it is *reducible*. It is easy to see that if $\{A, B\}$ is irreducible, then $A \cap B = \emptyset$ or $|A| = |B| = 1$.

We define the *length* of $\{A, B\}$ as

$$\ell(A, B) = |A| + |B|.$$

An irreducible pair $\{A, B\}$ is said to be *k-irreducible* if $\max(A \cup B) \leq k$. We define

$$\ell(k) = \max_{\{A,B\}} \ell(A, B), \tag{1}$$

where the maximum is taken over all k-irreducible pairs $\{A, B\}$.

For $k > 1$, let

$$A = \{\underbrace{k, \ldots, k}_{k-1}\} \text{ and } B = \{\underbrace{k-1, \ldots, k-1}_{k}\}. \tag{2}$$

Then $\{A, B\}$ is k-irreducible and $\ell(A, B) = 2k - 1$. This implies that $\ell(k) \geq 2k - 1$. El-Zanati, Seelinger, Sissokho, Spence, and Vanden Eynden introduced k-irreducible pairs in connection with their work on irreducible λ-fold partitions (e.g., see [2]). They also conjectured that $\ell(k) = 2k - 1$.

In our main theorem below, we prove a more general result which implies this conjecture.

Theorem 1. *If $\{A, B\}$ is an irreducible pair, then $|A| \leq \max(B)$ and $|B| \leq \max(A)$. Consequently, $\ell(k) = 2k - 1$ for any $k > 1$.*

One may naturally ask which k-irreducible pairs $\{A, B\}$ achieve the maximum possible length. We answer this question in the the following corollary.

Corollary 1. *Let $k > 1$ be an integer. A k-irreducible pair $\{A, B\}$ has (maximum possible) length $\ell(A, B) = 2k - 1$ if and only if $\{A, B\}$ is the pair shown in* (2).

A *zero-sum* sequence is a sequence of nonzero terms that sum to 0. A zero-sum sequence is said to be *irreducible* if it does not contain a proper nonempty zero-sum subsequence.

Let k be a positive integer, and let $[-k, k]$ denote the set of all nonzero integers between $-k$ and k. Given a zero-sum sequence τ with elements from $[-k, k]$, let A_τ be the multiset of all positive integers from τ, and let B_τ be the multiset containing the absolute values of all negative integers from τ. Then the sequence τ is irreducible if and only if the pair $\{A_\tau, B_\tau\}$ is irreducible. Moreover, the number $\ell(k)$, defined in (1), is also equal to the smallest integer ℓ such that any zero-sum sequence with elements from $[-k, k]$ and length greater than ℓ contains a proper nonempty zero-sum subsequence. It follows from Theorem 1 that $\ell(k) = 2k - 1$.

Let G be a finite (additive) abelian group of order n. The *Davenport constant* of G, denoted by $D(G)$, is the smallest integer m such that any sequence of elements from G with length m contains a nonempty zero-sum subsequence. Another key constant, $E(G)$, is the smallest integer m such that any sequence of elements from G with length m contains a zero-sum subsequence of length exactly n. The constant $E(G)$ was inspired by the well-known result of Erdös, Ginzburg, and Ziv [3], which states that $E(\mathbb{Z}/n\mathbb{Z}) = 2n - 1$. Subsequently, Gao [4] proved that $E(G) = D(G) + n - 1$. There is a large number of research papers dealing with the constants $D(G)$ and $E(G)$. We refer the interested reader to the survey papers of Caro [1] and Gao–Geroldinger [5] for further information.

Using the language of zero-sum sequence, we can view our main theorem as a zero-sum theorem. Whereas zero-sum sequences are traditionally studied for finite abelian groups such as $\mathbb{Z}/n\mathbb{Z}$, we consider in this paper zero-sum sequences over the infinite group $\mathbb{Z}$.

The rest of the paper is structured as follows. In Section 2, we prove our main results (Theorem 1 and Corollary 1), and in Section 3, we end with some concluding remarks.

2. Proofs of Theorem 1 and Corollary 1

Suppose we are given a k-irreducible pair $\{A, B\}$. We may assume that $A = \{x_1 \cdot a_1, \ldots, x_n \cdot a_n\}$ and $B = \{y_1 \cdot b_1, \ldots, y_m \cdot b_m\}$, where the a_i's and b_j's are all positive integers such that $1 \leq a_i, b_j \leq k$ for $1 \leq i \leq n$, $1 \leq j \leq m$. We also assume that the a_i's (resp. b_j's) are pairwise distinct. Moreover, $x_i > 0$ and $y_j > 0$ are the multiplicities of a_i and b_j respectively. For any pair (a_i, b_j), let

1. C be the multiset obtained from A by: (i) removing one copy of a_i, and (ii) introducing one copy of $a_i - b_j$ if $a_i > b_j$;
2. D be the multiset obtained from B by: (i) removing one copy of b_j, and (ii) introducing one copy of $b_j - a_i$ if $b_j > a_i$.

We say that $\{C, D\}$ is (a_i, b_j)-*derived* from $\{A, B\}$. We also call the above process an (a_i, b_j)-*derivation*. Consider the integers $p > 0$, $q > 0$, and $z_{ij} \geq 0$ for $p \leq i \leq q$ and $u \leq j \leq v$. We say that $\{C, D\}$ is $\prod_{i=p}^{q} \prod_{j=u}^{v} (a_i, b_j)^{z_{ij}}$-derived from $\{A, B\}$ if it is obtain by performing on $\{A, B\}$ an (a_i, b_j)-derivation z_{ij} times for each (i, j) pair. (If $z_{ij} = 0$, then we simply do not perform the corresponding (a_i, b_j)-derivation.)

We illustrate this operation with the following example. Let $A = \{3 \cdot 7, 2 \cdot 1\} = \{7, 7, 7, 1, 1\}$ and $B = \{3 \cdot 6, 5\} = \{6, 6, 6, 5\}$. Then $\{A, B\}$ is 7-irreducible. A $(7, 6)^2(7, 5)$-derivation of (A, B) yields the pair $\{C, D\}$, where $C = \{2, 1, 1, 1, 1\}$ and $D = \{6\}$. Note that $\{C, D\}$ is 6-irreducible (thus, 7-irreducible).

In general, the order in which the derivation is done makes a difference. For example, if $A = \{5, 5\}$ and $B = \{2, 2, 2, 2, 2\}$, then we can do a $(5, 2)$ derivation followed by a $(3, 2)$-derivation on $\{A, B\}$, but not in reverse order. However, all the derivation used in our proofs can be done in any order.

We will use the following lemma.

Lemma 1. *Let $A = \{x_1 \cdot a_1, \ldots, x_n \cdot a_n\}$ and $B = \{y_1 \cdot b_1, \ldots, y_m \cdot b_m\}$ be multisets, where the a_i's and b_i's are all positive integers such that $1 \leq a_i, b_j \leq k$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Moreover, $x_i > 0$ and $y_j > 0$ are the multiplicities of a_i and b_j respectively. Suppose that $\{A, B\}$ is a k-irreducible pair with length $|A| + |B| > 2$.*

(i) If $\{C, D\}$ is (a_i, b_j)-derived from (A, B), then it is k-irreducible.
(ii) Let $p > 0$, $q > 0$, and $z_{ij} \geq 0$ for $p \leq i \leq q$ and $u \leq j \leq v$, be integers. Assume that $\sum_{j=u}^{v} z_{ij} \leq x_i$ and $\sum_{i=p}^{q} z_{ij} \leq y_j$. If $\{C, D\}$ is $\prod_{i=p}^{q} \prod_{j=u}^{v} (a_i, b_j)^{z_{ij}}$-derived from $\{A, B\}$, then it is k-irreducible.

Proof. We first prove (i). Without loss of generality, we may assume that $a_i > b_j$ since the proof is similar for $a_i < b_j$. Let

$$C = (A - \{a_i\}) \cup \{a_i - b_j\} \text{ and } D = B - \{b_j\}. \tag{3}$$

Since $\{A, B\}$ is irreducible, we have

$$\Sigma A = \Sigma B \Rightarrow \Sigma C = \Sigma A - a_i + (a_i - b_j) = \Sigma B - b_j = \Sigma D.$$

Then C and D are nonempty since C is clearly nonempty by (3). Assume that $\{C, D\}$ is reducible. Then, there exist nonempty proper multisubsets $C' \subsetneq C$ and $D' \subsetneq D$ such that $\Sigma C' = \Sigma D'$. Let $\overline{C}' = C - C'$ and $\overline{D}' = D - D'$. Then $\overline{C}' \subsetneq C$ and $\overline{D}' \subsetneq D$ are also nonempty proper multisubsets that satisfy $\Sigma\overline{C}' = \Sigma\overline{D}'$. However, it follows from the definition of C in (3) that either C' or $\overline{C}'$ is a proper multisubset of A, because there is a copy of the element $a_i - b_j$ which cannot be in both C' and $\overline{C}'$. It also follows from the definition of D in (3) that both D' and $\overline{D}'$ are proper multisubsets of B. Thus, either the pair $\{C', D'\}$ or $\{\overline{C}', \overline{D}'\}$ is a witness to the reducibility of $\{A, B\}$. This contradicts the fact that $\{A, B\}$ is irreducible. Hence, if $\{A, B\}$ is irreducible, then $\{C, D\}$ is also irreducible. In addition, it follows from (3) that $\max(C) \leq \max(A)$ and $\max(D) \leq \max(B)$. Hence, if $\{A, B\}$ is k-irreducible, then $\{C, D\}$ is also k-irreducible.

To prove (ii), observe that we can apply (i) recursively by performing (in any order) on $\{A, B\}$ an (a_i, b_j)-derivation z_{ij} times for each (i, j) pair. The conditions on the z_{ij}'s guarantee that there are enough pairs (a_i, b_j) in $A \times B$ to independently perform all the (a_i, b_j)-derivations for $p \leq i \leq q$ and $u \leq j \leq v$. □

We will also need the following basic lemma.

Lemma 2. *Let x_i and y_j be positive integers, where $1 \leq i \leq n$ and $1 \leq j \leq t+1$. If*

$$\sum_{j=1}^{t} y_j \leq \sum_{i=1}^{n} x_i \text{ and } \sum_{j=1}^{t+1} y_j > \sum_{i=1}^{n} x_i,$$

then there exist integers $z_{ij} \geq 0$, $1 \leq i \leq n$ and $1 \leq j \leq t+1$, such that

$$\sum_{i=1}^{n} z_{ij} = y_j \text{ for } 1 \leq j \leq t, \;\; z_{i,t+1} = x_i - \sum_{j=1}^{t} z_{ij} \geq 0, \;\; \text{and } y_{t+1} > \sum_{i=1}^{n} z_{i,t+1}.$$

Proof. For each j, $1 \le j \le t$, consider y_j marbles of color j. For each i, $1 \le i \le n$, consider a bin with capacity x_i (i.e., it can hold x_i marbles). Since $M = \sum_{j=1}^{t} y_j \le \sum_{i=1}^{n} x_i = C$, we can distribute all the M marbles into the n bins (with total capacity C) without exceeding the capacity of any given bin. Since $M + y_{t+1} = \sum_{j=1}^{t+1} y_j > C$, we can use some of the y_{t+1} marbles to top off the bins that were not already full.

For $1 \le i \le n$ and $1 \le j \le t$, we define z_{ij} to be the number of marbles in bin i that have color j. Then the numbers $z_{i,t+1} = x_i - \sum_{j=1}^{t} z_{ij} \ge 0$ are well defined for all $1 \le i \le n$. Hence the z_{ij}'s satisfy the required properties. □

We now prove our main theorem.

Proof of Theorem 1.

Let $\{A, B\}$ be a k-irreducible pair. We write $A = \{x_1 \cdot a_1, \ldots, x_n \cdot a_n\}$ and $B = \{y_1 \cdot b_1, \ldots, y_m \cdot b_m\}$, where the a_i's and b_i's are all positive integers such that $a_i, b_j \le k$ for $1 \le i \le n$ and $1 \le j \le m$. Moreover, $x_i > 0$ and $y_j > 0$ are the multiplicities of a_i and b_j respectively. Consequently, we may assume that the a_i's (resp. b_j's) are pairwise distinct. Without loss of generality, we may also assume that

$$a_1 > \ldots > a_n \text{ and } b_1 > \ldots > b_m. \tag{4}$$

We shall prove by induction on $r = \max(A) + \max(B) \ge 2$ that

$$|A| \le \max(B) \quad \text{and} \quad |B| \le \max(A). \tag{5}$$

If $r = 2$, then $k \le 2$ and the only possible irreducible pair is $\{\{1\}, \{1\}\}$. Thus, the inductive statement (5) is clearly true.

If $a_i = b_j$ for some pair (i, j), then $A = \{a_i\} = B$. (Otherwise, $A' = \{a_i\} \subsetneq A$ and $B' = \{a_i\} \subsetneq B$ are nonempty proper subsets satisfying $\Sigma A' = \Sigma B'$, which contradicts the irreducibility of $\{A, B\}$.) Moreover, $|A| = |B| = 1 \le a_i = \max(A) = \max(B)$ holds. Since $k > 1$, we further obtain $\ell(A, B) = |A| + |B| = 2 < 2k - 1$.

So we can assume that $A \cap B = \emptyset$. Without loss of generality, we may also assume that $a_1 = \max(A) > \max(B) = b_1$.

Suppose that the theorem holds for all k-irreducible pairs $\{C, D\}$ with $2 \le r' = \max(C) + \max(D) < r$. To prove the inductive step, we consider two parts.

Part I: In this part, we show $|A| \le \max(B)$. We consider two cases.

`Case 1:` $y_1 > x_1$.

Since $y_1 > x_1$, we can perform a $(a_1, b_1)^{x_1}$-derivation from $\{A, B\}$ to obtain (by Lemma 1) the k-irreducible pair $\{C, D\}$, where

$$C = \{x_1 \cdot (a_1 - b_1), x_2 \cdot a_2, \ldots, x_n \cdot a_n\}$$

and $D = \{(y_1 - x_1) \cdot b_1, y_2 \cdot b_2, \ldots, y_m \cdot b_m\}$.

Since $r' = \max(C) + \max(D) = \max\{a_1 - b_1, a_2\} + b_1 < r$, it follows from the induction hypothesis that

$$|C| = \sum_{i=1}^{n} x_i \leq \max(D) = b_1. \tag{6}$$

It follows from (6) that $|A| = \sum_{i=1}^{n} x_i = |C| \leq b_1$ as required.

`Case 2:` $y_1 \leq x_1$.

Since $y_1 \leq x_1$, we can perform a $(a_1, b_1)^{y_1}$-derivation from $\{A, B\}$ to obtain (by Lemma 1) the k-irreducible pair $\{C, D\}$, where

$$C = \{(x_1 - y_1) \cdot a_1, y_1 \cdot (a_1 - b_1), x_2 \cdot a_2, \ldots, x_n \cdot a_n\}$$

and $D = \{y_2 \cdot b_2, \ldots, y_m \cdot b_m\}$.

Since $r' = \max(C) + \max(D) \leq a_1 + b_2 < r$, it follows from the induction hypothesis that

$$|C| = (x_1 - y_1) + y_1 + \sum_{i=2}^{n} x_i = \sum_{i=1}^{n} x_i \leq \max(D) = b_2. \tag{7}$$

It follows from (7) that $|A| = \sum_{i=1}^{n} x_i = |C| \leq b_2 < b_1$. This concludes the first part of the proof.

Part II: In this part, we show that $|B| \leq \max(A) = a_1$. Assume that $|B| > a_1$. Then since $a_1 > b_1$ and $|A| \leq b_1$ (by Part I), we obtain $|B| > |A|$. We now consider the cases $a_n > b_1$ and $b_1 > a_n$. (Recall that $b_1 \neq a_n$ since $A \cap B = \emptyset$.)

`Case 1:` $a_n > b_1$.

Then it follows from our general assumption (4) that

$$a_1 > \ldots > a_n > b_1 > \ldots > b_m.$$

We consider the following two subcases.

`Case 1.1:` $y_1 > \sum_{i=1}^{n} x_i$.

Then, we can perform a $\prod_{i=1}^{n}(a_i, b_1)^{x_i}$-derivation from $\{A, B\}$ to obtain (by Lemma 1) the k-irreducible pair $\{C, D\}$, where

$$C = \{x_1 \cdot (a_1 - b_1), x_2 \cdot (a_2 - b_1), \ldots, x_n \cdot (a_n - b_1)\},$$

and

$$D = \left\{\left(y_1 - \sum_{i=1}^{n} x_i\right) \cdot b_1, y_2 \cdot b_2, \ldots, y_m \cdot b_m\right\}.$$

Since $r' = \max(C) + \max(D) = (a_1 - b_1) + b_1 < r$, it follows from the induction hypothesis that

$$|C| = \sum_{i=1}^{n} x_i \leq \max(D) \text{ and } |D| = \sum_{j=1}^{m} y_j - \sum_{i=1}^{n} x_i \leq \max(C). \tag{8}$$

Thus, it follows from (8) that

$$|B| = \sum_{j=1}^{m} y_j = |C| + |D| \leq \max(C) + \max(D) = (a_1 - b_1) + b_1 = a_1.$$

`Case 1.2:` $y_1 \leq \sum_{i=1}^{n} x_i$.

Recall from the first paragraph in Part II that

$$\sum_{j=1}^{m} y_j = |B| > |A| = \sum_{i=1}^{n} x_i.$$

Consequently, the above inequality together with $y_1 \leq \sum_{i=1}^{n} x_i$ imply that there exists an integer t, $1 \leq t < m$, such that

$$\sum_{j=1}^{t} y_j \leq \sum_{i=1}^{n} x_i \text{ and } \sum_{j=1}^{t+1} y_j > \sum_{i=1}^{n} x_i. \tag{9}$$

Then it follows from Lemma 2 that there exist integers $z_{ij} \geq 0$, $1 \leq i \leq n$ and $1 \leq j \leq t+1$, such that

$$\sum_{i=1}^{n} z_{ij} = y_j \text{ for } 1 \leq j \leq t, \;\; z_{i,t+1} = x_i - \sum_{j=1}^{t} z_{ij} \geq 0, \text{ and } y_{t+1} > \sum_{i=1}^{n} z_{i,t+1}.$$

Thus, we can perform a $\prod_{i=1}^{n} \prod_{j=1}^{t+1} (a_i, b_j)^{z_{ij}}$-derivation from $\{A, B\}$ to obtain (by Lemma 1) the k-irreducible pair $\{C, D\}$, where

$$\begin{aligned} C = \{ & z_{11} \cdot (a_1 - b_1), \ldots, z_{1,t+1} \cdot (a_1 - b_{t+1}), \ldots, \\ & z_{i1} \cdot (a_i - b_1), \ldots, z_{i,t+1} \cdot (a_i - b_{t+1}), \ldots, \\ & z_{n1} \cdot (a_n - b_1), \ldots, z_{n,t+1} \cdot (a_n - b_{t+1})\}, \end{aligned}$$

and

$$D = \left\{ \left(y_{t+1} - \sum_{i=1}^{n} z_{i,t+1} \right) \cdot b_{t+1}, y_{t+2} \cdot b_{t+2}, \ldots, y_m \cdot b_m \right\}.$$

Since $a_1 > \ldots > a_n > b_1 > \ldots > b_m$, it follows that

$$\max(C) \leq \max(A) - \min_{1 \leq j \leq t+1} b_j = a_1 - b_{t+1} \text{ and } \max(D) = b_{t+1}.$$

Thus, $r' = \max(C) + \max(D) \le (a_1 - b_{t+1}) + b_{t+1} < r$ and it follows from the induction hypothesis that

$$|C| = \sum_{j=1}^{t}\sum_{i=1}^{n} z_{ij} + \sum_{i=1}^{n} z_{i,t+1} = \sum_{j=1}^{t} y_j + \sum_{i=1}^{n} z_{i,t+1} \le \max(D), \tag{10}$$

and

$$|D| = (y_{t+1} - \sum_{i=1}^{n} z_{i,t+1}) + \sum_{j=t+2}^{m} y_j \le \max(C). \tag{11}$$

From (10) and (11), we obtain

$$|B| = \sum_{j=1}^{m} y_j = |C| + |D| \le \max(C) + \max(D) \le a_1 - b_{t+1} + b_{t+1} = a_1,$$

as required.

`Case 2:` $b_1 > a_n$.

Let s be that smallest index such that $b_1 > a_s$. Since $a_1 > b_1 > a_n$, the integer s exists and $2 \le s \le n$. We consider the following two subcases.

`Case 2.1:` $y_1 \le \sum_{i=s}^{n} x_i$.

Since $y_1 \le \sum_{i=s}^{n} x_i$, there exist integers $z_i \ge 0$, $s \le i \le n$ such that $x_i \ge z_i$, and $y_1 = \sum_{i=s}^{n} z_i$. We can perform an $\prod_{i=s}^{n}(a_i, b_1)^{z_i}$-derivation from $\{A, B\}$ to obtain (by Lemma 1) the k-irreducible pair $\{C, D\}$, where

$$C = \{x_1 \cdot a_1, \ldots, x_{s-1} \cdot a_{s-1}, (x_s - z_s) \cdot a_s, \ldots, (x_n - z_n) \cdot a_n\},$$

and

$$D = \{z_s \cdot (b_1 - a_s), \ldots, z_n \cdot (b_1 - a_n), y_2 \cdot b_2, \ldots, y_m \cdot b_m\}.$$

Since $r' = \max(C) + \max(D) \le a_1 + \max\{b_1 - a_n, b_2\} < r$, it follows from the induction hypothesis that

$$|D| = \sum_{i=s}^{n} z_i + \sum_{j=2}^{m} y_j = y_1 + \sum_{j=2}^{m} y_j = \sum_{j=1}^{m} y_j \le \max(C) = a_1. \tag{12}$$

Thus, it follows from (12) that $|B| = \sum_{j=1}^{m} y_i = |D| \le a_1$ as required.

`Case 2.2:` $y_1 > \sum_{i=s}^{n} x_i$.

Since $y_1 > \sum_{i=s}^{n} x_i$, we can perform a $\prod_{i=s}^{n}(a_i, b_1)^{x_i}$-derivation from $\{A, B\}$ to obtain (by Lemma 1) the k-irreducible pair $\{A', B'\}$, where

$$A' = \{x_1 \cdot a_1, x_2 \cdot a_2, \ldots, x_{s-1} \cdot a_{s-1}\},$$

and

$$B' = \Big\{(y_1 - \sum_{i=s}^{n} x_i) \cdot b_1, x_s \cdot (b_1 - a_s), \ldots, x_n \cdot (b_1 - a_n), y_2 \cdot b_2, \ldots, y_m \cdot b_m\Big\}.$$

Note that $\max(B') = b_1$. We can now rename the distinct elements of the multiset B' as $b'_1, \ldots, b'_{m'}$ such that $\max(B') = b'_1 > \ldots > b'_{m'} = \min(B')$. Let y'_j be the multiplicity of b'_j for $1 \le j \le m'$. We also let $a'_i = a_i$ for $1 \le i \le s - 1 = n'$.

Recall from Part I that $|A| \le \max(B) = b_1$. Hence,

$$|A'| = \sum_{i=1}^{s-1} x_i \le \sum_{i=1}^{n} x_i = |A| \le b_1.$$

If $|B'| \le |A'|$, then $|B| = \sum_{j=1}^{m} y_j = |B'| \le |A'| \le b_1 < a_1$, and we are done. So, we may assume that $|B'| > |A'|$. Since $a'_{n'} = a_{s-1} > b_1 = b'_1$ (owing to the definition of s and the fact that $A \cap B = \emptyset$), it follows that

$$a'_1 > \ldots > a'_{n'} > b'_1 > \ldots > b'_{m'}.$$

We can now proceed as in Part II (Case 1) to infer that

$$|B'| \le \max(A') \Longrightarrow |B| = \sum_{j=1}^{m} y_j = |B'| \le \max(A') = a_1.$$

This concludes the second part of the proof.

We conclude from Part I and Part II that

$$|A| \le \max(B) = b_1 \quad \text{and} \quad |B| \le \max(A) = a_1.$$

Moreover, these inequalities imply that

$$\ell(A, B) = |A| + |B| \le b_1 + a_1 \le 2k - 1,$$

where the last inequality follows from the fact that $1 \le b_1 < a_1 \le k$. Finally, since $\ell(k) \ge 2k-1$ (see the example in (2) from Section 1), it follows that $\ell(k) = 2k-1$. □

We now prove the corollary.

Proof of Corollary 1. Let $A = \{x_1 \cdot a_1, \ldots, x_n \cdot a_n\}$ and $B = \{y_1 \cdot b_1, \ldots, y_m \cdot b_m\}$ be multisets, where the a_i's and b_i's are all positive integers such that $a_i, b_j \le k$ for $1 \le i \le n$ and $1 \le j \le m$. Moreover, $x_i > 0$ and $y_j > 0$ are the multiplicities of a_i and b_j respectively. We also assume that the a_i's (resp. b_j's) are pairwise distinct. Without loss of generality, we may also assume that

$$A \cap B = \emptyset;\ a_1 > \ldots > a_n;\ b_1 > \ldots > b_m;\ \text{and}\ a_1 > b_1.$$

Suppose that $\{A, B\}$ is a k-irreducible pair such that $\ell(A, B) = 2k - 1$. Then it follows from Theorem 1 (and the above setup) that

$$|A| = \max(B) = b_1 = k - 1 \text{ and } |B| = \max(A) = a_1 = k. \tag{13}$$

For a proof by contradiction assume that the pair $\{A, B\}$ is different from the pair $\{\{k \cdot (k-1)\}, \{(k-1) \cdot k\}\}$. We consider two cases.

`Case 1:` $x_1 \geq y_1$.

We perform a $(a_1, b_1)^{y_1}$-derivation from $\{A, B\}$ to obtain (by Lemma 1) the k-irreducible pair $\{C, D\}$, where

$$C = \{(x_1 - y_1) \cdot a_1, y_1 \cdot (a_1 - b_1), x_2 \cdot a_2, \ldots, x_n \cdot a_n\}$$

and $D = \{y_2 \cdot b_2, \ldots, y_m \cdot b_m\}$.

Since $a_1 > b_1$, $y_1 > 0$, and $\sum A = \sum B$, we have $m > 1$, so that $b_2 \in D$. Hence, C and D are both nonempty. We now use Theorem 1 on the irreducible pair $\{C, D\}$ to infer that

$$|C| = (x_1 - y_1) + y_1 + \sum_{i=2}^{n} x_i = \sum_{i=1}^{n} x_i \leq \max(D) = b_2. \tag{14}$$

It follows from (14) that $|A| = \sum_{i=1}^{n} x_i = |C| \leq b_2 < b_1$. This contradicts the fact that $|A| = b_1$ (see (13)).

`Case 2:` $y_1 > x_1$.

We perform a $(a_1, b_1)^{x_1}$-derivation from $\{A, B\}$ to obtain (by Lemma 1) the k-irreducible pair $\{C, D\}$, where

$$C = \{x_1 \cdot (a_1 - b_1), x_2 \cdot a_2, \ldots, x_n \cdot a_n\} = \{x_1 \cdot 1, x_2 \cdot a_2, \ldots, x_n \cdot a_n\},$$

and $D = \{(y_1 - x_1) \cdot b_1, y_2 \cdot b_2, \ldots, y_m \cdot b_m\}$.

If $n = 1$, then $x_1 = |A| = k - 1$. So $y_1 > x_1$ and $\ell(A, B) = 2k - 1$ imply $y_1 = k$, contradicting that $\{A, B\}$ is different from $\{\{k \cdot (k-1)\}, \{(k-1) \cdot k\}\}$. Thus we may assume that $n \geq 2$, that is, $a_2 \in C$.

Since $a_2 \neq b_1 = k - 1$, we must have $z = b_1 - a_2 > 0$. If $z < x_1$ also holds, then $C' = \{a_2, z \cdot 1\} \subsetneq C$ and $D' = \{b_1\} \subsetneq D$ form a witness for the reducibility of $\{C, D\}$, which is a contradiction. Thus, we must have $z = b_1 - a_2 \geq x_1$. We now use Theorem 1 on the irreducible pair $\{C, D\}$ to infer that

$$|D| = (y_1 - x_1) + \sum_{j=2}^{m} y_j = -x_1 + \sum_{j=1}^{m} y_j \leq \max(C) = a_2 \leq b_1 - x_1. \tag{15}$$

It follows from (15) that $|B| = \sum_{j=1}^{m} y_j = x_1 + |D| \leq b_1 = k - 1$. This contradicts the fact that $|B| = a_1 = k$ (see (13)). $\square$

3. Concluding Remarks

One may wonder if our results can be extended to other infinite abelian groups. For instance, consider irreducible pairs $\{A, B\}$, where A and B are multisets of rational numbers. Is there a suitable (and interesting) definition of irreducibility that would guarantee the finiteness of $\ell(A, B)$?

Finally, we remark that Theorem 1 can be used to bound the number of λ-*fold vector space partitions* (e.g., see [2]). We shall address this application in a subsequent paper.

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