A NOTE ON A CONJECTURE OF ERDŐS, GRAHAM, AND SPENCER

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Received: 2/3/13, Revised: 9/6/13, Accepted: 11/12/13, Published: 11/27/13

Abstract

Let $1 \leq a_1 \leq a_2 \leq \cdots \leq a_s$ be integers with $\sum_{i=1}^{s} 1/a_i \geq n + 9/31$. In this paper, we prove that this sum can be decomposed into n parts so that all partial sums are greater than or equal to 1.

1. Introduction

Erdős, Graham and Spencer [1] posed the conjecture that if $1 \leq a_1 \leq a_2 \leq \cdots \leq a_s$ are integers with $\sum_{i=1}^{s} 1/a_i < n - 1/30$, then this sum can be decomposed into n parts so that all partial sums are less than or equal to 1. A counterexample given by Guo [4], as shown by $a_1 = 2, a_2 = a_3 = 3, a_4 = 4, a_5 = \cdots = a_{11n-12} = 11$, tells us that we should replace $1/30$ by $5/132$ or a larger quantity.

On the other hand, Sándor [3] proved that the Erdős-Graham-Spencer conjecture is true for $\sum_{i=1}^{s} 1/a_i \leq n - 1/2$, and recently, Chen [5], Fang and Chen [2] replace $1/2$ by $1/3$ and $2/7$, respectively.

In this paper, instead of improving the bound, we consider the following similar problem.

Problem 1. Find the least positive number $\eta^+ = \eta^+(n)$ such that when $1 \leq a_1 \leq a_2 \leq \cdots \leq a_s$ are integers with $\sum_{i=1}^{s} 1/a_i \geq n + \eta^+$, then this sum can be decomposed into n parts so that all partial sums are ≥ 1.

We get the following result.

Theorem 2. *Let n given, and let $\eta^+ = \eta^+(n)$ be defined as in Problem 1. Then*

$$\frac{5}{156} < \eta^+ \leq \frac{9}{31}.$$

Let $\eta^- = \eta^-(n)$ be the least positive number such that when $1 \leq a_1 \leq a_2 \leq \cdots \leq a_s$ are integers with $\sum_{i=1}^{s} 1/a_i \leq n - \eta^-$, then this sum can be decomposed into n parts so that all partial sums are less than or equal to 1. By the results of Guo [4], Fang and Chen [2] we know $\frac{5}{132} < \eta^- \leq \frac{2}{7}$. We have the following problem.

Problem 3. Is there any relationship between η^- and η^+?

In order to prove the theorem, we only need to consider those sequences such that each term is more than 1 and no partial sum (of two or more terms) is the inverse of a positive integer; otherwise, we may replace the partial sum by the inverse of the integer. We call a sequence $1 \leq a_1 \leq a_2 \leq \cdots \leq a_s$ *primitive* if no partial sum of $\sum_{i=1}^{s} 1/a_i$ is the inverse of a positive integer. In this paper, we consider multisets (i.e., sets with repetitions allowed) of positive integers. Let A be a multiset, and $T(A) = \sum_{i=1}^{s} 1/a_i$, then A is primitive if $1 \notin A$ and there is no multisubset A_1 of A with the cardinality of $A_1 \geq 2$ and $T(A_1)^{-1}$ being an integer.

2. Notation

For a multiset A and a positive real number x, let $m_A(a)$ denote the multiplicity of a in A, let $m(A)$ denote the cardinality of A and let

$$A(x) = \{a : a \in A, a < x\}.$$

For example, if $A = \{2, 3, 3, 4, 5, 5\}$, $B = \{3, 4, 5\}$, then $m_A(1) = 0$, $m_A(2) = 1$, $m_A(3) = 2$, $m_A(4) = 1$, $m_A(5) = 2$, $m(A) = 6$, and

$$A(4) = \{2, 3, 3\}, \quad A \setminus B = \{2, 3, 5\}.$$

With this notation, we say that A has an n^+-*quasiunit partition* if A can be decomposed into n multisubsets $A_1, A_2, \cdots, A_n$, with $T(A_i) \geq 1$ $(1 \leq i \leq n)$ and $m_{A_1}(a) + m_{A_2}(a) + \cdots + m_{A_n}(a) = m_A(a)$ for all integers a. In the following discussion, if we write $A = \cup_{i=1}^{n} A_i$, we mean that $\sum_{i=1}^{n} m_{A_i}(a) = m_A(a)$ for every $a \in A$, and, without loss of generality, we assume $n \geq 2$.

3. Preliminaries

Similar to Lemma 2 of [5], we have the following lemma.

Lemma 4. *Let η be a positive real number and n a positive integer. If for any positive integer $k \le n$, any finite primitive multiset A with $T(A) \ge k + \eta$ has a k^+-quasiunit partition, then any finite multiset A with $T(A) \ge n + \eta$ has an n^+-quasiunit partition.*

Proof. Let A be a finite multiset of positive integers. From Lemma 1 of [5] we know there exists an effective constructible finite primitive multiset A' and a nonnegative integer k such that $T(A) = k + T(A')$, and then the lemma follows. □

Lemma 5. *Let η be a positive real number and let A be a finite multiset with $T(A) = n + \eta$ and $A((n-1)/\eta) = B_1 \cup B_2 \cup \cdots \cup B_n$, and such that $\sum_{i=1}^{n} m_{B_i}(a) = m_A(a)$ for all integers a. Then A has an n^+- quasiunit partition if one of the following conditions holds:*

(i) $T(B_i) \ge 1$ for $1 \le i \le n$;
(ii) $T(B_i) \le 1 + \frac{\eta}{n-1}$ for $1 \le i \le n$;
(iii) $T(A(\frac{n-1}{\eta})) \le n - \frac{2}{7}$.

Proof. (i) It is obvious in this case.

(ii) If for every $1 \le i \le n$, one has $T(B_i) \ge 1$, then it is just case (i). If there exists $1 \le j \le n$, such that $T(B_j) < 1$, then

$$T\left(\cup_{i=1}^{s} B_i\right) = \sum_{i=1}^{s} T(B_i) < n + \frac{(n-1)\eta}{n-1} = n + \eta = T(A).$$

Thus $A \setminus A((n-1)/\eta) \ne \varnothing$. Add $a \in A \setminus A((n-1)/\eta)$ to B_j. If $T(B_j) \ge 1$, we have $1 \le T(B_j) < 1 + \eta/(n-1)$. Otherwise, repeat this process for $A \setminus A((n-1)/\eta) \setminus \{a\}$. Since A is a finite set, we may get $T(B_i) \ge 1$ after finite steps, and then the result follows.

(iii) It follows from (ii) and the result of [2]. □

Let $B = \{b_1, b_2, \cdots, b_r\}$, $C = \{c_1, c_2, \cdots, c_s\}$ be multisets. We write $F(B) \ge F(C)$ if $r \ge s$ and $c_i \ge b_i$ for $1 \le i \le s$.

Lemma 6. *Let B, C be multisets with $F(B) \ge F(C)$ and $\mu_i > 0$ for $1 \le i \le n$. If $B = \cup_{i=1}^{n} B_i$ and $T(B_i) \le \mu_i$, then C can be decomposed into $C = \cup_{i=1}^{n} C_i$ with $T(C_i) \le \mu_i$.*

Proof. Let $B = \{b_1, b_2, \cdots, b_r\}$, $C = \{c_1, c_2, \cdots, c_s\}$ be multisets and $c_i \ge b_i$ for $1 \le i \le s$. Let $B_i \cap \{b_1, b_2, \cdots, b_s\} = \{b_{i_1}, b_{i_2}, \cdots, b_{i_k}\}$ and put $C_i = \{c_{i_1}, c_{i_2}, \cdots, c_{i_k}\}$. Then $T(C_i) \le T(B_i) \le \mu_i$. □

4. The Proof

We are now ready to give the proof of Theorem 2.

Proof. We begin by proving $\eta^+ > 5/156$. Let $n \geq 2$ and

$$a_1 = 2, \quad a_2 = a_3 = 3, \quad a_4 = 4, \quad a_5 = \cdots = a_{13n-14} = 13;$$

then

$$\sum_{i=1}^{13n-14} \frac{1}{a_i} = n + \frac{5}{156}.$$

Let $A = \{a_1, a_2, \cdots, a_{13n-14}\} = B_1 \cup B_2 \cup \cdots \cup B_n$. Then there exists $1 \leq j \leq n$ such that $T(B_j) < 1$, which yields $\eta^+ > 5/156$. In fact, suppose that $T(B_j) \geq 1$ for every $1 \leq j \leq n$. Without loss of generality, we assume $2 \in B_1$.

If $\{3, 3\} \subset B_1$, then

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{3} = 1 + \frac{1}{6} > 1 + \frac{5}{156},$$

which is impossible.

If $3 \in B_1$, then

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 1 + \frac{1}{12} > 1 + \frac{5}{156},$$

and we know that $4 \notin B_1$; thus

$$T(B_1) = \frac{1}{2} + \frac{1}{3} + \frac{r}{13},$$

which is impossible since

$$\frac{1}{2} + \frac{1}{3} + \frac{2}{13} = \frac{77}{78} < 1, \qquad \frac{1}{2} + \frac{1}{3} + \frac{3}{13} = 1 + \frac{5}{78} > 1 + \frac{5}{156}.$$

Therefore we have

$$T(B_1) = \frac{1}{2} + \frac{r}{13},$$

which is also impossible since

$$\frac{1}{2} + \frac{6}{13} = \frac{25}{26} < 1, \qquad \frac{1}{2} + \frac{7}{13} = 1 + \frac{1}{26} > 1 + \frac{5}{156}.$$

We have proved the left inequality in Theorem 2, and we proceed to prove the right inequality in it, that is, $\eta^+ \leq 9/31$.

By Lemma 4, we assume A is primitive with $T(A) \geq n + 9/31$. By Lemmas 3-5 in [2] and the proof of the main theorem in the same paper, we know that $T(A(31(n-1)/9)) \leq T(A(7(n-1)/2)) \leq n - 1/3$ for $n = 2, 3, 4$ and $n \geq 11$. Then from Lemma 5(iii), A has an n^+- quasiunit partition.

To finish the proof, we treat the cases $5 \le n \le 10$. First we have the following equalities:

$$\frac{1}{3}+\frac{1}{6}=\frac{1}{2}, \quad \frac{1}{4}+\frac{1}{12}=\frac{1}{3}, \quad \frac{2}{5}+\frac{1}{10}=\frac{1}{2}, \quad \frac{1}{5}+\frac{2}{15}=\frac{1}{3},$$
$$\frac{1}{5}+\frac{1}{20}=\frac{1}{4}, \quad \frac{3}{7}+\frac{1}{14}=\frac{1}{2}, \quad \frac{2}{7}+\frac{1}{21}=\frac{1}{3}, \quad \frac{1}{8}+\frac{1}{24}=\frac{1}{6},$$
$$\frac{1}{9}+\frac{1}{18}=\frac{1}{6}, \quad \frac{5}{11}+\frac{1}{22}=\frac{1}{2}, \quad \frac{1}{15}+\frac{1}{30}=\frac{1}{10}, \quad \frac{6}{13}+\frac{1}{26}=\frac{1}{2}.$$

Since A is primitive, the two fractions in the left-hand side of each equality above cannot exist in $T(A)$ at the same time.

Below, the notation $b(v)$ in the multiset B means that $m_B(b) = v$.

(1) $n = 5$

Let $B = \{2, 3(2), 4, 5(4), 7(6), 8, 9(2), 11(10), 13(12)\}$; then $F(B) \ge F(A(124/9))$. By Lemma 5(ii) and Lemma 6, A has a 5^+- quasiunit partition since $B = \cup_{i=1}^{5} B_i$ with $B_1 = \{2, 3, 9(2)\}$, $B_2 = \{3, 4, 5(2), 13\}$, $B_3 = \{5(2), 7(4), 11\}$, $B_4 = \{7(2), 8, 11(7)\}$, $B_5 = \{11(2), 13(11)\}$ and $T(B_i) < 1 + \frac{9}{124}$ for $1 \le i \le 5$.

(2) $n = 6$

Let $B = \{2, 3(2), 4, 5(4), 7(6), 8, 9(2), 11(10), 12, 13(12), 15, 16, 17(16)\}$; then $F(B) \ge F(A(155/9))$. Let $B = \cup_{i=1}^{6} B_i$ with $B_1 = \{2, 3, 9(2)\}$, $B_2 = \{3, 4, 17(8)\}$, $B_3 = \{5(4), 8, 15, 16\}$, $B_4 = \{7(6), 13, 17(2)\}$, $B_5 = \{11(9), 17(4)\}$, $B_6 = \{11, 13(11), 17(2)\}$. Then $T(B_i) < 1 + \frac{9}{155}$ for $1 \le i \le 6$, which implies that A has a 6^+- quasiunit partition by Lemma 5 and Lemma 6.

(3) $n = 7, 8$

Let $B = \{2, 3(2), 4, 5(4), 7(6), 8, 9(2), 11(10), 13(12), 15, 16, 17(16), 19(18)\}$, $C = B \cup \{23(22)\}$. Then $B = \cup_{i=1}^{7} B_i$, $C = \cup_{i=1}^{8} C_i$ with $B_1 = \{2, 13(7)\}$, $B_2 = \{3(2), 4\}$, $B_3 = \{5(4), 11, 13(2)\}$, $B_4 = \{7(6), 9, 13\}$, $B_5 = \{11(9), 9, 19(2)\}$, $B_6 = \{8, 13(2), 15, 16, 17(2), 19\}$, $B_7 = \{17(4), 19(15)\}$, $C_1 = B_1$, $C_2 = B_2$, $C_3 = \{5(4), 13(3)\}$, $C_4 = \{7(6), 11(2)\}$, $C_5 = \{9(2), 11(8), 23(2)\}$, $C_6 = B_6$, $C_7 = B_7$, $C_8 = \{19(2), 23(20)\}$ and $T(B_i) < 1 + \frac{3}{62}$ for $1 \le i \le 7$, $T(C_i) \le 1 + \frac{9}{217}$ for $1 \le i \le 8$. It is easy to see that $F(T(A(62/3))) \le F(B)$ and $F(T(A(217/9))) \le F(C)$. From Lemma 5(ii) and Lemma 6, we know that A has a 7^+- quasiunit partition for $n = 7$ and an 8^+- quasiunit partition for $n = 8$.

(4) $n = 9, 10$

Let $B = \{2, 3(2), 4, 5(4), 7(6), 8, 9(2), 11(10), 13(12), 15, 16, 17(16), 19(18), 23(22), 25(4), 27(2)\}$, $C = B \cup \{28, 29(28)\}$. Then $B = \cup_{i=1}^{9} B_i$, $C = \cup_{i=1}^{10} C_i$ with $B_1 = \{2, 4, 8, 27(2)\}$, $B_2 = \{3(2), 9(2)\}$, $B_3 = \{5(4), 15\}$, $B_4 = \{7(6), 16\}$, $B_5 = \{11(10), 25(2)\}$, $B_6 = \{13(12), 25\}$, $B_7 = \{17(16), 25\}$, $B_8 = \{19(18)\}$, $B_9 = \{23(22)\}$, $C_1 = B_1 \cup \{28)\}$, $C_i = B_i$ for $2 \le i \le 9$, $C_{10} = \{29(28)\}$, and for all i, $T(B_i), T(C_i)$ are less than 1. Since $F(T(A(248/9))) \le F(B)$, $F(T(A(31))) \le F(C)$, from Lemma 5(ii) and Lemma 6, we get $\eta^+ \le \frac{9}{31}$ for $n = 9, 10$. □

Acknowledgments. This work was supported by the Guangdong Provincial Natural Science Foundation (No. S2012040007653) and NSF of China (No. 11271142). We would like to thank the referee and the editor for their valuable suggestions.

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