#G1 INTEGERS 13 (2013)

CAPTURED-REVERSIBLE MOVES AND STAR DECOMPOSITION DOMINATION IN HEX

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Received: 4/16/10, Revised: 8/17/12, Accepted: 1/15/13, Published: 1/25/13

Abstract

By applying the combinatorial game theory notions of dominated move and reversible move, and by exploiting graph-theoretic properties of Hex board decompositions, we identify two new types of inferior Hex move.

1. Introduction

Hex is the classic two-person alternate-turn board game invented by Piet Hein [10] and John Nash [14, 15, 16].

The players are Left and Right. The board is an $m \times n$ array of hexagonal cells; usually $m = n$, in which case the board has the shape of a rhombus. The color black and two opposite sides of the board are assigned to Left; the color white and the other two sides of the board are assigned to Right. A move consists of coloring an uncolored cell. The game ends when a player completes a path of their color joining their two sides; this player is the winner.

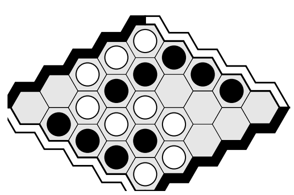

Figure 1: A game won by Right (white).

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Hex has several elegant properties:

- Hex is *monotonic*, or *regular* [20]: it is never disadvantageous to move,
- Hex never ends in a draw [5],
- Hex on an $n \times n$ board is a first-player win [2, 16],
- solving Hex — determing the winner of an abitrary position — is PSPACE-complete[17].

The last result hints that there might be no efficient algorithm to find a winning move whenever one exists. However, there are efficient methods that can identify some kinds of inferior move, and pruning such moves often improves the efficiency of finding a winning move [2, 4, 8, 9, 18].

Hex is not a combinatorial game in the strictest sense, since a Hex game ends when a player connects their two sides, whereas a combinatorial game ends when a player has no legal moves. But many combinatorial game theory (CGT) concepts — e.g., dominated and reversible moves — can be applied to Hex. Indeed, Hex is often included when CGT is discussed in a broad context, for example in Albert, Nowakowski and Wolfe's *Lessons in Play* [1].

In this paper[3], we consider inferior Hex moves in the context of CGT outcome classes. In the process, we identify two new kinds of inferior Hex cell that allow efficient pruning of the corresponding move.

In the rest of this section we present our notation. In §2 we reformulate previous Hex inferior cell analysis in terms of CGT. In §3 we identify a new class of Hex reversible moves. In §4 we identify a new class of Hex dominated moves.

1.1. Notation

Definition 1.1. A *(Hex) position* is defined by specifying the color status — uncolored, black, or white — of each board cell.

Definition 1.2. A *(Hex) state* is defined by a position and the player to move next. Given a position H and player to move next X, we denote the associated state as $H[X]$.

In CGT, the *outcome (class)* of a combinatorial game G is the result that can be achieved with perfect play. A game G is *positive* if Left wins regardless of who moves next; *negative* if Right wins regardless of who moves next; *fuzzy* if the first player wins; *zero* if the second player wins. In these four cases, we write $G > 0$, $G < 0$, $G \,||\, 0$, $G = 0$ respectively. A state is *Left-win* if Left wins and *Right-win* if Right wins.

Hex has no draws, so:

[3]See [19] for further discussion on modelling Hex with CGT.

Observation 1.3. *A Hex state is Left-win or Right-win.*

Hex is monotonic, so a position cannot have outcome zero, so:

Observation 1.4. *A Hex position is positive, negative, or fuzzy.*

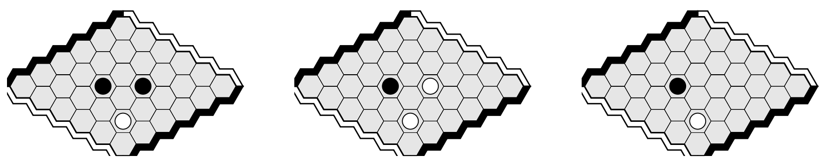

Figure 2: Three Hex positions: positive, negative, fuzzy.

In this paper, we consider (in)equality among states and positions not with respect to general CGT game values but only with respect to CGT outcome classes. Following CGT convention, we compare outcome classes of different states or positions from the perspective of Left.

Definition 1.5. For Hex states S_1, S_2, we write $S_1 \geq S_2$ if S_1 is at least as good for Left as S_2, namely if Left wins S_1 whenever Left wins S_2. Similarly, for Hex positions H_1, H_2, we write $H_1 \geq H_2$ if $H_1[L] \geq H_2[L]$ and $H_1[R] \geq H_2[R]$.

Corollary 1.6. *For Hex states S and S', $S \geq S'$ if and only if S is Left-win and/or S' is Right-win. For positions H and H', $H \geq H'$ if and only if at least one of these conditions holds: $H > 0$, $H' < 0$, or $H \,||\, 0$ and $H' \,||\, 0$.*

Definition 1.7. Hex positions (or states) A and B are *equal* if the associated sets of uncolored/black/white color assignments are equal (and the player to move is equal), in which case we write $A = B$. Hex positions (or states) A and B are *equivalent* if they have the same outcomes, in which case we write $A \equiv B$.

Definition 1.8. An *L-move* is a move by the Hex player Left, and an *L(c)-move* is an L-move to an uncolored cell c. For a position H with an uncolored cell c, a set of uncolored cells C, and a set of colored cells D:

- $H +$ *L(c)* is the position obtained from H by coloring c black,
- $H +$ *L(C)* is the position obtained from H by coloring all cells in C black,
- $H - D$ is the position obtained from H by uncoloring all cells in D.

These terms are defined similarly for Right and white.

Caveat: in CGT, "+" usually indicates the sum of games; in this paper, "+" is used only as defined above.

Definition 1.9. For a Hex position H, $\chi_U(H)$, $\chi_L(H)$, $\chi_R(H)$ denote the sets of all cells of H that are respectively uncolored, black, or white. For a set D of cells of a Hex position H, $\chi_U(H, D)$, $\chi_L(H, D)$, $\chi_R(H, D)$ denote the sets of all cells of D that are respectively uncolored, black, or white.

Definition 1.10. For Hex positions H_1 and H_2, H_2 is a *continuation* of H_1 if $\chi_L(H_1) \subseteq \chi_L(H_2)$ and $\chi_R(H_1) \subseteq \chi_R(H_2)$. A continuation with no uncolored cells is a *completion*.

In CGT a move *dominates* another if it is at least as good as the other, while a move *reverses* a previous move if it renders the previous move useless. In Hex, we define these notions in terms of the associated uncolored cells and restrict them according to outcome class.

Definition 1.11. For a Hex position H with uncolored cells c_1 and c_2,

- c_1 *Left-dominates* c_2 if $H + L(c_1) \geq H + L(c_2)$,
- c_1 is *Left-reversible* if $H \geq H + L(c_1) + R(c_2)$, in which case c_2 is a *Right-reverser* of c_1.

As explained in *Winning Ways for your Mathematical Plays* by Berlekamp, Conway, and Guy, when determining a game's value, dominated moves can be pruned (as long as one dominating move remains) and reversible moves can be bypassed, and — in its simplest form — a combinatorial game has no dominated or reversible moves [3].

The same result holds for outcome classes: when determining a game's outcome class, dominated moves can be pruned (as long as one dominating move remains) and reversible moves can be bypassed; this follows by simple case analysis from the minimax calculation of outcome classes.

We are interested in simplifying Hex positions. In the next section we rephrase some previous Hex results in terms of dominated and reversible moves.

2. Previously Known Inferior Cell Analysis

Following observations by Beck et al. [2, pp. 327-339] and Schensted and Titus [18], Hayward and van Rijswijck defined a class of provably useless Hex cells, called dead cells:

Definition 2.1. ([9]) For a Hex position H, an uncolored cell c is *live* if H has a completion H' in which changing c's color changes the winner of H'; otherwise, c is *dead*.

Observation 2.2. *([4]) For a position H, a cell c is live if and only if c is in a set S of uncolored cells of H such that some coloring of S yields the winning condition, but no coloring of a proper subset of S yields the winning condition.*

Thus determining whether a cell is live reduces to determining in a graph whether a given vertex is on a minimal path joining two other given vertices. This problem is NP-complete for general graphs, although its complexity on graphs that arise from Hex positions is unknown [4].

Observation 2.3. *([9]) Coloring a dead cell does not change a position's outcome.*

Observation 2.4. *A dead cell remains dead in all continuations in which it is uncolored.*

Some dead cells can be recognized by matching patterns of neighboring cells. For example, for each pattern in Figure 3, the uncolored cell is dead [8].

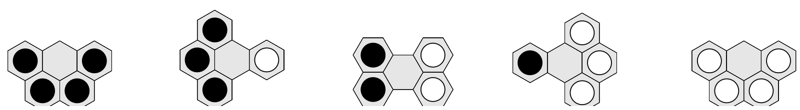

Figure 3: Some dead patterns. For any containing position, the uncolored cell is dead: coloring the cell cannot change the position's outcome.

In CGT terms, Hex is a hot game — for any position H with uncolored cell c, $H + L(c) \geq H$ — so it is never disadvantageous to have the next move. However, a move to a dead cell is equivalent to a pass move, so:

Observation 2.5. *([9]) A Hex player with a winning strategy has a winning strategy with no move to a dead cell.*

Definition 2.6. ([13]) For a Hex position and a player, a move to a cell c is *vulnerable* if the opponent has a move to a cell k that makes that cell dead; k is a *killer* of c.

E.g., in a position H, an uncolored cell c is Left-vulnerable if there is an uncolored cell k such that c is dead in $H + R(k)$.

We now redefine vulnerability in CGT terms:

Definition 2.7. For a Hex position and a player, a move to a cell c is *dead-reversible* if it is vulnerable.

Our first result is that dead-reversible cells are reversible:

Lemma 2.8. *Let H be a Hex position with a cell c that is Left-dead-reversible to killer k. Then c is Left-reversible in H, with Right-reverser k.*

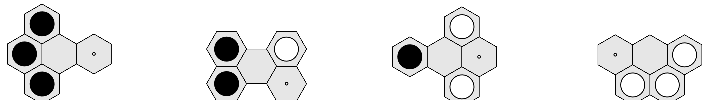

Figure 4: Some Left-dead-reversible patterns. For any containing position, the empty cell is killed by white-coloring the dotted cell.

Proof. $H \geq H + R(k)$ by monotonicity. $H + R(k) \equiv H + L(c) + R(k)$ by Observation 2.3. Thus $H \geq H + L(c) + R(k)$. □

Recall Observation 2.4: in a continuation, a dead cell remains dead for as long as it is uncolored. By contrast, a dead-reversible cell need not remain dead-reversible:

Observation 2.9. *Let H_1 be a Hex position with a Left-dead-reversible cell c and Right-reverser k, and let H_2 be a continuation of H_1 obtained by Right-coloring some cell(s) other than c. Then in H_2, c is either dead or Left-dead-reversible with Right-reverser k.*

The following rephrases a result of Hayward and van Rijswijck.

Theorem 2.10. *([9]) A Hex player with a winning strategy has a winning strategy with no move to a dead or dead-reversible cell.* □

Thus, dead-reversible Hex moves can be pruned. This is stronger than what is guaranteed by CGT, namely that reversible moves can be bypassed.

As noted in Observation 2.3, dead cells can be assigned to either player without changing a position's outcome. We now identify a class of cells that can be assigned to one particular player without changing a position's outcome.

Definition 2.11. For a position H, a set U of uncolored cells is *Left-captured* if Left has a second-player strategy on U such that, for each leaf position F in the strategy tree, each cell in $\chi_R(F, U)$ is dead in position $F - \chi_R(F, U)$. In H, any cell in such a set U is *Left-captured*.

Notice that, for each leaf position F in Definition 2.11, it follows by Observation 2.3 that

$$F \equiv F - \chi_R(F, U) \equiv (F - \chi_R(F, U)) + L(\chi_R(F, U)).$$

In other words, if Right ever plays in a Left-captured set, then Left has a replying strategy that guarantees no possible benefit to Right.

Consider any pattern in Figure 5. Left has a second-player strategy on the uncolored cell pair — if Right colors one, Left colors the other — that kills the cell just colored by Right. Thus the uncolored cell pair is Left-captured.

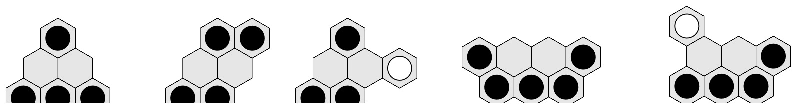

Figure 5: Some Left-captured patterns. For any containing position, black-coloring the pattern's uncolored cells cannot alter the position's outcome.

Observation 2.12. *Let H be a position with a Left-captured set S. Then S remains Left-captured in any continuation of H in which S remains uncolored.*

The observation holds because the uncolored cells outside S neither affect the capturing strategy nor revive any cell that is dead in a leaf position. Thus if H is a Hex position with a Left-captured set S, then $H \equiv H + L(S)$. Moreover, combining a captured set strategy for S with a winning strategy on the reduced board $H+L(S)$ yields a winning strategy in the original position [7].

Our second result shows that playing in one's own captured set is equivalent to a pass move:

Lemma 2.13. *Let H be a position with a cell c that is both Left-captured in H and a winning move in $H[L]$. Then $H > 0$.*

Proof. Let c be in a set F of Left-captured cells of H. Then $H \equiv H + L(F)$, and $H + L(F) \geq H + L(c) \geq H$ by monotonicity, so $H \equiv H + L(c)$. But c is a winning move in $H[L]$, so $(H + L(c))[R]$ is Left-win, so $H + L(c) \equiv H > 0$. □

Definition 2.14. An uncolored cell set F of a position H is *Left-fillin* if F partitions into cell sets $F_1, \ldots, F_t$ such that, for $1 \leq j \leq t$, each F_j is dead or Left-captured in position $H + L(F_1) + \ldots + L(F_{j-1})$.

This follows by induction on the fillin partition:

Observation 2.15. *For a position H with Left-fillin F, $H \equiv H + L(F)$.*

Definition 2.16. For uncolored cells c and f of a position H, c *Left-fillin-dominates* f if f is in some Left-fillin set F of $H + L(c)$.

This follows by monotonicity [6]:

Observation 2.17. *For a position H with uncolored cell c such that $H + L(c)$ has Left-fillin F, the cell c Left-dominates all cells f in F; namely,*

$$H + L(c) \equiv H + L(c) + L(F) \geq H + L(f).$$

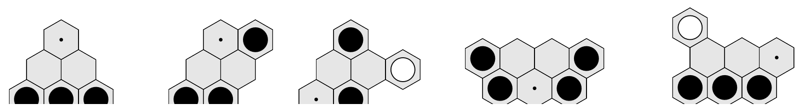

Figure 6: Some Left-domination patterns. For any containing position, black-coloring a dotted cell fillin-dominates black-coloring either of the other two uncolored cells.

In Hex, another form of domination arises when a cell is on all induced winning paths of another cell [11, 13]. When we wish to distinguish between these two forms of domination, we call the former *fillin domination* and the latter *induced path domination*. In this paper we use only fillin domination.

Thus far we have discussed inferior cells that can be pruned. Now we examine certain decompositions of Hex positions.

Definition 2.18. A *chain* is a maximal set of connected same-colored cells. A *Left chain* is a maximal set of connected black cells.

Definition 2.19. Two opposite-colored chains C_1, C_2 *touch* if there are cells c_1, c_2 in C_1, C_2 respectively such that c_1 and c_2 are adjacent or form an opposite-colored bridge pattern as shown in Figure 7.

During a Hex game, chains can form that decompose the board.

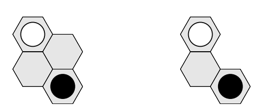

Figure 7: A bridge and a half-bridge between opposite-coloured cells. In the former, regarding the empty cells as non-adjacent enables decompositions.

Lemma 2.20. *Let H be a Hex position with an opposite-colored half-bridge composed of cells c_1, c_2, c_3 that are respectively black, white, uncolored, and let C_1, C_2 be the chains containing c_1, c_2 respectively. Then C_1 and C_2 touch in H.*

Proof. Let c_4 be the unique cell in H adjacent to c_1, c_2, and c_3. If c_4 is black then it is in C_1, and c_4 in C_1 is adjacent to c_2 in C_2. Similarly, the conclusion holds if c_4 is white. Lastly, if c_4 is uncolored, then C_1 and C_2 touch via the opposite-colored bridge $\{c_1 \ldots c_4\}$. □

Definition 2.21. A *Left-splitting decomposition* is a Left chain that touches both of Right's sides.

In a position with such a decomposition, if Left has a winning strategy, then Left has a connection strategy for each of the two subgames (connect the chain to one Left side; connect the chain to the other Left side), and combining these two strategies yields a winning Left strategy for the whole board [12].

Definition 2.22. A *four-sided decomposition* is a 4-cycle of consecutively touching chains. The *carrier* of this decomposition is the set of uncolored cells bounded by the four chains and their touching points.

If Left has a second-player strategy within the carrier of a four-sided decomposition that guarantees connection of the decomposition's two bounding Left chains, then the decomposition carrier is Left-captured [12].

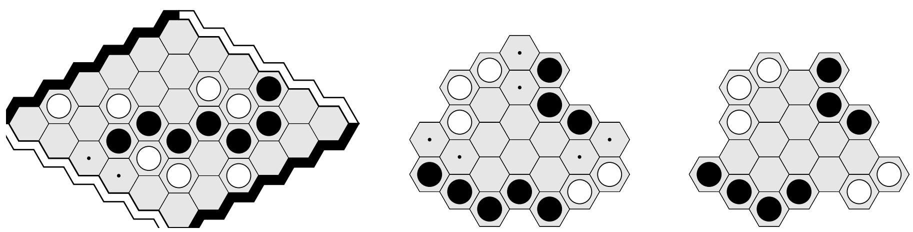

Figure 8: A Left-split decomposition, a four-sided decomposition, and the same four-sided decomposition using half-bridges. Dotted cells show opposite-coloured bridges of the decomposition. In the rightmost figure, all empty cells are black-captured.

3. Captured-Reversible Moves

Just as 'dead' leads to 'dead-reversible', so 'captured' leads to 'captured-reversible'. The following definition may seem counter-intuitive, since the *opponent's* move — not the player's — yields the *player's* fillin.

Definition 3.1. A cell c in position H is *Left-captured-reversible* if there is a cell r such that $H + R(r)$ has Left-fillin F containing c.

Lemma 3.2. *Let cell c be Left-captured-reversible in Hex position H. Then cell c is Left-reversible in H.*

Proof. By Definition 3.1, some Right-move r in H yields Left-fillin F containing c. By monotonicity, $H + L(F) + R(r) \geq H + L(c) + R(r) \geq H + R(r)$. By Observation 2.15, $H + R(r) \equiv H + L(F) + R(r)$, so $H + R(r) \equiv H + L(c) + R(r)$. By monotonicity, $H \geq H + R(r)$, so $H \geq H + R(r) \equiv H + L(c) + R(r)$, satisfying the definition of Left-reversible. □

Captured-reversible cells are reversible, so they can be bypassed. However, we would like to prune them from consideration completely, as is done with dead-reversible cells. It is an open question whether all capture-reversible cells can be pruned while preserving a position's outcome. However, we now show sufficient conditions for such pruning.

Definition 3.3. For a Left-captured-reversible move m with Right-reverser r and Left-fillin F, we call F a *Left-captured-reversible carrier* of m. With respect to carriers F_1, F_2 and Right-reversers r_1, r_2 of Left-captured-reversible moves m_1, m_2, we say that m_1 and m_2 *interfere* if r_1 is in F_2 or r_2 is in F_1. The *Left-captured-reversible graph* $G_{\gamma(H,L)}$ of position H is defined as follows:

- for each Left-captured-reversible move m_j in H, select a Right-reverser r_j and carrier F_j,
- vertices of $G_{\gamma(H,L)}$ correspond to the moves m_j,
- vertices are adjacent if and only if their corresponding moves interfere.

An independent vertex set in $G_{\gamma(H,L)}$ is called an *independent Left-captured-reversible set* in position H.

Lemma 3.4. *Let H_1 be a position with an uncolored cell c and an independent Left-captured-reversible set $I_1 = \{m_1, \ldots, m_n\}$, where each captured-reversible cell m_j has selected Right-reverser r_j and carrier F_j. Then in position $H_2 = H_1 + R(c)$ the set $I_2 = \{m_j \in I_1 : c \notin \{r_j\} \cup F_j\}$ is an independent Left-captured-reversible set.*

Proof. Each m_j in I_2 is Left-captured-reversible in H_2 since r_j remains a legal move for Right, and F_j remains fillin for any continuation of $H_1 + R(r_j)$ in which all cells in F_j are uncolored. In defining the Left-captured-reversible graph $G_{\gamma(H_2,L)}$, we can select the same reversers and carriers for all cells in I_2 to guarantee independence. □

Theorem 3.5. *Let H be a Hex position with a set of dead cells D, a set of Left-dead-reversible cells V, and an independent Left-captured-reversible set I. If Left wins $H[L]$, then either Left has a winning move not in D or V or I, or Left wins $H[R]$.*

Proof. Proof by contradiction. Let H be a counterexample with the smallest number of uncolored cells. Thus, Left has a winning move in $H[L]$, but each such move is in D or V or I, and Right wins $H[R]$. For $H[L]$, let $W = \{m_1, \ldots, m_n\}$ be the set of Left-winning moves not in D or V. Pruning dead and dead-reversible moves cannot eliminate all winning moves (Theorem 2.10), so W is a nonempty subset of I.

For some $m_j \in W$, let $T_j = H + R(r_j)$, where r_j is the Right-reverser of m_j. By the definition of captured-reversible, it follows that $T_j \equiv H + L(m_j) + R(r_j)$: see the proof of Lemma 3.2. Left wins $(H + L(m_j))[R]$, so Left wins $(H + L(m_j) + R(r_j))[L] \equiv T_j[L]$. By monotonicity $H \geq T_j$, so Right wins $T_j[R]$.

Since T_j has fewer uncolored cells than H, it is not a counterexample to this theorem. Thus in position T_j, for any dead cell set D_j, Left-dead-reversible cell set V_j, and independent Left-captured-reversible set I_j, Left has a winning move not in D_j or V_j or I_j.

Dead cells are dead in all continuations (Observation 2.4), and Left-dead-reversible cells are dead or Left-dead-reversible in all continuations in which only Right-colored cells are added (Observation 2.9). Thus we can select D_j, V_j such that $D_j \cup V_j \supseteq (D \cup V) \setminus \{r_j\}$. Also, we can select I_j to be the set of cells in I whose Right-reverser is not r_j. By Lemma 3.4, I_j is an independent Left-captured-reversible set in T_j. Thus some winning Left-move m of $T_j[L]$ is not in D_j or V_j or I_j.

Since $T_j = H + R(r_j)$, m is also winning in $H[L]$. Thus, by our assumption, m is in $(D \cup V \cup I) \setminus (D_j \cup V_j \cup I_j \cup \{r_j\}) \subseteq I \setminus I_j$. Thus m is Left-captured-reversible with Right-reverser r_j in H, meaning that m is Left-captured in T_j. By Lemma 2.13, $T_j > 0$, contradicting the fact that R wins $T_j[R]$. □

If Left wins $H[R]$, then any legal move in state $H[L]$ is Left-winning. Thus we can apply Theorem 3.5 as follows: given a Hex position in which we are trying to find a Left-winning move, we can identify dead cells, Left-dead-reversible cells, and an independent Left-captured-reversible set, and prune all these inferior cells from consideration with the caveat that we consider at least one legal move.

Using known captured patterns allows us to identify captured-reversible patterns.

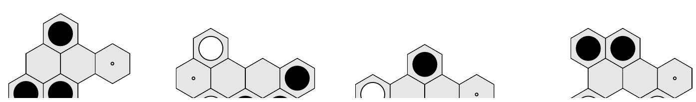

Figure 9: Some Left-captured-reversible patterns. For any containing position, white-coloring the dotted cell black-captures the other two uncolored cells.

4. Star Decomposition Domination

As mentioned in §2, the carrier C of a four-sided decomposition is Left-captured if Left has a second-player connection strategy joining the two bounding Left-chains within C. Thus if a Left-move m creates such a four-sided decomposition, then m fillin-dominates all moves in C.

Since Hex has no draws, it follows that every four-sided decomposition falls into one of three cases: the carrier is Left-captured, the carrier is Right-captured, or each player has a first-player strategy to connect their bounding chains within the carrier. In this last case, each player has a move available that captures the carrier for themselves and — by fillin domination — no other move in the carrier need be considered. Due to the strategic resemblance to the nimber $* = \{0|0\}$, we call such four-sided decompositions *star decompositions*.

Definition 4.1. A four-sided decomposition is a *star decomposition* if each player has a first-player connection strategy to join their two bounding chains within the carrier.

Unlike the situation with captured-carrier decompositions, a move that creates a star decomposition need not dominate all cells inside the carrier. However, some domination can be deduced by determining if coloring cells inside the carrier does not alter the decomposition's outcome.

Theorem 4.2. *Let H be a position such that a Left-move m yields a star decomposition with carrier C. Let $D \subseteq C$ be a set of cells on which Right has a first-player strategy to connect his two bounding chains. Then in H, m Left-dominates every cell in $C \setminus D$.*

Proof. If we can show that $H + L(m) \equiv H + L(m) + L(C \setminus D)$, then the result follows immediately by monotonicity. Since the cells in $C \setminus D$ are within the star decomposition, they can only affect Right's strategy within the decomposition.

If Left is the first to play inside the star decomposition carrier, then C becomes Left-captured, so the Left-coloring of cells $C \setminus D$ does not alter the position's outcome as they would be assigned to Left in either case.

If Right is the first to play inside the star decomposition carrier, then D becomes Right-captured since Left's additional cells do not obstruct Right's connection strategy on D. Let X be a continuation of $H + L(m)$ such that $\chi_U(X) \supseteq C$. In the position $X + R(D)$, the cells $C \setminus D$ are dead. So $X + L(C \setminus D) + R(D) \equiv X + R(D) \equiv X + R(C)$ since dead cells can be assigned any color without altering a position's outcome. Thus once again the Left-coloring of cells $C \setminus D$ does not alter the position's outcome. □

In other words, Right's star decomposition strategy is not adversely affected by Left-coloring $C \setminus D$.

Corollary 4.3. *Let H be a Hex position such that a Left-move m creates a star decomposition with carrier C. Let $I \subseteq C$ be the set of cells intersecting all of Right's* *first-player strategies to connect his two bounding chains within C. Then in H, m Left-dominates every cell in $C \setminus I$.*

Proof. Repeatedly apply Theorem 4.2 to every such Right first-player strategy. □

Star decompositions often allow the pruning of moves that cannot be pruned by the other techniques mentioned in this paper.

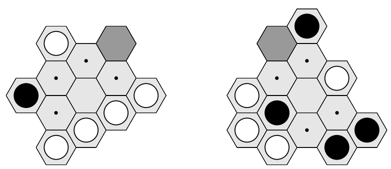

Figure 10: Star decomposition domination. For any containing position, black-coloring a pattern's shaded cell forms a star decomposition and dominates black-coloring any of the pattern's dotted cells.

5. Conclusions

By examining previous inferior cell analysis in terms of CGT, we have identified two new classes of inferior cells for the game of Hex. It remains an open question whether captured-reversible cells can be unconditionally pruned. Also, it would be interesting to know whether decomposition domination exists in other combinatorial games.

Acknowledgements. We thank Broderick Arneson, Martin Müller for helpful comments on an early version of this work, and David Spies for a detailed critique of a later version.

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