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EHRHART POLYNOMIALS OF INTEGRAL SIMPLICES WITH PRIME VOLUMES

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Abstract

For an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ of dimension d, we denote by $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ the δ-vector of $\mathcal{P}$, and $\text{vol}(\mathcal{P}) = \sum_{i=0}^{d} \delta_i$ its normalized volume. In this paper, we will establish new equalities and inequalities on δ-vectors for integral simplices whose normalized volumes are prime. Moreover, by using those, we will classify all the possible δ-vectors of integral simplices with normalized volume 5 and 7.

1. Introduction

One of the most fascinating problems in enumerative combinatorics is the characterization of the δ-vectors of integral convex polytopes.

Let $\mathcal{P} \subset \mathbb{R}^N$ be an *integral* convex polytope of dimension d, that is, a convex polytope any of whose vertices has integer coordinates. Let $\partial\mathcal{P}$ denote the boundary of $\mathcal{P}$. Given a positive integer n, we define

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N|, \qquad i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|,$$

where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ and $|X|$ is the cardinality of a finite set X. The enumerative function $i(\mathcal{P}, n)$ has the following fundamental properties, which were studied originally in the work of Ehrhart [2]:

- $i(\mathcal{P}, n)$ is a polynomial in n of degree d;
- $i(\mathcal{P}, 0) = 1$;

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- (loi de réciprocité) $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$ for every integer $n > 0$.

This polynomial $i(\mathcal{P}, n)$ is called the *Ehrhart polynomial* of $\mathcal{P}$. We refer the reader to [1, Chapter 3], [3, Part II] or [12, pp. 235–241] for the introduction to the theory of Ehrhart polynomials.

We define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1 - \lambda)^{d+1} \sum_{n=0}^{\infty} i(\mathcal{P}, n)\lambda^n = \sum_{i=0}^{\infty} \delta_i \lambda^i. \tag{1}$$

Then, from a fundamental result on generating functions ([12, Corollary 4.3.1]), we know that $\delta_i = 0$ for $i > d$. We call the integer sequence

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d),$$

which appears in (1), the *δ-vector* of $\mathcal{P}$. The δ-vector has the following properties:

- $\delta_0 = 1$, $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d + 1)$ and $\delta_d = |(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|$. Hence, $\delta_1 \geq \delta_d$. In particular, when $\delta_1 = \delta_d$, $\mathcal{P}$ must be a simplex.
- Each δ_i is nonnegative ([11]).
- If $(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N$ is nonempty, then one has $\delta_1 \leq \delta_i$ for every $1 \leq i \leq d - 1$ ([4]).
- The leading coefficient $(\sum_{i=0}^{d} \delta_i)/d!$ of $i(\mathcal{P}, n)$ is equal to the usual volume of $\mathcal{P}$ ([12, Proposition 4.6.30]). In particular, the positive integer $\mathrm{vol}(\mathcal{P}) = \sum_{i=0}^{d} \delta_i$ is said to be the *normalized volume* of $\mathcal{P}$.

Recently, the δ-vectors of integral convex polytopes have been studied intensively. For example, see [7, 14, 15].

There are two well-known inequalities on δ-vectors. Let $s = \max\{i : \delta_i \neq 0\}$. One is

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i}, \quad 0 \leq i \leq s. \tag{2}$$

This is proved by Stanley [13]. Another one is

$$\delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_1 + \delta_2 + \cdots + \delta_i + \delta_{i+1}, \quad 0 \leq i \leq d - 1. \tag{3}$$

This appears in the work of Hibi [4, Remark (1.4)].

On the classification problem on δ-vectors of integral convex polytopes, the above inequalities (2) and (3) characterize the possible δ-vectors completely when $\sum_{i=0}^{d} \delta_i \leq 3$ ([6, Theorem 0.1]). Moreover, when $\sum_{i=0}^{d} \delta_i = 4$, the possible δ-vectors are determined completely by (2) and (3) together with an additional condition ([5, Theorem 5.1]). Furthermore, from the proofs of [6, Theorem 0.1] and [5, Theorem

5.1], we know that all the possible δ-vectors can be realized as the δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i \leq 4$. However, unfortunately, this is no longer true when $\sum_{i=0}^{d} \delta_i = 5$. (See [5, Remark 5.2].) Hence, for the further classifications of δ-vectors, it is natural to study δ-vectors of integral simplices at first. Even for non-simplex cases, every convex polytope can be triangulated into finitely many simplices and we can compute the δ-vector of an integral convex polytope from its triangulation. Hence, the investigations of δ-vectors of integral simplices are an essential and important work.

In this paper, we establish new constraints on δ-vectors for integral simplices whose normalized volumes are prime numbers. The following theorem is our main result of this paper.

Theorem 1.1. *Let $\mathcal{P}$ be an integral simplex of dimension d and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ its δ-vector. Suppose that* $\mathrm{vol}(\mathcal{P}) = \sum_{i=0}^{d} \delta_i = p$ *is an odd prime number. Let $i_1, \ldots, i_{p-1}$ be the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_{p-1}}$ with $1 \leq i_1 \leq \cdots \leq i_{p-1} \leq d$. Then,*

(a) (cf. [7, Theorem 2.3]) *one has*

$$i_1 + i_{p-1} = i_2 + i_{p-2} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} \leq d + 1;$$

(b) one has

$$i_k + i_\ell \geq i_{k+\ell} \ \textit{ for } \ 1 \leq k \leq \ell \leq p-1 \ \textit{ with } \ k + \ell \leq p-1.$$

We give a proof of Theorem 1.1 in Section 2. We also see in Lemma 2.3 that the number of the inequalities in Theorem 1.1 (b) can be reduced by using the relations in (a).

Now, we remark that the part (a) of Theorem 1.1 is not a new result in some sense. In [7, Theorem 2.3], the author proved that for an integral simplex $\mathcal{P}$ with prime normalized volume, if $i_1 + i_{p-1} = d + 1$, then $\mathcal{P}$ is shifted symmetric, i.e., we have $i_1 + i_{p-1} = i_2 + i_{p-2} = \cdots = i_{(p-1)/2} + i_{(p+1)/2}$. Moreover, it follows from an easy observation that every integral simplex with prime normalized volume is either a simplex with $i_1 + i_{p-1} = d + 1$ or a polytope which is obtained by an iteration of taking pyramid at height 1 over such a simplex. Since taking such a pyramid does not change the normalized volume and the polynomial $1 + \sum_{j=1}^{p-1} t^{i_j}$, the equalities $i_1 + i_{p-1} = i_2 + i_{p-2} = \cdots = i_{(p-1)/2} + i_{(p+1)/2}$ also hold for the case where $i_1 + i_{p-1} < d + 1$. On the other hand, in this paper, we give an another proof of this statement. More precisely, we provide an elementary proof of Theorem 1.1 (a) in terms of some abelian groups associated with integral simplices.

In addition, as an application of Theorem 1.1, we give a complete characterization of the possible δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i = 5$ and 7.

Theorem 1.2. *Given a sequence* $(\delta_0, \delta_1, \ldots, \delta_d)$ *of nonnegative integers, where* $\delta_0 = 1$ *and* $\sum_{i=0}^{d} \delta_i = 5$*, there exists an integral simplex* $\mathcal{P}$ *of dimension* d *whose* δ*-vector coincides with* $(\delta_0, \delta_1, \ldots, \delta_d)$ *if and only if* $i_1, \ldots, i_4$ *satisfy* $i_1 + i_4 = i_2 + i_3 \leq d+1$ *and* $i_k + i_\ell \geq i_{k+\ell}$ *for* $1 \leq k \leq \ell \leq 4$ *with* $k + \ell \leq 4$*, where* $i_1, \ldots, i_4$ *are the positive integers such that* $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_4}$ *with* $1 \leq i_1 \leq \cdots \leq i_4 \leq d$*.*

Theorem 1.3. *Given a sequence* $(\delta_0, \delta_1, \ldots, \delta_d)$ *of nonnegative integers, where* $\delta_0 = 1$ *and* $\sum_{i=0}^{d} \delta_i = 7$*, there exists an integral simplex* $\mathcal{P}$ *of dimension* d *whose* δ*-vector coincides with* $(\delta_0, \delta_1, \ldots, \delta_d)$ *if and only if* $i_1, \ldots, i_6$ *satisfy* $i_1 + i_6 = i_2 + i_5 = i_3 + i_4 \leq d + 1$ *and* $i_k + i_\ell \geq i_{k+\ell}$ *for* $1 \leq k \leq \ell \leq 6$ *with* $k + \ell \leq 6$*, where* $i_1, \ldots, i_6$ *are the positive integers such that* $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_6}$ *with* $1 \leq i_1 \leq \cdots \leq i_6 \leq d$*.*

The "Only if" parts of Theorem 1.2 and 1.3 follow from Theorem 1.1. A proof of the "If" part of Theomre 1.2 is given in Section 3 and that of Theorem 1.3 is given in Section 4. Moreover, in Section 5, we note some problems towards the classification of δ-vectors of integral convex polytopes with any normalized volume.

2. A Proof of Theorem 1.1

The goal of this section is to give a proof of Theorem 1.1.

First of all, we recall the well-known combinatorial technique used to compute the δ-vector of an integral simplex. Given an integral simplex $\mathcal{F} \subset \mathbb{R}^N$ of dimension d with the vertices $v_0, v_1, \ldots, v_d \in \mathbb{Z}^N$, we set

$$\mathrm{Box}(\mathcal{F}) = \left\{ \alpha \in \mathbb{Z}^{N+1} : \alpha = \sum_{i=0}^{d} r_i (v_i, 1), \ \ 0 \leq r_i < 1 \right\}.$$

We define the degree of $\alpha = \sum_{i=0}^{d} r_i(v_i, 1) \in \mathrm{Box}(\mathcal{F})$ to be $\deg(\alpha) = \sum_{i=0}^{d} r_i$, i.e., the last coordinate of α. Then we have the following:

Lemma 2.1 (cf. [3, Proposition 27.7]). *Let* $\delta(\mathcal{F}) = (\delta_0, \delta_1, \ldots, \delta_d)$*. Then each* δ_i *is equal to the number of integer points* $\alpha \in \mathrm{Box}(\mathcal{F})$ *with* $\deg(\alpha) = i$*.*

For $\alpha = \sum_{i=0}^{d} r_i(v_i, 1)$ and $\beta = \sum_{i=0}^{d} s_i(v_i, 1)$ in $\mathrm{Box}(\mathcal{F})$, we define an operation in $\mathrm{Box}(\mathcal{F})$ by setting $\alpha \oplus \beta := \sum_{i=0}^{d} \{r_i + s_i\}(v_i, 1)$, where $\{r\} = r - \lfloor r \rfloor$ denotes the fractional part of a rational number r. Then it is clear that $0 \leq \{r_i + s_i\} < 1$ and $\sum_{i=0}^{d} \{r_i + s_i\}(v_i, 1) \in \mathbb{Z}^{N+1}$. Moreover, when we set $\alpha' = \sum_{i=0}^{d} \{1 - r_i\}(v_i, 1)$, one has $\alpha' \in \mathrm{Box}(\mathcal{F})$ and $\alpha \oplus \alpha' = (0, \ldots, 0)$. This means that $\mathrm{Box}(\mathcal{F})$ has the structure of an abelian group with respect to $\oplus$ whose unit is $\mathbf{0} = (0, \ldots, 0) \in \mathrm{Box}(\mathcal{F})$. (Throughout this paper, in order to distinguish the operation in $\mathrm{Box}(\mathcal{F})$ from the usual addition, we use the notation $\oplus$, which is not a direct sum.) For a positive integer m and $g \in \mathrm{Box}(\mathcal{F})$, let mg denote $\underbrace{g \oplus \cdots \oplus g}_{m}$.

We prove Theorem 1.1 using the above notation.

Proof of Theorem 1.1. Let $v_0, v_1, \ldots, v_d$ be the vertices of the integral simplex $\mathcal{P}$ and $\mathrm{Box}(\mathcal{P})$ the abelian group as above. Since $\sum_{i=0}^{d} \delta_i = \mathrm{vol}(\mathcal{P}) = p$ is prime, it follows from Lemma 2.1 that $|\mathrm{Box}(\mathcal{P})|$ is also prime. Hence, $\mathrm{Box}(\mathcal{P}) \cong \mathbb{Z}/p\mathbb{Z}$.

(a) Write $g_1, \ldots, g_{p-1}$ for $(p-1)$ distinct elements belonging to $\mathrm{Box}(\mathcal{P}) \setminus \{\mathbf{0}\}$ with $\deg(g_j) = i_j$ for $1 \le j \le p-1$, that is, $\mathrm{Box}(\mathcal{P}) = \{\mathbf{0}, g_1, \ldots, g_{p-1}\}$. Then, for each g_j, there exists its inverse $-g_j$ in $\mathrm{Box}(\mathcal{P}) \setminus \{\mathbf{0}\}$. Let $g'_j = -g_j$. If $g_j = \sum_{q=0}^{d} r_q (v_q, 1)$, where $0 \le r_q < 1$, then $g'_j = \sum_{q=0}^{d} \{1 - r_q\}(v_q, 1)$. Thus, one has

$$\deg(g_j) + \deg(g'_j) = \sum_{q=0}^{d} (r_q + \{1 - r_q\}) \le \sum_{q=0}^{d} (r_q + 1 - r_q) = d + 1$$

for all $1 \le j \le p-1$.

For $1 \le j_1 \ne j_2 \le p-1$, let $g_{j_1} = \sum_{q=0}^{d} r_q^{(1)} (v_q, 1)$ and $g_{j_2} = \sum_{q=0}^{d} r_q^{(2)} (v_q, 1)$. Since $\mathrm{Box}(\mathcal{P}) \cong \mathbb{Z}/p\mathbb{Z}$, we know that g_{j_1} generates $\mathrm{Box}(\mathcal{P})$. This implies that g_{j_2} and g'_{j_2} can be written as $g_{j_2} = t g_{j_1}$ and $g'_{j_2} = t g'_{j_1}$ for some integer $t \in \{2, \ldots, p-1\}$, respectively. Thus, we have

$$\begin{aligned}\sum_{q=0}^{d} (r_q^{(2)} + \{1 - r_q^{(2)}\}) &= \deg(g_{j_2}) + \deg(g'_{j_2}) = \deg(t g_{j_1}) + \deg(t g'_{j_1}) \\ &= \sum_{q=0}^{d} (\{t r_q^{(1)}\} + \{t(1 - r_q^{(1)})\}).\end{aligned}$$

Moreover, since $p g_{j_1} = \mathbf{0}$, we have $\{p r_q^{(1)}\} = 0$ for $0 \le q \le d$. This means that the denominator of each rational number $r_q^{(1)}$ must be p. Hence, if $0 < r_q^{(1)} < 1$, then $0 < \{t r_q^{(1)}\} < 1$ and $0 < \{t(1 - r_q^{(1)})\} < 1$, so $r_q^{(1)} + \{1 - r_q^{(1)}\} = \{t r_q^{(1)}\} + \{t(1 - r_q^{(1)})\} = 1$. In addition, obviously, if $r_q^{(1)} = \{1 - r_q^{(1)}\} = 0$, then $\{t r_q^{(1)}\} = \{t(1 - r_q^{(1)})\} = 0$, so $r_q^{(1)} + \{1 - r_q^{(1)}\} = \{t r_q^{(1)}\} + \{t(1 - r_q^{(1)})\} = 0$. Thus, $\deg(g_{j_1}) + \deg(g'_{j_1}) = \deg(g_{j_2}) + \deg(g'_{j_2})$. Let $i'_j = \deg(g'_j)$. Then we obtain

$$i_1 + i'_1 = \cdots = i_{(p-1)/2} + i'_{(p-1)/2} = i_{(p+1)/2} + i'_{(p+1)/2} = \cdots = i_{p-1} + i'_{p-1} \le d + 1. \tag{4}$$

By our assumption, we have the inequalities $i_1 \le \cdots \le i_{p-1}$. Moreover, from (4), we also have $i'_{p-1} \le \cdots \le i'_1$. Thus, we conclude that $i'_j = i_{p-j}$ for $1 \le j \le (p-1)/2$. Therefore, we obtain the desired

$$i_1 + i_{p-1} = i_2 + i_{p-2} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} \le d + 1.$$

(b) Let k and ℓ be integers satisfying $1 \leq k \leq \ell \leq p-1$ and $k+\ell \leq p-1$. Write $g_1, \ldots, g_\ell \in \mathrm{Box}(\mathcal{P}) \setminus \{\mathbf{0}\}$ for ℓ distinct elements with $\deg(g_j) = i_j$ for $1 \leq j \leq \ell$ and set $A = \{g_1, \ldots, g_\ell\} \cup \{\mathbf{0}\}$ and $B = \{g_1, \ldots, g_k\} \cup \{\mathbf{0}\}$. Now, the Cauchy–Davenport Theorem (cf. [10, Theorem 2.2]) guarantees that $|A \oplus B| \geq \min\{p, |A|+|B|-1\}$, where $A \oplus B = \{a \oplus b : a \in A, b \in B\}$. Clearly, $\mathbf{0}$ belongs to $A \oplus B$. Moreover, since $|A|+|B|-1 = k+\ell+1 \leq p$, it follows that $A \oplus B$ contains at least $(k+\ell)$ distinct elements in $\mathrm{Box}(\mathcal{P}) \setminus \{\mathbf{0}\}$. In addition, for each $g \in A \oplus B$, g satisfies $\deg(g) \leq i_k + i_\ell$. Indeed, for non-zero elements $g_j \in A$ and $g_{j'} \in B$, if they have expressions $g_j = \sum_{q=0}^{d} r_q(v_q, 1)$ and $g_{j'} = \sum_{q=0}^{d} r'_q(v_q, 1)$, then one has

$$\deg(g_j \oplus g_{j'}) = \sum_{q=0}^{d} \{r_q + r'_q\} \leq \sum_{q=0}^{d} (r_q + r'_q) = i_j + i_{j'} \leq i_k + i_\ell.$$

Hence, from the definition of $i_1, \ldots, i_{p-1}$, we obtain the inequalities $i_k + i_\ell \geq i_{k+\ell}$ for $1 \leq k \leq \ell \leq p-1$ with $k+\ell \leq p-1$, as desired. $\square$

Here we notice that some of the inequalities in Theorem 1.1 follow from (2) and (3).

Proposition 2.2. *Let $\mathcal{P}$ be an integral convex polytope of dimension d with its δ-vector $(\delta_0, \delta_1, \ldots, \delta_d)$ and $i_1, \ldots, i_{m-1}$ the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_{m-1}}$ with $1 \leq i_1 \leq \cdots \leq i_{m-1} \leq d$, where $m = \sum_{i=0}^{d} \delta_i$.*

(a) *The inequalities $i_j + i_{m-j-1} \geq i_{m-1}$, where $1 \leq j \leq m-2$, are equivalent to* (2).

(b) *The inequalities $i_j + i_{m-j} \leq d+1$, where $1 \leq j \leq m-1$, are equivalent to* (3).

Proof. (a) For each $1 \leq j \leq m-1$, the inequality $\delta_0 + \cdots + \delta_{i_j} \leq \delta_s + \cdots + \delta_{s-i_j}$ follows from (2). Then its left-hand side is at least $j+1$ by the definition of i_j. Thus, in particular, $j+1 \leq \delta_s + \cdots + \delta_{s-i_j}$. On the other hand, if we suppose that $s - i_j = i_{m-1} - i_j > i_{m-j-1}$, then we have $\delta_s + \cdots + \delta_{s-i_j} \leq \delta_{i_{m-1}} + \cdots + \delta_{i_{m-j-1}+1} \leq j$ because of the nonnegativity of δ_i and the definition of i_j, a contradiction. Thus one has $i_{m-1} - i_j \leq i_{m-j-1}$, which means $i_j + i_{m-j-1} \geq i_{m-1}$. On the contrary, assume that $i_j + i_{m-j-1} \geq i_{m-1}$. For each k with $0 \leq k \leq i_{m-1} = s$, there exists a unique j with $0 \leq j \leq m-1$ such that $i_j \leq k < i_{j+1}$, where we let $i_0 = 0$ and $i_m = d+1$. Thus,

$$\begin{aligned}\delta_s + \cdots + \delta_{s-k} - (\delta_0 + \cdots + \delta_k) &= \delta_{i_{m-1}} + \cdots + \delta_{i_{m-1}-k} - (j+1) \\ &\geq \delta_{i_{m-1}} + \cdots + \delta_{i_{m-1}-i_j} - (j+1) \geq \delta_{i_{m-1}} + \cdots + \delta_{i_{m-j-1}} - (j+1) \geq 0.\end{aligned}$$

(b) For each $1 \leq j \leq m-1$, the inequality $\delta_1 + \cdots + \delta_{d+1-i_j} \geq \delta_d + \cdots + \delta_{i_j}$ follows from (3). Then its right-hand side is at least $m-j$. Thus, it must be

$d + 1 - i_j \geq i_{m-j}$, which means $i_j + i_{m-j} \leq d + 1$. On the contrary, assume that $i_j + i_{m-j} \leq d+1$. For each k with $1 \leq k \leq d$, there exists a unique j with $1 \leq j \leq m$ such that $i_{j-1} < k \leq i_j$. Thus,

$$\begin{aligned}\delta_1 + \cdots + \delta_{d+1-k} - (\delta_d + \cdots + \delta_k) &\geq \delta_1 + \cdots + \delta_{d+1-i_j} - (m-j)\\ &\geq \delta_1 + \cdots + \delta_{i_{m-j}} - (m-j) \geq 0,\end{aligned}$$

as required. □

As is shown above, the inequalities $i_j + i_{m-j-1} \geq i_{m-1}$ and $i_j + i_{m-j} \leq d + 1$ are not new. However, the inequalities $i_k + i_\ell \geq i_{k+\ell}$ include a lot of new ones. See Remark 4.1 and Example 4.2 below.

Moreover, it is easy to see that we can reduce the number of the inequalities in Theorem 1.1 (b) by using the equalities $i_1 + i_{p-1} = i_2 + i_{p-2} = \cdots = i_{(p-1)/2} + i_{(p+1)/2}$. We claim the following:

Lemma 2.3. *Assume $p > 3$. Given Theorem 1.1(a), Theorem 1.1 (b) is equivalent to*

$$i_k + i_\ell \geq i_{k+\ell} \ \textit{ for } \ 1 \leq k \leq \left\lfloor \frac{p-1}{3} \right\rfloor \ \textit{ and } \ k \leq \ell \leq \left\lfloor \frac{p-k}{2} \right\rfloor.$$

Proof. When $k > \lfloor (p-1)/3 \rfloor$, since $p > 3$, we have $k > p/3$. Thus, $k + 2\ell \geq 3k > p$, equivalently, $p - k - \ell < \ell$. By using $i_{k+\ell} + i_{p-k-\ell} = i_\ell + i_{p-\ell}$, we obtain $i_k + i_\ell - i_{k+\ell} = i_k + i_{p-k-\ell} - i_{p-\ell} \geq 0$, which is $i_k + i_{p-k-\ell} \geq i_{p-\ell}$, where $p - k - \ell < \ell$. Similarly, when $\ell > \lfloor (p-k)/2 \rfloor$, we have $k + 2\ell > p$. Thus, we can deduce $i_k + i_{p-k-\ell} \geq i_{p-\ell}$. □

3. The Possible δ-Vectors of Integral Simplices With Normalized Volume 5

In this section, we give a proof of the "If" part of Theorem 1.2. Namely, we show that if the integers $i_1, \ldots, i_4$ satisfy the conditions in Theorem 1.1 (a) and (b), then the integer sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ defined from $i_1, \ldots, i_4$ is the δ-vector of an integral simplex of dimension d with normalized volume 5.

Let $(\delta_0, \delta_1, \ldots, \delta_d)$ be a nonnegative integer sequence with $\delta_0 = 1$ and $\sum_{i=0}^{d} \delta_i = 5$ which satisfies

$$i_1 + i_4 = i_2 + i_3 \leq d + 1, \ 2i_1 \geq i_2 \text{ and } i_1 + i_2 \geq i_3,$$

where $i_1, \ldots, i_4$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_4}$ with $1 \leq i_1 \leq \cdots \leq i_4 \leq d$. We note that the inequalities $2i_1 \geq i_2$ and $i_1 + i_2 \geq i_3$

come from Lemma 2.3 instead of the inequalities in Theorem 1.1 (b), which are necessary conditions for $(\delta_0, \delta_1, \ldots, \delta_d)$ to be a δ-vector of some integral simplex. By the conditions $\delta_0 = 1$, $\sum_{i=0}^{d} \delta_i = 5$ and $i_1 + i_4 = i_2 + i_3 \leq d+1$, the only possible sequences look like

(i) $(1, \underbrace{0, \ldots, 0}_{q_1}, 4, \underbrace{0, \ldots, 0}_{q_2})$ with $q_1 \leq q_2$;

(ii) $(1, \underbrace{0, \ldots, 0}_{q_1}, 2, 0, \ldots, 0, 2, \underbrace{0, \ldots, 0}_{q_2})$ with $q_1 \leq q_2$;

(iii) $(1, \underbrace{0, \ldots, 0}_{q_1}, 1, \underbrace{0, \ldots, 0}_{q_2}, 2, \underbrace{0, \ldots, 0}_{q_3}, 1, \underbrace{0, \ldots, 0}_{q_4})$ with $q_1 \leq q_4$ and $q_2 = q_3$;

(iv) $(1, \underbrace{0, \ldots, 0}_{q_1}, 1, \underbrace{0, \ldots, 0}_{q_2}, 1, 0, \ldots, 0, 1, \underbrace{0, \ldots, 0}_{q_3}, 1, \underbrace{0, \ldots, 0}_{q_4})$ with $q_1 \leq q_4$ and $q_2 = q_3$.

Our work is to find integral simplices whose δ-vectors are of the above forms also satisfying the remaining two inequalities $2i_1 \geq i_2$ and $i_1 + i_2 \geq i_3$, which will be used below.

In order to construct integral simplices, we prepare the following. For positive integers d and m and nonnegative integers $d_1, \ldots, d_{m-1}$ satisfying $d_1 + \cdots + d_{m-1} \leq d-1$, we define the following $d \times d$ integer matrix:

$$A_m(d_1, \ldots, d_{m-1}) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ * & \cdots & \cdots & * & m \end{pmatrix}, \tag{5}$$

where there are d_1 1's between the first and the d_1th entries and d_j j's between the $(d_{j-1}+1)$th and the d_jth entries for $2 \leq j \leq m-1$ among $*$'s, and the rest entries are all 0. For example, if $d = 5$, then $A_7(1, 1, 0, 2, 0, 0)$ is the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 4 & 4 & 7 \end{pmatrix}$. By determining integers $m, d_1, \ldots, d_{m-1}$, we obtain an integer matrix $A_m(d_1, \ldots, d_{m-1})$ and we define the integral simplex $\mathcal{P}_m(d_1, \ldots, d_{m-1})$ of dimension d from the matrix (5) by setting

$$\mathcal{P}_m(d_1, \ldots, d_{m-1}) = \mathrm{conv}(\{\mathbf{0}, v_1, \ldots, v_d\}) \subset \mathbb{R}^d,$$

where v_i is the ith row vector of (5). The following lemma enables us to compute $\delta(\mathcal{P}_m(d_1, \ldots, d_{m-1}))$ easily.

Lemma 3.1 ([5, Corollary 3.1]). *Let $\delta(\mathcal{P}_m(d_1,\ldots,d_{m-1})) = (\delta_0,\delta_1,\ldots,\delta_d)$. Then one has $\sum_{i=0}^{d}\delta_i t^i = 1+\sum_{i=1}^{m-1} t^{1-s_i}$, where $s_i = \left\lfloor \frac{i}{m} - \sum_{j=1}^{m-1}\left\{\frac{ij}{m}\right\} d_j \right\rfloor$ for $i = 1,\ldots,m-1$.*

Let $m = 5$. In the sequel, in each case of (i) – (iv) above, by giving concrete values of $d_1,\ldots,d_4$, we obtain the matrix $A_5(d_1,\ldots,d_4)$ and hence the integral simplex $\mathcal{P}_5(d_1,\ldots,d_4)$ whose δ-vector looks like each of (i) – (iv). The δ-vectors of such simplices can be computed by using Lemma 3.1.

3.1. The Case (i)

First, let us consider the case (i), namely, the nonnegative integer sequence like $(1,\underbrace{0,\ldots,0}_{q_1},4,\underbrace{0,\ldots,0}_{q_2})$ with $q_1 \le q_2$, which means that $i_1 = i_2 = i_3 = i_4$ and $q_1 = i_1 - 1 \le d - i_1 = q_2$. Set $i_1 = \cdots = i_4 = i$. Then we can define the polytope $\mathcal{P}_5(0,i-1,i-1,0)$. Indeed, one has, of course, $i-1 \ge 0$ and $i-1 \le d-i$, that is, $(i-1)+(i-1) \le d-1$. We can also calculate each s_i as follows:

$$s_1 = \left\lfloor \frac{1}{5} - \sum_{j=1}^{4}\left\{\frac{j}{5}\right\} d_j \right\rfloor = \left\lfloor \frac{1}{5} - \frac{2}{5}(i-1) - \frac{3}{5}(i-1) \right\rfloor = -i+1;$$
$$s_2 = \left\lfloor \frac{2}{5} - \sum_{j=1}^{4}\left\{\frac{2j}{5}\right\} d_j \right\rfloor = \left\lfloor \frac{2}{5} - \frac{4}{5}(i-1) - \frac{1}{5}(i-1) \right\rfloor = -i+1;$$
$$s_3 = \left\lfloor \frac{3}{5} - \sum_{j=1}^{4}\left\{\frac{3j}{5}\right\} d_j \right\rfloor = \left\lfloor \frac{3}{5} - \frac{1}{5}(i-1) - \frac{4}{5}(i-1) \right\rfloor = -i+1;$$
$$s_4 = \left\lfloor \frac{4}{5} - \sum_{j=1}^{4}\left\{\frac{4j}{5}\right\} d_j \right\rfloor = \left\lfloor \frac{4}{5} - \frac{3}{5}(i-1) - \frac{2}{5}(i-1) \right\rfloor = -i+1.$$

This implies that $\delta(\mathcal{P}_5(0,i-1,i-1,0))$ coincides with $(1,0,\ldots,4,0,\ldots,0)$ from Lemma 3.1, where $\delta_i = 4$.

Similar discussions can be applied to the rest cases (ii) – (iv).

3.2. The Case (ii)

In this case, we have $i_1 = i_2$, $i_3 = i_4$ and $i_1 - 1 \le d - i_3$. Let $i_1 = i_2 = i$ and $i_3 = i_4 = j$, where $i < j$. Then one has $2i \ge j$ and $i + j \le d+1$, which come from $i_1 + i_2 \ge i_3$ and $i_1 + i_4 \le d+1$. Thus we can define $\mathcal{P}_5(0, i, 2i-j, 2j-2i-2)$. Indeed, we have $2i-j \ge 0$, $2j-2i-2 \ge 0$ and $i+j-2 \le d-1$. Moreover, we obtain that its δ-vector coincides with (ii) since $s_1 = s_2 = -j+1$ and $s_3 = s_4 = -i+1$.

3.3. The Case (iii)

In this case, let $i_1 = i, i_2 = i_3 = j$ and $i_4 = k$. Then we can define $\mathcal{P}_5(0, 2i-j, i, 3j-3i-2)$ from $2i-j \geq 0$, $3j-3i-2 \geq 0$ and $2j-2 \leq d-1$ since $2i_1 \geq i_2$, $i < j$ and $i_2+i_3 \leq d+1$. Thus its δ-vector coincides with (iii) since $s_1 = -2j+i+1 = -k+1$, $s_2 = s_3 = -j+1$ and $s_4 = -i+1$.

3.4. The Case (iv)

In this case, we can define $\mathcal{P}_5(0, 2i_1 - i_2, i_1 + i_2 - i_3, i_2 + 2i_3 - 3i_1 - 2)$ from $2i_1-i_2 \geq 0, i_1+i_2-i_3 \geq 0, i_2+2i_3-3i_1-2 \geq 0$ and $i_2+i_3-2 \leq d-1$. Thus its δ-vector coincides with (iv) since $s_1 = i_1 - i_2 - i_3 + 1 = -i_4 + 1, s_2 = -i_3 + 1, s_3 = -i_2 + 1$ and $s_4 = -i_1 + 1$.

Remark 3.2. (a) The classification of the case (iv) is essentially given in [7, Lemma 4.3]. (b) Since we know $i_1 + i_4 = i_2 + i_3$, the inequality $2i_1 \geq i_2$ (resp. $i_1 + i_2 \geq i_3$) is equivalent to $i_1 + i_3 \geq i_4$ (resp. $2i_2 \geq i_4$). Thus, these two inequalities can be obtained from (2) (see Proposition 2.2 (a)). Hence, the possible δ-vectors of integral simplices with normalized volume 5 can be characterized only by Theorem 1.1 (a) and the inequalities (2).

4. The Possible δ-Vectors of Integral Simplices With Normalized Volume 7

In this section, similar to the previous section, we give a proof of the "If" part of Theorem 1.3.

Let $(\delta_0, \delta_1, \ldots, \delta_d)$ be a nonnegative integer sequence with $\delta_0 = 1$ and $\sum_{i=0}^{d} \delta_i = 7$ which satisfies

$$i_1 + i_6 = i_2 + i_5 = i_3 + i_4 \leq d+1,\ i_1 + i_l \geq i_{l+1} \text{ for } l = 1,2,3 \text{ and } 2i_2 \geq i_4,$$

where $i_1, \ldots, i_6$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_6}$ with $1 \leq i_1 \leq \cdots \leq i_6 \leq d$. We note that the inequalities $i_1 + i_l \geq i_{l+1}, l = 1,2,3$ and $2i_2 \geq i_4$ come from Lemma 2.3. By the conditions $\delta_0 = 1$, $\sum_{i=0}^{d} \delta_i = 7$ and $i_1 + i_6 = i_2 + i_5 = i_3 + i_4 \leq d+1$, only the possible sequences look like

(i) $(1, \underbrace{0, \ldots, 0}_{q_1}, 6, \underbrace{0, \ldots, 0}_{q_2})$ with $q_1 \leq q_2$;

(ii) $(1, \underbrace{0, \ldots, 0}_{q_1}, 3, 0, \ldots, 0, 3, \underbrace{0, \ldots, 0}_{q_2})$ with $q_1 \leq q_2$;

(iii) $(1,\underbrace{0,\ldots,0}_{q_1},1,\underbrace{0,\ldots,0}_{q_2},4,\underbrace{0,\ldots,0}_{q_3},1,\underbrace{0,\ldots,0}_{q_4})$ with $q_1 \le q_4$ and $q_2 = q_3$;

(iv) $(1,\underbrace{0,\ldots,0}_{q_1},2,\underbrace{0,\ldots,0}_{q_2},2,\underbrace{0,\ldots,0}_{q_3},2,\underbrace{0,\ldots,0}_{q_4})$ with $q_1 \le q_4$ and $q_2 = q_3$;

(v) $(1,\underbrace{0,\ldots,0}_{q_1},1,\underbrace{0,\ldots,0}_{q_2},2,0,\ldots,0,2,\underbrace{0,\ldots,0}_{q_3},1,\underbrace{0,\ldots,0}_{q_4})$ with $q_1 \le q_4$ and $q_2 = q_3$;

(vi) $(1,\underbrace{0,\ldots,0}_{q_1},2,\underbrace{0,\ldots,0}_{q_2},1,0,\ldots,0,1,\underbrace{0,\ldots,0}_{q_3},2,\underbrace{0,\ldots,0}_{q_4})$ with $q_1 \le q_4$ and $q_2 = q_3$;

(vii) $(1,\underbrace{0,\ldots,0}_{q_1},1,\underbrace{0,\ldots,0}_{q_2},1,\underbrace{0,\ldots,0}_{q_3},2,\underbrace{0,\ldots,0}_{q_4},1,\underbrace{0,\ldots,0}_{q_5},1,\underbrace{0,\ldots,0}_{q_6})$ with $q_1 \le q_6$, $q_2 = q_5$ and $q_3 = q_4$;

(viii) $(1,\underbrace{0,\ldots,0}_{q_1},1,\underbrace{0,\ldots,0}_{q_2},1,\underbrace{0,\ldots,0}_{q_3},1,0,\ldots,0,1,\underbrace{0,\ldots,0}_{q_4},1,\underbrace{0,\ldots,0}_{q_5},1,\underbrace{0,\ldots,0}_{q_6})$ with $q_1 \le q_6, q_2 = q_5$ and $q_3 = q_4$.

4.1. The Case (i)

Let $i_1 = \cdots = i_6 = i$. Then we can define $\mathcal{P}_7(0,0,i-1,i-1,0,0)$ from $i-1 \ge 0$ and $2i-2 \le d-1$. Thus, by Lemma 3.1, $\delta(\mathcal{P}_7(0,0,i-1,i-1,0,0))$ coincides with (i) since $s_1 = \cdots = s_6 = -i+1$.

4.2. The Case (ii)

Let $i_1 = \cdots = i_3 = i$ and $i_4 = \cdots = i_6 = j$. Then we can define $\mathcal{P}_7(0,j-i,2i-j,2i-j,0,2j-2i-2)$ from $j-i \ge 0, 2i-j \ge 0, 2j-2i-2 \ge 0$ and $i+j-2 \le d-1$. We also obtain that its δ-vector coincides with (ii) since $s_1 = s_2 = s_3 = -j+1$ and $s_4 = s_5 = s_6 = -i+1$.

4.3. The Case (iii)

Let $i_1 = i, i_2 = \cdots = i_5 = j$ and $i_6 = k$. Then we can define $\mathcal{P}_7(i+j-k,k-j,k-i-1,0,0,i-1)$ from $i+j-k \ge 0, k-j \ge 0, k-i-1 \ge 0, i-1 \ge 0$ and $i+k-2 \le d-1$. We also obtain that its δ-vector coincides with (iii) since $s_1 = \lfloor(-4i+j-4k+10)/7\rfloor = -j+1, s_2 = \lfloor(-i+2j-8k+13)/7\rfloor = -k+1, s_3 = \lfloor(-5i+3j-5k+9)/7\rfloor = -j+1, s_4 = \lfloor(-2i-3j-2k+12)/7\rfloor = -j+1, s_5 = \lfloor(-6i-2j+k+8)/7\rfloor = -i+1$ and $s_6 = \lfloor(-3i-j-3k+11)/7\rfloor = -j+1$.

4.4. The Case (iv)

Let $i_1 = i_2 = i, i_3 = i_4 = j$ and $i_5 = i_6 = k$. Then we can define $\mathcal{P}_7(0, 0, i-1, i+j-k, 0, 3k-3j-1)$ from $i-1 \geq 0, i+j-k \geq 0, 3k-3j-1 \geq 0$ and $2i-2j+2k-2 = i+k-2 \leq d-1$. We also obtain that its δ-vector coincides with (iv) since $s_1 = s_2 = -i+2j-2k+1 = -k+1, s_3 = s_4 = -i+j-k+1 = -j+1$ and $s_5 = s_6 = -i+1$.

4.5. The Case (v)

Let $i_1 = k_1, i_2 = i_3 = k_2, i_4 = i_5 = k_3$ and $i_6 = k_4$. Then we can define $\mathcal{P}_7(0, 2k_1-k_2, 0, k_2-k_1, k_1+k_2-k_3, 2k_3-2k_1-2)$ from $2k_1-k_2 \geq 0, k_2-k_1 \geq 0, k_1+k_2-k_3 \geq 0, 2k_3-2k_1-2 \geq 0$ and $k_2+k_3-2 \leq d-1$. We also obtain that its δ-vector coincides with (v) since $s_1 = k_1-k_2-k_3+1 = -k_4+1, s_2 = s_3 = -k_3+1, s_4 = s_5 = -k_2+1$ and $s_6 = -k_1+1$.

4.6. The Case (vi)

Let $i_1 = i_2 = k_1, i_3 = k_2, i_4 = k_3$ and $i_5 = i_6 = k_4$. Then we can define $\mathcal{P}_7(0, k_3-k_2-1, k_1+k_2-k_3, 2k_1-k_3, 0, k_2+2k_3-3k_1-1)$ from $k_3-k_2-1 \geq 0, k_1+k_2-k_3 \geq 0, 2k_1-k_3 \geq 0, k_2+2k_3-3k_1-1 \geq 0$ and $k_2+k_3-2 \leq d-1$. We also obtain that its δ-vector coincides with (vi) since $s_1 = s_2 = k_1-k_2-k_3+1 = -k_4+1, s_3 = -k_3+1, s_4 = -k_2+1$ and $s_5 = s_6 = -k_1+1$.

4.7. The Case (vii)

Let $i_1 = k_1, i_2 = k_2, i_3 = i_4 = k_3, i_5 = k_4$ and $i_6 = k_5$. Then we can define $\mathcal{P}_7(0, 0, 2k_1-k_2, k_1+k_2-k_3, k_2-k_1, 3k_3-2k_1-k_2-2)$ from $2k_1-k_2 \geq 0, k_1+k_2-k_3 \geq 0, k_2-k_1 \geq 0, 3k_3-2k_1-k_2-2 \geq 0$ and $2k_3-2 \leq d-1$. We also obtain that its δ-vector coincides with (vii) since $s_1 = k_1-2k_3+1 = -k_5+1, s_2 = k_2-2k_3+1 = -k_4+1, s_3 = s_4 = -k_3+1, s_5 = -k_2+1$ and $s_1 = -k_1+1$.

4.8. The Case (viii)

In this case, if $i_1+i_3 \geq 2i_2$, then we can define $\mathcal{P}_7(0, i_1+i_2-i_3, i_1+i_3-2i_2, 0, 2i_2-i_4, i_3+2i_4-2i_1-i_2-2)$ from $i_1+i_2-i_3 \geq 0, i_1+i_3-2i_2 \geq 0, 2i_2-i_4 \geq 0, i_3+2i_4-2i_1-i_2-2 \geq 0$ and $i_3+i_4-2 \leq d-1$. Similarly, if $i_1+i_3 \leq 2i_2$, then we can define $\mathcal{P}_7(0, 2i_1-i_2, 0, 2i_2-i_1-i_3, i_1+i_3-i_4, i_3+2i_4-2i_1-i_2-2)$. Moreover, each of their δ-vectors coincides with (viii) since $s_1 = i_1-i_3-i_4+1 = -i_6+1$, $s_2 = i_2-i_3-i_4+1 = -i_5+1$, $s_3 = -i_4+1$, $s_4 = -i_3+1$, $s_5 = -i_2+1$ and $s_6 = -i_1+1$.

Remark 4.1. Although the inequalities $i_1 + i_l \geq i_{l+1}, l = 1, 2, 3$, which are equivalent to $i_l + i_{6-l} \geq i_6, l = 1, 2, 3$, can be obtained from (2) (Proposition 2.2 (a)), the inequality $2i_2 \geq i_4$ comes from neither (2) nor (3). Moreover, when we discuss the cases (vi) and (viii), we need this new $2i_2 \geq i_4$. For example, the sequence $(1,0,2,0,1,1,0,2,0)$ cannot be the δ-vector of an integral simplex, while this satisfies $i_1 + i_l \geq i_{l+1}, l = 1,2,3$. Similarly, the sequence $(1,0,1,0,1,1,1,1,0,1,0)$ is also impossible to be the δ-vector of an integral simplex, while this satisfies $i_1 + i_l \geq i_{l+1}, l = 1,2,3$.

More generally, the following example shows that a lot of new inequalities are required to verify whether a given integer sequence is the δ-vector of some integral simplex.

Example 4.2. For a prime number p with $p \geq 7$, fix positive integers k and ℓ satisfying

$$1 \leq k \leq \left\lfloor \frac{p-1}{3} \right\rfloor \text{ and } k \leq \ell \leq \left\lfloor \frac{p-k}{2} \right\rfloor.$$

Let us consider the integer sequence

$$(\delta_0, \delta_1, \ldots, \delta_d) = (1, 0, \ell, 0, \underbrace{1, \ldots, 1}_{p-2\ell-1}, 0, \ell, 0) \in \mathbb{Z}^{d+1},$$

where $d = p-2\ell+5$. Then $i_1 = \cdots = i_\ell = 2$, $i_j = j-\ell+3$ for $j = \ell+1, \ldots, p-\ell-1$ and $i_{p-\ell} = \cdots = i_{p-1} = p-2\ell+4$, where $i_1, \ldots, i_{p-1}$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_{p-1}}$ with $1 \leq i_1 \leq \cdots \leq i_{p-1} \leq d$. Thus, one has $i_k + i_\ell = 4$ but $i_{k+\ell} = k+3$ or $i_{k+\ell} = p-2\ell+4$. In fact, since

$$p - \ell - (k+\ell) \geq p - k - 2\lfloor (p-k)/2 \rfloor \geq p - k - p + k = 0,$$

we have $\ell + 1 \leq k + \ell \leq p - \ell - 1$ or $k + \ell = p - \ell$. Hence, this integer sequence satisfies none of the inequalities $i_k + i_\ell \geq i_{k+\ell}$ when $k \geq 2$. On the other hand, this satisfies both $i_j + i_{p-j} = d+1$ for $1 \leq j \leq p-1$ and $i_j + i_{p-j-1} \geq i_{p-1}$ for $1 \leq j \leq p-2$. By Proposition 2.2, this sequence satisfies both (2) and (3).

We remark that since $\delta_1 = 0$, if there exists an integral convex polytope of dimension d whose δ-vector equals this sequence, then it must be a simplex. Therefore, thanks to Theorem 1.1 (b) (or equivalently Lemma 2.3), we see that there exists no integral convex polytope whose δ-vector equals this sequence, while we cannot determine whether this integer sequence is the δ-vector of some integral convex polytope only from (2) and (3).

5. Towards the Classification of δ-Vectors With Any Normalized Volume

Finally, we note some future problems on the classification of δ-vectors of integral convex polytopes.

5.1. Higher Prime Case

We remark that we cannot characterize the possible δ-vectors of integral simplices with higher prime normalized volumes only by Theorem 1.1, that is, Theorem 1.1 is not sufficient. In fact, since the volume of an integral convex polytope containing a unique integer point in its interior has an upper bound (see, e.g., [9]), if p is a sufficiently large prime number, then the integer sequence $(1, 1, p-3, 1)$ cannot be a δ-vector of any integral simplex of dimension 3, although $(1, 1, p-3, 1)$ satisfies all the inequalities of Theorem 1.1.

5.2. Non-Prime Case

We also remark that Theorem 1.1 is not true in general when $\sum_{i=0}^{d} \delta_i$ is not prime. For example, there exists an integral simplex of dimension 5 whose δ-vector is $(1, 1, 0, 2, 0, 0)$ ([5, Theorem 5.1]), while this satisfies neither $i_1 + i_3 = i_2 + i_2$ nor $2i_1 \geq i_2$, where $i_1 = 1$ and $i_2 = i_3 = 3$.

More generally, for a non-prime integer $m = gq$, where $g > 1$ is the least prime divisor of m, let $d = m+1$ and $\mathcal{P} = \mathcal{P}_m(0, \ldots, 0, \underbrace{d-1}_{d_g}, 0, \ldots, 0)$. Then, from Lemma 3.1, we have $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$, where

$$\delta_i = \begin{cases} 1 & i = 0, \\ g-1 & i = 1, \\ g & i = g+1, 2g+1, \ldots, (q-1)g+1. \end{cases}$$

Then one has $i_j = \lfloor j/g \rfloor g + 1$ for $j = 1, \ldots, gq-1$. This δ-vector satisfies neither $i_1 + i_{gq-1} = i_g + i_{(q-1)g}$ nor $i_1 + i_{g-1} \geq i_g$.

On the other hand, Proposition 2.2 is true even for the non-prime normalized volume case and we also know an analogue of Theorem 1.1 for such case as follows.

Proposition 5.1. *Let $\mathcal{P}$ be an integral simplex of dimension d with its δ-vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ and $i_1, \ldots, i_{m-1}$ the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_{m-1}}$ with $1 \leq i_1 \leq \cdots \leq i_{m-1} \leq d$, where $\sum_{i=0}^{d} \delta_i = m$ is not prime. Then one has*

$$i_k + i_\ell \geq i_{k+\ell} \quad \textit{for} \quad 1 \leq k \leq \ell \leq g-1 \quad \textit{with} \quad k + \ell \leq g-1,$$

where g is the least prime divisor of m.

Proof. By applying [8, Theorem 13], a proof of this statement is given in the same way as the proof of Theorem 1.1 (b). □

It is immediate that the above example satisfies $i_k + i_\ell \geq i_{k+\ell}$ for $1 \leq k \leq \ell \leq g-1$ with $k + \ell \leq g - 1$.

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