# SUMS OF GENERALIZED HARMONIC SERIES

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**Abstract**

We prove two results on sums of generalized harmonic series.

## 1. Introduction

For nonnegative integers $a_1, a_2, \ldots, a_k$ with $a_1 + \cdots + a_k \geq 2$, define

$$H(a_1, a_2 \ldots, a_k) = \sum_{n=1}^{\infty} \frac{1}{n^{a_1}(n+1)^{a_2} \cdots (n+k-1)^{a_k}}. \tag{1}$$

It is easy to show that this series converges, and in fact (by considering the partial-fractions decomposition of $n^{-a_1}(n+1)^{-a_2} \cdots (n+k-1)^{-a_k}$) that it can be written as a rational number plus a sum of values $\zeta(m)$ of the Riemann zeta function for $2 \leq m \leq \max\{a_1, \ldots, a_k\}$. For example,

$$H(2,3) = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^3} = \sum_{n=1}^{\infty} \left[ -\frac{3}{n} + \frac{1}{n^2} + \frac{3}{n+1} + \frac{2}{(n+1)^2} + \frac{1}{(n+1)^3} \right]$$

$$= -3 \sum_{n=1}^{\infty} \left[ \frac{1}{n} - \frac{1}{n+1} \right] + \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{2}{(n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^3} = -6 + 3\zeta(2) + \zeta(3).$$

We call the quantity (1) an $H$-series of length $k$ and weight $a_1 + \cdots + a_k$. In §2 below we establish various formulas for $H$-series, in which Stirling numbers of the first kind are prominently involved. These are used in §3 to prove the following result.

**Theorem 1.** *For any positive integers $k, m$ with $m \geq 2$,*

$$\sum_{a_1+a_2+\cdots+a_k=m} H(a_1, \ldots, a_k) = k\zeta(m),$$

*where the sum is over $k$-tuples of nonnegative integers.*

The $H$-series can be put into the framework of Shintani zeta functions [5]. For an $m \times k$ matrix $A = (a_{ij})$ and a $k$-vector $\vec{x} = (x_1, \ldots, x_k)$, the Shintani zeta function of $(s_1, \ldots, s_k)$ is

$$\zeta(s_1, \ldots, s_k; A; \vec{x}) = \sum_{n_1,\ldots,n_m \geq 0} \frac{1}{(x_1 + \sum_{i=1}^{m} a_{i1}n_i)^{s_1} \cdots (x_k + \sum_{i=1}^{m} a_{ik}n_i)^{s_k}}. \quad (2)$$

Then an $H$-series is just the special case of (2) with $m = 1$, $A = [1\ 1\ \cdots\ 1]$, and $\vec{x} = (1, 2, \ldots, k)$. Several other special cases of Shintani zeta functions have been extensively studied. These include the Barnes zeta functions [1], which is the case $k = 1$; and the multiple zeta values [4, 7], which is the case where $m = k$, $A$ is the $k \times k$ matrix with 1's on and below the main diagonal and 0's above, and $\vec{x} = (k, k-1, \ldots, 1)$. In the latter case the series (2) becomes

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1,\ldots,n_k \geq 0} \frac{1}{(k + \sum_{i=1}^{k} n_i)^{s_1}(k - 1 + \sum_{i=2}^{k} n_i)^{s_2} \cdots (1 + n_k)^{s_k}}$$
$$= \sum_{m_1 > m_2 > \cdots > m_k \geq 1} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_k^{s_k}}.$$

In fact, Theorem 1 is superficially similar to the famous "sum conjecture" for multiple zeta values, i.e.,

$$\sum_{a_1+\cdots+a_k=m,\ a_i \geq 1,\ a_1 > 1} \zeta(a_1, a_2, \ldots, a_k) = \zeta(m) \quad (3)$$

for $m \geq 2$. The multiple zeta values sum conjecture (3) was made by the second author (see [4]), and was proved in full generality by A. Granville [2]. Note, however, that $\zeta(a_1, \ldots, a_k)$ is a $k$-fold sum while $H(a_1, \ldots, a_k)$ is a single sum. Also, the sum on the left-hand side of (3) is over $k$-tuples of positive integers rather than nonnegative integers.

It is also natural to ask about the sum of all $H$-series $H(a_1, \ldots, a_k)$ of weight $m$ over all $k$-tuples $(a_1, \ldots, a_k)$ of positive integers of fixed weight. As indicated by the example

$$H(1, 1) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1,$$

the answer is a rational number. More precisely, we have the following result, which follows easily from the machinery developed in §2.

**Theorem 2.** *For $m \geq k \geq 2$,*

$$\sum_{a_1+\cdots+a_k=m,\ a_i\geq 1} H(a_1,\ldots,a_k) = \frac{1}{(k-1)!}\sum_{i=0}^{k-2}\binom{k-2}{i}\frac{(-1)^i}{(i+1)^{m-k+1}}.$$

## 2. *H*-Series and Stirling Numbers of the First Kind

Henceforth we assume $m$ to be a positive integer greater than 1. We have the following result.

**Lemma 1.** *Let $1 \leq i < j \leq p$, and let $a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{j-1},a_{j+1},\ldots,a_p$ be fixed nonnegative integers. Then*

$$\sum_{k=1}^{m-1} H(a_1,\ldots,a_{i-1},k,a_{i+1},\ldots,a_{j-1},m-k,a_{j+1},\ldots,a_p) = \frac{1}{j-i}[H(a_1,\ldots,a_{i-1},m-1,a_{i+1},\ldots,a_{j-1},0,a_{j+1},\ldots,a_p) - H(a_1,\ldots,a_{i-1},0,a_{i+1},\ldots,a_{j-1},m-1,a_{j+1},\ldots,a_p)].$$

*Proof.* Evidently

$$\frac{1}{(n+i-1)(n+j-1)} = \frac{1}{j-i}\left[\frac{1}{n+i-1} - \frac{1}{n+j-1}\right],$$

so that

$$H(a_1,\ldots,a_{i-1},k,a_{i+1},\ldots,a_{j-1},m-k,a_{j+1},\ldots,a_p) = \frac{1}{j-i}[H(a_1,\ldots,a_{i-1},k,a_{i+1},\ldots,a_{j-1},m-k-1,a_{j+1},\ldots,a_p) - H(a_1,\ldots,a_{i-1},k-1,a_{i+1},\ldots,a_{j-1},m-k,a_{j+1},\ldots,a_p)].$$

Now sum on $k$ to obtain the conclusion. □

Using the preceding result, we can obtain a formula for the sum of all $H$-series of fixed weight and length having nonzero entries at specified locations.

**Lemma 2.** *For $k \geq 1$ and $1 \leq i_0 < i_1 < \cdots < i_k \leq n$,*

$$\sum_{a_{i_0}+a_{i_1}+\cdots+a_{i_k}=m,\ a_j\neq 0 \text{ iff } j=i_q \text{ for some } q} H(a_1,\ldots,a_n) = \sum_{j=1}^{k}\frac{(-1)^{j-1}H^{(m-k)}_{i_0,i_j-1}}{(i_j-i_0)(i_j-i_1)\cdots(i_j-i_{j-1})(i_{j+1}-i_j)\cdots(i_k-i_j)}, \quad (4)$$

*where for* $p \leq q$

$$H_{p,q}^{(n)} = \sum_{j=p}^{q} \frac{1}{j^n}.$$

*Proof.* For convenience, denote by $S(i_0, i_1, \ldots, i_k; m)$ the left-hand side of equation (4). We proceed by induction on $k$, using Lemma 1. First, note that the case $k = 1$ of the result, i.e.,

$$S(i_0, i_1; m) = \frac{H_{i_0, i_1-1}^{(m-1)}}{i_1 - i_0},$$

follows immediately from Lemma 1. Now suppose the result holds for $k$, and consider $S(i_0, \ldots, i_k, i_{k+1}; m)$, i.e., the sum of all $H(a_1, \ldots, a_n)$ with nonzero entries $a_{i_0}$, $a_{i_1}, \ldots, a_{i_{k+1}}$ adding up to $m$. Fix $a_{i_0}, \ldots, a_{i_{k-1}}$, and sum the $H(a_1, \ldots, a_n)$ from $a_{i_k} = 1$ to $a_{i_k} = m - a_{i_0} - \cdots - a_{i_{k-1}} - 1$, keeping the sum of $a_{i_k}$ and $a_{i_{k+1}}$ equal to $m - a_{i_0} - \cdots - a_{i_k}$. By Lemma 1 this gives us, after summation on $a_{i_0}, \ldots, a_{i_{k-1}}$,

$$\frac{1}{i_{k+1} - i_k} \left[ S(i_0, \ldots, i_{k-1}, i_k; m-1) - S(i_0, \ldots, i_{k-1}, i_{k+1}; m-1) \right]. \qquad (5)$$

By the induction hypothesis, $S(i_0, \ldots, i_k; m-1)$ is

$$\sum_{j=1}^{k-1} \frac{(-1)^{j-1} H_{i_0, i_j - 1}^{(m-k-1)}}{(i_j - i_0) \cdots (i_j - i_{j-1})(i_{j+1} - i_j) \cdots (i_k - i_j)} + \frac{(-1)^{k-1} H_{i_0, i_k - 1}^{(m-k-1)}}{(i_k - i_0) \cdots (i_k - i_{k-1})}$$

and $S(i_0, \ldots, i_{k-1}, i_{k+1}; m-1)$ is

$$\sum_{j=1}^{k-1} \frac{(-1)^{j-1} H_{i_0, i_j - 1}^{(m-k-1)}}{(i_j - i_0) \cdots (i_j - i_{j-1})(i_{j+1} - i_j) \cdots (i_{k-1} - i_j)(i_{k+1} - i_j)} + \frac{(-1)^{k-1} H_{i_0, i_{k+1} - 1}^{(m-k-1)}}{(i_{k+1} - i_0) \cdots (i_{k+1} - i_{k-1})}.$$

Hence the quantity (5) is

$$\sum_{j=1}^{k-1} \frac{(-1)^{j-1} H_{i_0, i_j - 1}^{(m-k-1)}}{(i_j - i_0) \cdots (i_j - i_{j-1})(i_{j+1} - i_j) \cdots (i_{k-1} - i_j)} \cdot \frac{\frac{1}{i_k - i_j} - \frac{1}{i_{k+1} - i_j}}{i_{k+1} - i_k}$$
$$+ \frac{(-1)^{k-1} H_{i_0, i_k - 1}^{(m-k-1)}}{(i_k - i_0) \cdots (i_k - i_{k-1})(i_{k+1} - i_k)} + \frac{(-1)^{k} H_{i_0, i_{k+1} - 1}^{(m-k-1)}}{(i_{k+1} - i_0) \cdots (i_{k+1} - i_{k-1})(i_{k+1} - i_k)}$$
$$= \sum_{j=1}^{k+1} \frac{(-1)^{j-1} H_{i_0, i_j - 1}^{(m-k-1)}}{(i_j - i_0) \cdots (i_j - i_{j-1})(i_{j+1} - i_j) \cdots (i_{k+1} - i_j)}.$$

□

For $0 \le k \le n-1$, let $C(k,n;m)$ be the sum of all $H(a_1,\dots,a_n)$ of weight $m$ with exactly $k+1$ of the $a_i$ nonzero. Since each of the $H^{(m-k)}_{i_0,i_j-1}$ in the preceding result is a sum of quantities $\frac{1}{p^{m-k}}$ with $1 \le i_0 \le p \le i_j - 1 \le n-1$, for $k \ge 1$ we can write

$$C(k,n;m) = \sum_{j=1}^{n-1} \frac{c^{(n)}_{k,j}}{j^{m-k}} \tag{6}$$

where $c^{(n)}_{k,j}$ is rational. Then the numbers $c^{(n)}_{k,j}$ have a symmetry/antisymmetry property.

**Lemma 3.** *For $1 \le k, j \le n-1$, $c^{(n)}_{k,n-j} = (-1)^{k-1} c^{(n)}_{k,j}$.*

*Proof.* Borrowing the notation used in the proof of Lemma 2, we have

$$C(k,n;m) = \sum_{1 \le i_0 < i_1 < \cdots < i_k \le n} S(i_0, i_1 \dots, i_k; m).$$

Now note that Lemma 2 implies that if

$$S(i_0, i_1, \dots, i_k; m) = p_1 + p_2 \frac{1}{2^{m-k}} + \cdots + p_{n-1} \frac{1}{(n-1)^{m-k}},$$

then

$$S(n+1-i_k, n+1-i_{k-1}, \dots, n+1-i_0; m) = \\ (-1)^{k-1} \left( p_{n-1} + p_{n-2} \frac{1}{2^{m-k}} + \cdots + p_1 \frac{1}{(n-1)^{m-k}} \right).$$

□

The first (and last) columns of the numbers $c^{(n)}_{k,j}$ can be written in terms of (unsigned) Stirling numbers of the first kind: we write $\left[{n \atop k}\right]$ for the number of permutations of $\{1, 2 \dots, n\}$ with exactly $k$ disjoint cycles. This notation follows [3], which is also a useful reference on these numbers.

**Lemma 4.** *For $1 \le k \le n-1$,*

$$c^{(n)}_{k,1} = (-1)^{k-1} c^{(n)}_{k,n-1} = \frac{1}{(n-1)!} \left[{n \atop k+1}\right].$$

*Proof.* It suffices to prove the formula for $c^{(n)}_{k,n-1}$, as that for $c^{(n)}_{k,1}$ then follows from Lemma 3. Now Lemma 2 says that

$$c^{(n)}_{k,n-1} = \sum_{1 \le i_0 < i_1 < \cdots < i_{k-1} < n} \frac{(-1)^{k-1}}{(n-i_0)(n-i_1)\cdots(n-i_{k-1})},$$

so to prove the result we need to show

$$\sum_{1\le i_0<i_1<\cdots<i_{k-1}<n} \frac{(n-1)!}{(n-i_0)(n-i_1)\cdots(n-i_{k-1})} = \begin{bmatrix} n \\ k+1 \end{bmatrix}.$$

The left-hand side is evidently the sum of all products of $n-k-1$ distinct factors from the set $\{1,2,\ldots,n-1\}$, and that this is the Stirling number $\left[{n \atop k+1}\right]$ follows from consideration of the generating function

$$x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad n \ge 1. \tag{7}$$

□

The preceding result generalizes as follows.

**Lemma 5.** *For* $1 \le k, j \le n-1$,

$$c_{k,j}^{(n)} = \sum_{q=1}^{j} \sum_{p=1}^{q} \frac{(-1)^{p-1} \left[{q \atop p}\right] \left[{n+1-q \atop k+2-p}\right]}{(q-1)!(n-q)!}. \tag{8}$$

*Proof.* We need to show that

$$c_{k,j}^{(n)} - c_{k,j-1}^{(n)} = \sum_{p=1}^{j} \frac{(-1)^{p-1} \left[{j \atop p}\right] \left[{n+1-j \atop k+2-p}\right]}{(j-1)!(n-j)!}. \tag{9}$$

Suppose first that $j \le k$. Using Lemma 2, all terms that contribute to $c_{k,n-j+1}^{(n)}$ also contribute to $c_{k,n-j}^{(n)}$, and the terms that contribute to $c_{k,n-j}^{(n)}$ and not to $c_{k,n-j+1}^{(n)}$ are of the forms

$$\frac{(-1)^{k-1} H_{i_0,n-j}^{(m-k)}}{(n+1-j-i_0)\cdots(n+1-j-i_{k-1})}, \; i_0 < i_1 < \cdots < i_{k-1} < n+1-j,$$
$$\frac{(-1)^{k-2} H_{i_0,n-j}^{(m-k)}}{(n+1-j-i_0)\cdots(n+1-j-i_{k-2})(i_k-n-1+j)},$$
$$i_0 < i_1 < \cdots < i_{k-2} < n+1-j < i_k, \ldots$$
$$\frac{(-1)^{k-j} H_{i_0,n-j}^{(m-k)}}{(n+1-j-i_0)\cdots(n+1-j-i_{k-j})(i_{k-j+2}-n-1+j)\cdots(i_k-n-1+j)},$$
$$i_0 < i_1 < \cdots < i_{k-j} < n+1-j < i_{k-j+2} < \cdots < i_k$$

(where we assume throughout that $i_0, \ldots, i_k$ are integers between 1 and $n$), which we refer to as classes $1, 2, \dots j$. The contribution from class 1 is

$$\sum_{i_0<\cdots<i_{k-1}<n+1-j} \frac{(-1)^{k-1}}{(n+1-j-i_0)(n+1-j-i_1)\cdots(n+1-j-i_{k-1})}$$
$$= (-1)^{k-1} \frac{\left[{n+1-j \atop k+1}\right]}{(n-j)!} = (-1)^{k-1} \frac{\left[{j \atop 1}\right]}{(j-1)!} \frac{\left[{n+1-j \atop k+1}\right]}{(n-j)!}.$$

The contribution from class 2 is

$$\sum_{n+1-j<i_k} \frac{1}{i_k-n-1+j} \sum_{i_0<\cdots<i_{k-2}<n+1-j} \frac{(-1)^{k-2}}{(n+1-j-i_0)\cdots(n+1-j-i_{k-2})}$$
$$= \sum_{s=1}^{j-1} \frac{1}{s} \sum_{i_0<\cdots<i_{k-2}<n+1-j} \frac{(-1)^{k-2}}{(n+1-j-i_0)\cdots(n+1-j-i_{k-2})}$$
$$= (-1)^{k-1} \frac{(-1)\left[{j \atop 2}\right]}{(j-1)!} \frac{\left[{n+1-j \atop k}\right]}{(n-j)!},$$

and so forth. Note that for class $j$ we must have $i_{k-j+2} = n+2-j$, $i_{k-j+3} = n+3-j$, $\dots, i_k = n$ and the contribution from this class is

$$\sum_{i_0<\cdots<i_{k-j}<n+1-j} \frac{(-1)^{k-j}}{(n+1-j-i_0)(n+1-j-i_1)\cdots(n+1-j-i_{k-j})(j-1)!}$$
$$= (-1)^{k-1} \frac{(-1)^{j-1}\left[{j \atop j}\right]}{(j-1)!} \frac{\left[{n+1-j \atop k-j+2}\right]}{(n-j)!}.$$

Adding together the contributions, we see that $c_{k,n-j}^{(n)} - c_{k,n-j+1}^{(n)}$ is

$$(-1)^{k-1} \frac{\left[{j \atop 1}\right]\left[{n+1-j \atop k+1}\right] - \left[{j \atop 2}\right]\left[{n+1-j \atop k}\right] + \cdots + (-1)^{j-1}\left[{j \atop j}\right]\left[{n+1-j \atop k-j+2}\right]}{(j-1)!(n-j)!}, \tag{10}$$

and equation (9) follows from Lemma 3.

Now if $j > k$, then classes $k+1, k+2, \dots, j$ are evidently empty. All the corresponding terms in (10) are zero, except for

$$(-1)^{k-1} \frac{(-1)^k \left[{j \atop k+1}\right]\left[{n+1-j \atop 1}\right]}{(j-1)!(n-j)!} = -\frac{\left[{j \atop k+1}\right]}{(j-1)!}.$$

This quantity is accounted for by terms that contribute to $c_{k,n-j+1}^{(n)}$ and not to

$c_{k,n-j}^{(n)}$, provided

$$\sum_{n-j+1<i_1<\cdots<i_k}\sum_{p=1}^{k}\frac{(-1)^{p-1}}{(i_p-n+j-1)(i_p-i_1)\cdots(i_p-i_{p-1})(i_{p+1}-i_p)\cdots(i_k-i_p)} = \frac{\left[{j \atop k+1}\right]}{(j-1)!}. \quad (11)$$

By equating the expressions given for $c_{k,1}^{(j)}$ by Lemmas 2 and 4 respectively, we have

$$\sum_{1<i_1<\cdots<i_k\le j}\sum_{p=1}^{k}\frac{(-1)^{p-1}}{(i_p-1)(i_p-i_1)\cdots(i_p-i_{p-1})(i_{p+1}-i_p)\cdots(i_k-i_p)} = \frac{\left[{j \atop k+1}\right]}{(j-1)!},$$

from which equation (11) follows. □

## 3. Proofs of the Theorems

First we need one more lemma.

**Lemma 6.** *For positive integers $n$ and $j$,*

$$\sum_{q=1}^{j}\binom{n+j-q}{j}\binom{j-1}{q-1}(-1)^q = -n. \quad (12)$$

*Proof.* We use the "snake oil method" of H. Wilf [6]. Let

$$F(x) = \sum_{n=1}^{\infty}x^n\sum_{q=1}^{j}\binom{n+j-q}{j}\binom{j-1}{q-1}(-1)^q.$$

Then $F(x)$ can be rewritten

$$\sum_{q=1}^{j}\binom{j-1}{q-1}(-1)^q\sum_{n=1}^{\infty}\binom{n+j-q}{j}x^n = \sum_{q=1}^{j}\binom{j-1}{q-1}(-1)^q\sum_{m=j}^{\infty}\binom{m}{j}x^{m-j+q}$$
$$= \sum_{q=1}^{j}\binom{j-1}{q-1}(-1)^q\frac{x^q}{(1-x)^{j+1}} = \frac{-x}{(1-x)^{j+1}}\sum_{p=0}^{j-1}\binom{j-1}{p}(-x)^p$$
$$= \frac{-x}{(1-x)^{j+1}}(1-x)^{j-1} = -\frac{x}{(1-x)^2} = -x-2x^2-3x^3-\cdots,$$

and equation (12) follows. □

Now we prove Theorems 1 and 2.

*Proof of Theorem 1.* We have

$$\sum_{a_1+a_2+\cdots+a_n=m} H(a_1,\ldots,a_n) = \sum_{k=0}^{n-1} C(k,n;m).$$

Now

$$\begin{aligned} C(0,n;m) &= H(m,0,0\ldots,0) + H(0,m,0,\ldots,0) + \cdots + H(0,\ldots,0,m) \\ &= \zeta(m) + \zeta(m) - 1 + \cdots + \zeta(m) - 1 - \frac{1}{2^m} - \cdots - \frac{1}{(n-1)^m} \\ &= n\zeta(m) - (n-1) - \frac{n-2}{2^m} - \cdots - \frac{1}{(n-1)^m} \end{aligned}$$

and

$$C(k,n;m) = \sum_{j=1}^{n-1} \frac{c_{k,j}^{(n)}}{j^{m-k}}, \quad k \geq 1,$$

so to prove the result we need

$$\sum_{i=1}^{n-1} j^i c_{i,j}^{(n)} = n - j$$

for all $1 \leq j \leq n-1$. By Lemma 5 this means we must show

$$\sum_{i=1}^{n-1}\sum_{q=1}^{j}\sum_{p=1}^{q} j^i \frac{(-1)^{p-1}\left[{q \atop p}\right]\left[{n+1-q \atop i+2-p}\right]}{(q-1)!(n-q)!} = n - j. \tag{13}$$

We rewrite the left-hand side of equation (13) as

$$\sum_{q=1}^{j}\sum_{p=1}^{q} \frac{(-1)^{p-1}\left[{q \atop p}\right]}{(q-1)!} \cdot \frac{1}{(n-q)!} \sum_{i=1}^{n-1} \left[{n+1-q \atop i+2-p}\right] j^i. \tag{14}$$

If $p = 1$, the inner sum in (14) is

$$\sum_{i=1}^{n-1} \left[{n+1-q \atop i+1}\right] j^i = \sum_{i=2}^{n} \left[{n+1-q \atop i}\right] j^{i-1} = \frac{(n+j-q)!}{j(j-1)!} - \left[{n+1-q \atop 1}\right],$$

where we have used equation (7). Hence

$$\frac{1}{(n-q)!}\sum_{i=1}^{n-1} \left[{n+1-q \atop i+1}\right] j^i = \binom{n+j-q}{j} - 1$$

since $\left[{n+1-q \atop 1}\right] = (n-q)!$. If $p \geq 2$, then the inner sum in (14) is

$$\sum_{i=1}^{n-1} \left[{n+1-q \atop i+2-p}\right] j^i = \sum_{i\geq 1} \left[{n+1-q \atop i}\right] j^{i+p-2} = j^{p-2}\frac{(n+j-q)!}{(j-1)!} = j^{p-1}\frac{(n+j-q)!}{j!},$$

again using equation (7), from which follows

$$\frac{1}{(n-q)!}\sum_{i=1}^{n-1}\begin{bmatrix}n+1-q\\ i+2-p\end{bmatrix}j^i = j^{p-1}\binom{n+j-q}{j}.$$

Thus, the sum (14) can be written

$$\begin{aligned}\sum_{q=1}^{j}\left[\binom{n+j-q}{j}-1\right] &+ \sum_{q=2}^{j}\sum_{p=2}^{q}\frac{(-1)^{p-1}\left[{q \atop p}\right]}{(q-1)!}\binom{n+j-q}{j}j^{p-1}\\ &= -j+\sum_{q=1}^{j}\binom{n+j-q}{j}+\sum_{q=2}^{j}\binom{n+j-q}{j}\frac{1}{(q-1)!}\sum_{p=2}^{q}(-j)^{p-1}\begin{bmatrix}q\\ p\end{bmatrix}.\end{aligned}$$

If we use equation (7), this becomes

$$\begin{aligned}&-j+\sum_{q=1}^{j}\binom{n+j-q}{j}+\sum_{q=2}^{j}\binom{n+j-q}{j}\frac{1}{(q-1)!}\left[(1-j)\cdots(q-1-j)-\begin{bmatrix}q\\ 1\end{bmatrix}\right]\\ &= -j+\sum_{q=1}^{j}\binom{n+j-q}{j}-\sum_{q=2}^{j}\binom{n+j-q}{j}+\sum_{q=2}^{j}\binom{n+j-q}{j}\binom{j-1}{q-1}(-1)^{q-1}\\ &= -j+\binom{n+j-1}{j}+\sum_{q=2}^{j}\binom{n+j-q}{j}\binom{j-1}{q-1}(-1)^{q-1}\\ &= -j-\sum_{q=1}^{j}\binom{n+j-q}{j}\binom{j-1}{q-1}(-1)^{q}.\end{aligned}$$

Then equation (13) follows from Lemma 6. □

*Proof of Theorem 2.* It suffices to show that

$$c_{n-1,j}^{(n)} = \frac{(-1)^{j-1}}{(n-1)!}\binom{n-2}{j-1}.$$

From Lemma 5 we have

$$\begin{aligned}c_{n-1,j}^{(n)} &= \sum_{q=1}^{j}\sum_{p=1}^{q}\frac{(-1)^{p-1}\left[{q \atop p}\right]\left[{n+1-q \atop n+1-p}\right]}{(q-1)!(n-q)!} = \sum_{q=1}^{j}\frac{(-1)^{q-1}\left[{q \atop q}\right]\left[{n+1-q \atop n+1-q}\right]}{(q-1)!(n-q)!}\\ &= \frac{1}{(n-1)!}\sum_{q=1}^{j}(-1)^{q-1}\binom{n-1}{q-1} = \frac{(-1)^{j-1}}{(n-1)!}\binom{n-2}{j-1}.\end{aligned}$$

□

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