A COMBINATORIAL PROOF OF A FAMILY OF MULTINOMIAL-FIBONACCI IDENTITIES

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Abstract

In this paper we generalize completely a class of multinomial-Fibonacci identities. They are given in a variety of forms, including one associated with the Zeckendorf representation of an integer as the sum of non-consecutive Fibonacci numbers, which may be regarded as the canonical form for these identities. Our proofs are obtained via both combinatorial and algebraic methods.

1. Introduction

There are a number of well-known identities involving sums of products of binomial coefficients and Fibonacci numbers, the latter of which may be defined by way of the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, where $F_0 = 0$ and $F_1 = 1$ [4, 5]. For example,

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}, \tag{1}$$

$$\sum_{k=0}^{n} \binom{n}{k} F_{3k} = 2^n F_{2n}, \tag{2}$$

$$\sum_{k=0}^{n} \binom{n}{k} F_{4k} = 3^n F_{2n} \tag{3}$$

may be found in [1, 6, 7, 9, 11], [1, 8, 11] and [1, 11], respectively. In addition, the two identities

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} F_{5k} = 5^n F_{2n}, \tag{4}$$

$$\sum_{k=0}^{n}\binom{n}{k}3^{n-k}F_{6k}=8^nF_{2n}, \tag{5}$$

appear in [11], whilst

$$\sum_{k=0}^{n}\binom{n}{k}\left(F_{q-2}\right)^{n-k}F_{qk}=\left(F_q\right)^nF_{2n}, \tag{6}$$

which is valid for $q \geq 3$, can be seen in [1].

We show here how identities of these forms may be generalized completely to give an infinite family of multinomial-Fibonacci identities. Both combinatorial and algebraic proofs of our results are provided. Of further interest is the way that this work gives rise to a natural application of Zeckendorf's theorem. Finally, we consider briefly special cases of the theorem that take on a particularly simple form.

2. Preliminaries

In this section we set up some of the mathematical machinery that will allow us to provide combinatorial proofs of the main theorems in this paper. A first point to note is that the definition of the Fibonacci numbers may in fact be extended to negative indices, in which case it is straightforward to show that $F_{-n}=(-1)^{n+1}F_n$ for all $n \in \mathbb{N}$. Of particular relevance to this paper, $F_{-1}=1$ and $F_{-2}=-1$.

Next, it is well-known [1, 2] that the number of ways of tiling a $1 \times n$ board using 1×1 squares and 1×2 dominoes is F_{n+1}. For ease of notation, we use s and d to denote a square and a domino, respectively. Let $\mathbb{T}$ be the set of finite words in the alphabet $\{s, d\}$. Then there is an obvious one-to-one correspondence between the elements of $\mathbb{T}$ and those of the set of finite tilings. For example, the word *ssdsddd* comprising 7 letters corresponds to a particular tiling of length 11. In keeping, however, with some previous combinatorial proofs associated with Fibonacci numbers [1, 2, 3, 12], we take the liberty throughout this paper of calling both s and d "tiles", and words comprising them "tilings". Furthermore, in order to avoid confusion, "length" is to be interpreted in the sense of tilings, so that both *ssdsddd* and its corresponding tiling are said to have length 11.

Definition 1. A tiling $\mathcal{T} \in \mathbb{T}$ is, for $k \geq 1$, a *k-block* if, and only if, it is either a tiling of length k or a tiling of length $k+1$ that ends in a domino. For $k=0$ and $k=-1$, a k-block is defined to be the tiling of length 0, which we denote by t_0.

Note that if, from a conceptual point of view, we regard a (-1)-block as a tiling of length 0 that ends in a domino, it is straightforward to show from Definition 1 that, for $k \geq -1$, the total number of possible k-blocks is F_{k+2}, of which F_{k+1} have length k and F_k have length $k+1$ but end in a domino. The significance of 0-blocks and (-1)-blocks is discussed further at various points in the current section.

Definition 2. We let $\mathbb{B}_k \subseteq \mathbb{T}$ denote the set of k-blocks. Note, from the comments above, that $|\mathbb{B}_k| = F_{k+1} + F_k = F_{k+2}$. In particular, $|\mathbb{B}_0| = F_2 = 1$ and $|\mathbb{B}_{-1}| = F_1 = 1$, with the sets $\mathbb{B}_0$ and $\mathbb{B}_{-1}$ each containing just t_0.

Definition 3. Let $K = (k_1, k_2, \ldots, k_m)$ denote a fixed, finite sequence of integers k_i such that $k_i \geq -1$, $1 \leq i \leq m$. Given some $\mathcal{T} \in \mathbb{T}$ of sufficient length, we define both the *K-block decomposition* of $\mathcal{T}$ and its *tail* as follows: Proceed from the left of $\mathcal{T}$ one tile at a time until the first instance at which the length of the tiling thus far is at least k_1; this gives a k_1-block, which we take to be the first block in the decomposition. Then, starting afresh from the next tile, proceed one tile at a time until the first point at which the length of the tiling thus far is at least k_2, giving a k_2-block; the second block in the decomposition. This process continues until all m blocks have been constructed. The tail is what remains of $\mathcal{T}$ after the mth block, and the K-block decomposition of $\mathcal{T}$ comprises the m blocks and the tail.

By way of an example, the $(-1, 3, 5, 2, 0, 5)$-block decomposition of

$$\mathcal{T} = dsdddsdssdsdsdds$$

is given by

$$t_0 \mid ds \mid ddd \mid sd \mid t_0 \mid ssds \mid dsdds,$$

where the tail is $dsdds$.

Thus far we have been regarding the square tiles as indistinguishable, and similarly for the dominoes. We call tilings arising from such squares and dominoes *uncolored*. Let us now suppose that each tile may be assigned a color. Consider the set

$$\mathbb{C}_k = \{s_i : 1 \leq i \leq F_{k+1}\} \cup \{d_i : 1 \leq i \leq F_k\}$$

comprising F_{k+1} distinctly-colored squares and F_k distinctly-colored dominoes. Note that it is not necessary to assume that s_i and d_j have the same color whenever $i = j$. Indeed, for the arguments used in this paper, it makes no difference as to whether or not the set of colors used for the squares coincides with that used for the dominoes. From the definitions of $\mathbb{C}_k$ and $\mathbb{B}_k$ we know that there exists at least one bijection b_k from $\mathbb{C}_k$ to $\mathbb{B}_k$ such that colored squares and dominoes get mapped to the k-blocks of lengths k and $k+1$, respectively. For example, we see below a possible bijection for the case $k = 4$:

s_1	$\longleftrightarrow$	$ssss$
s_2	$\longleftrightarrow$	ssd
s_3	$\longleftrightarrow$	sds
s_4	$\longleftrightarrow$	dss
s_5	$\longleftrightarrow$	dd
d_1	$\longleftrightarrow$	$sssd$
d_2	$\longleftrightarrow$	sdd
d_3	$\longleftrightarrow$	dsd

Denoting this bijection by $b_4 : \mathbb{C}_4 \mapsto \mathbb{B}_4$, we have $b_4(s_1) = ssss$, $b_4(d_1) = sssd$, and so on.

Note in the above example, that when using b_4 to map a colored square in $\mathbb{C}_4$ to an uncolored tiling in $\mathbb{B}_4$, the tiling increases in length by 3. The same increase in length occurs when b_4 is used to map a colored domino $\mathbb{C}_4$ to a tiling in $\mathbb{B}_4$. It may be seen in general that when using some bijection b_k to map the colored squares and dominoes to the k-blocks of lengths k and $k+1$, respectively, the increase in length is $k-1$ per tile.

It is worth mentioning two particular cases in this regard. First, since $\mathbb{C}_0 = \{s_1\}$, the mapping b_0 simply maps a colored square to t_0, noting that the increase in length is $0-1=-1$ so that we actually lose length 1 here. Second, since $F_{-1} = 1$, it follows that $\mathbb{C}_{-1} = \{d_1\}$, from which we see that the mapping b_{-1} simply maps a colored domino to t_0. The increase in length is $-1-1=-2$, so we in fact lose length 2 in this case.

3. A Combinatorial Proof of a General Identity

We now state and prove our first result. It generalizes completely all of the identities given in the Introduction.

Theorem 1. *We have*

$$\sum_{\Sigma_{i=0}^r a_i = n} \binom{n}{a_0, \ldots, a_r} \left(\sum_{i=1}^r F_{x_i-2}\right)^{a_0} F_{\Sigma_{i=1}^r a_i x_i} = \left(\sum_{i=1}^r F_{x_i}\right)^n F_{2n} \tag{7}$$

when

$$\sum_{i=1}^r F_{x_i-2} \geq 1,$$

where $n \geq 0$, $r \geq 1$, $a_i \geq 0$ for $0 \leq i \leq r$ and $x_i \geq 0$ for $1 \leq i \leq r$.

Proof. Consider tiling a $1 \times (2n-1)$ board with 1×1 squares and 1×2 dominoes such that each of the leftmost n tiles is colored in one of

$$c = \sum_{i=1}^r F_{x_i}$$

colors. The expression on the right-hand side of (7) enumerates all possible such tilings. In order to establish the truth of the theorem, we will construct a bijection between the set of these partially-colored tilings of length $2n-1$ and a set of tilings that, as will be seen in due course, are counted by the expression on the left-hand side of (7).

First, the complete set of tile types $\{s_1, \ldots, s_c, d_1, \ldots, d_c\}$ is partitioned into disjoint sets A_i for $0 \le i \le r$, as follows. We start by letting $A_0 = \{d_1, \ldots, d_{\beta_0}\}$ where

$$\beta_0 = \sum_{i=1}^{r} F_{x_i - 2}.$$

Then, for $1 \le i \le r$, we let $A_i = \left\{s_{\alpha_{i-1}+1}, \ldots, s_{\alpha_i}, d_{\beta_{i-1}+1}, \ldots, d_{\beta_i}\right\}$, where $\alpha_0 = 0$, $\alpha_i = \alpha_{i-1} + F_{x_i}$ and $\beta_i = \beta_{i-1} + F_{x_i - 1}$. Note that this is indeed a partitioning of the tile types, since

$$\alpha_r = \sum_{i=1}^{r} F_{x_i} = c,$$

and

$$\beta_r = \sum_{i=1}^{r} F_{x_i - 2} + \sum_{i=1}^{r} F_{x_i - 1} = \sum_{i=1}^{r} F_{x_i} = c.$$

For $1 \le i \le r$, the structure of each set A_i may be regarded as isomorphic to that of $\mathbb{C}_{x_i - 1}$. It is also worth making the point here that A_i is equal to the singleton $\{s_{\alpha_i}\}$ if, and only if, $x_i = 1$. Similarly, A_i is equal to the singleton $\{d_{\beta_i}\}$ if, and only if, $x_i = 0$.

Now let $(a_0, a_1, \ldots, a_r)$ be a fixed sequence of $r + 1$ non-negative integers such that

$$\sum_{i=0}^{r} a_i = n.$$

We consider words of length n in the alphabet $\{A_0, \ldots, A_r\}$ for which A_i appears exactly a_i times, $0 \le i \le r$. Suppose that $\mathcal{A} = A_{q_1} A_{q_2} \ldots A_{q_n}$ is one such word of length n, noting that for each i satisfying $0 \le i \le r$, it must be the case that exactly a_i terms of the sequence $(q_1, q_2, \ldots, q_n)$ are equal to i.

Next, let $\mathcal{T}$ be an uncolored tiling of length

$$\sum_{i=1}^{r} a_i x_i - 1.$$

We obtain the K-block decomposition of $\mathcal{T}$ given by

$$K = (x_{q_1} - 1, x_{q_2} - 1, \ldots, x_{q_n} - 1),$$

where x_0 is defined to be 0. As mentioned in Section 2, it needs to be borne in mind that some of the resultant blocks may be t_0, namely those for which either $x_{q_j} = 0$ or $x_{q_j} = 1$. Let $\mathcal{U}$ denote the tail of this K-block decomposition of $\mathcal{T}$, so that $\mathcal{T} \backslash \mathcal{U}$ represents its pre-tail part.

We know, for each $k \ge -1$, that there exists a bijection b_k from $\mathbb{C}_k$ to $\mathbb{B}_k$ such that colored squares and dominoes get mapped to the k-blocks of lengths k and

$k+1$, respectively. Thus, since, for $1 \le i \le r$, the structure of A_i may be regarded as isomorphic to that of $\mathbb{C}_{x_i-1}$, there exists a bijection $\overline{b}_i : \mathbb{B}_{x_i-1} \mapsto A_i$ for which the (x_i-1)-blocks of lengths x_i-1 and x_i in $\mathcal{T}\backslash\mathcal{U}$ get mapped to colored squares and colored dominoes, respectively.

Note, however, that A_0 is not isomorphic to $\mathbb{C}_{x_0-1} = \mathbb{C}_{-1}$, and hence to $\mathbb{B}_{-1}$, when $\beta_0 > 1$. Instead, we may define $\overline{b}_0$ to be a function from $\mathbb{B}_{-1}$ to A_0, mapping t_0 to one of the β_0 elements of A_0. We thus have β_0 possible choices for the function $\overline{b}_0$ for each of the a_0 appearances of a (-1)-block in $\mathcal{T}\backslash\mathcal{U}$.

By way of the bijections $\overline{b}_i$, $1 \le i \le r$, and the possible choices of functions $\overline{b}_0$, the pair $(\mathcal{A},\mathcal{T})$ induces a total of $(\beta_0)^{a_0}$ distinct mappings from $\mathcal{T}\backslash\mathcal{U}$ to the set of colored tilings of length n using elements from the set $\{s_1,\ldots,s_c,d_1,\ldots,d_c\}$.

Next, suppose that $\mathcal{V}$ denotes the uncolored section that remains of the partially-colored tiling after the color and type of each of the leftmost n tiles have been specified via one of the mappings induced by the pair $(\mathcal{A},\mathcal{T})$. We will show that $\mathcal{U}$ and $\mathcal{V}$ have the same length. To this end, let k be the number of squares amongst the first n tiles of the partially-colored tiling, noting that k must satisfy $1 \le k \le n$. Then, remembering that the partially-colored tiling has length $2n-1$, the length of $\mathcal{V}$ is thus

$$2n-1-(k+2(n-k)) = k-1.$$

On the other hand, the length of $\mathcal{T}\backslash\mathcal{U}$ is given by

$$\begin{aligned}
\sum_{i=1}^{n}(x_{q_i}-1) &= \sum_{j=0}^{r} a_j\,(x_j-1) + (n-k) \\
&= \sum_{j=0}^{r} a_j x_j - \sum_{j=0}^{r} a_j + (n-k) \\
&= \sum_{j=0}^{r} a_j x_j - n + (n-k) \\
&= \sum_{j=0}^{r} a_j x_j - 1 - (k-1),
\end{aligned}$$

from which we see that $\mathcal{U}$ has length $k-1$. We are thus able to extend our induced mappings to include the identity mapping from the tail of $\mathcal{T}$ to the uncolored section of the partially-colored tiling.

For any fixed sequence $(a_0,a_1,\ldots,a_r)$ of $r+1$ non-negative integers satisfying $\sum_{i=0}^{r} a_i = n$, the number of words of length n in the alphabet $\{A_0,\ldots,A_r\}$ for which A_i appears exactly a_i times is given by the multinomial coefficient

$$\binom{n}{a_0,\ldots,a_r}.$$

Furthermore, the number of possible uncolored tilings of length $\sum_{i=1}^{r} a_i x_i - 1$ is equal to $F_{\Sigma_{i=1}^{r} a_i x_i}$. Thus, for a particular sequence $(a_0, a_1, \ldots, a_r)$, there are

$$\binom{n}{a_0, \ldots, a_r} F_{\Sigma_{i=1}^{r} a_i x_i}$$

possible pairs $(\mathcal{A}, \mathcal{T})$, and hence

$$\binom{n}{a_0, \ldots, a_r} \left(\sum_{i=1}^{r} F_{x_i - 2} \right)^{a_0} F_{\Sigma_{i=1}^{r} a_i x_i}$$

possible mappings. On summing over all possible sequences $(a_0, a_1, \ldots, a_r)$ of $r+1$ non-negative integers such that $\sum_{i=0}^{r} a_i = n$, the result follows. □

A specialization of Theorem 1 leads to identity (4), as we now show. On setting $r = 1$ and $x_1 = 5$ in Theorem 1, (7) becomes

$$\sum_{a_0 + a_1 = n} \binom{n}{a_0, a_1} (F_{5-2})^{a_0} F_{5a_1} = (F_5)^n F_{2n}.$$

Then, on noting that $F_{5-2} = F_3 = 2$ and $F_5 = 5$, result (4) follows. Similarly, (5) is obtained by setting $r = 1$ and $x_1 = 6$. Identity (6), which is a generalization of both (4) and (5), arises when we set $r = 1$ and $x_1 = q$.

Further specializations of Theorem 1 give rise to identities that would appear to be new. For example, with $r = n$ and $x_i = i$ for $i = 1, 2, \ldots, n$, we obtain

$$\sum_{\Sigma_{i=0}^{n} a_i = n} \binom{n}{a_0, \ldots, a_n} (F_n)^{a_0} F_{\Sigma_{i=1}^{n} i a_i} = (F_{n+2} - 1)^n F_{2n},$$

where use has been made of the identity $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ [4].

In the statement and proof of Theorem 1, it was assumed that $\beta_0 \geq 1$. However, it is worth mentioning the special case in which $x_1 = x_2 = \cdots = x_r = 2$, giving $\beta_0 = 0$. In this situation the summands on the left-hand side of (7) are each equal to 0 unless $a_0 = 0$, in which case we define

$$\left(\sum_{i=1}^{r} F_{x_i - 2} \right)^{a_0}$$

to be equal to 1. It may be seen then that this special case corresponds to the following trivial identity,

$$\sum_{\Sigma_{i=1}^{r} a_i = n} \binom{n}{a_1, \ldots, a_r} F_{2n} = r^n F_{2n},$$

which will also be discussed briefly at a later point in this paper.

4. Accounting for Multiplicities

The terms F_{x_i} given in the statement of Theorem 1 are not necessarily distinct. It is, however, possible to account for any multiplicities explicitly. Suppose, for example, the sum appearing on the right-hand side of (7) is given by

$$F_7 + F_3 + F_4 + F_3 + F_7 + F_7 + F_7 + F_9 + F_7 + F_{12}.$$

This may also be expressed as

$$2F_3 + F_4 + 5F_7 + F_9 + F_{12} = \sum_{i=1}^{5} p_i F_{x_i},$$

where $x_1 = 3, x_2 = 4, x_3 = 7, x_4 = 9, x_5 = 12$ and p_i denotes the multiplicity of F_{x_i} in this sum. In this particular case, $p_1 = 2, p_2 = 1, p_3 = 5, p_4 = 1, p_5 = 1$. We now give a result, corresponding to Theorem 1, which does indeed incorporate this potential multiplicity.

Theorem 2. *We have*

$$\sum_{\Sigma_{i=0}^{r} a_i = n} \binom{n}{a_0, \ldots, a_r} \left(\sum_{i=1}^{r} p_i F_{x_i - 2} \right)^{a_0} \left(\prod_{i=1}^{r} p_i^{a_i} \right) F_{\Sigma_{i=1}^{r} a_i x_i} = \left(\sum_{i=1}^{r} p_i F_{x_i} \right)^{n} F_{2n}, \tag{8}$$

when

$$\sum_{i=1}^{r} p_i F_{x_i - 2} \geq 1,$$

where $n \geq 0$*,* $r \geq 1$*,* $a_i \geq 0$ *for* $0 \leq i \leq r$*,* $p_i \geq 1$ *and* $x_i \geq 0$ *for* $1 \leq i \leq r$*.*

Proof. We provide merely an outline proof here, since most of the key ideas have already been encapsulated in the proof of Theorem 1. A $1 \times (2n-1)$ board is tiled with 1×1 squares and 1×2 dominoes such that each of the leftmost n tiles is colored in one of

$$c = \sum_{i=1}^{r} p_i F_{x_i}$$

colors. The expression on the right-hand side of (8) enumerates all possible such tilings.

The complete set of tile types $\{s_1, \ldots, s_c, d_1, \ldots, d_c\}$ is now partitioned into disjoint sets A_0 and $A_{i,j}$ for $1 \leq i \leq r$ and $1 \leq j \leq p_i$, as follows. We start by letting $A_0 = \{d_1, \ldots, d_{\beta_0}\}$ where

$$\beta_0 = \sum_{i=1}^{r} p_i F_{x_i - 2}.$$

Then, with $\alpha_0 = 0$, $\alpha_i = \alpha_{i-1} + F_{x_i}$, $\beta_i = \beta_{i-1} + F_{x_i-1}$, $1 \le i \le r$, we set

$$\begin{aligned}
A_{1,1} &= \{s_1, \ldots, s_{\alpha_1}, d_1, \ldots, d_{\beta_1}\}, \\
A_{1,2} &= \{s_{\alpha_1+1}, \ldots, s_{2\alpha_1}, d_{\beta_1+1}, \ldots, d_{2\beta_1}\}, \\
&\vdots \\
A_{1,p_1} &= \left\{s_{(p_1-1)\alpha_1+1}, \ldots, s_{p_1\alpha_1}, d_{(p_1-1)\beta_1+1}, \ldots, d_{p_1\beta_1}\right\},
\end{aligned}$$

and, in general,

$$A_{i,j} = \{s_{i,j}(1), \ldots, s_{i,j}(\alpha_i), d_{i,j}(1), \ldots, d_{i,j}(\beta_i)\},$$

where

$$\begin{aligned}
s_{i,j}(1) &= \sum_{k=1}^{i-1} p_k\alpha_k + (j-1)\alpha_i + 1, \\
s_{i,j}(\alpha_i) &= \sum_{k=1}^{i-1} p_k\alpha_k + j\alpha_i, \\
d_{i,j}(1) &= \sum_{k=1}^{i-1} p_k\beta_k + (j-1)\beta_i + 1, \\
d_{i,j}(\beta_i) &= \sum_{k=1}^{i-1} p_k\beta_k + j\beta_i.
\end{aligned}$$

Note that for any particular i such that $1 \le i \le r$, the structure of set $A_{i,j}$ may be regarded as isomorphic to that of $\mathbb{C}_{x_i-1}$ for each j satisfying $1 \le j \le p_i$.

Once more we let $(a_0, a_1, \ldots, a_r)$ be a fixed sequence of $r+1$ non-negative integers such that

$$\sum_{i=0}^{r} a_i = n.$$

Now, however, we consider words of length n in the alphabet $\{A_0\} \cup \{A_{i,j} : 1 \le i \le r, 1 \le j \le p_i\}$ for which A_0 appears exactly a_0 times, and, for each i such that $1 \le i \le r$, letters of the form $A_{i,j}$ appear exactly a_i times. Let $\mathcal{A} = A_{q_1,j_1}A_{q_2,j_2}\ldots A_{q_n,j_n}$ be one such word. For each k such that $1 \le k \le n$, it is the case that if $q_k = i$ then $1 \le j_k \le p_i$.

We let $\mathcal{T}$ be an uncolored tiling of length $\sum_{i=1}^{r} a_i x_i - 1$, with $\mathcal{U}$ denoting the tail of the K-block decomposition of $\mathcal{T}$, and, as in the proof of Theorem 1, $K = (x_{q_1} - 1, x_{q_2} - 1, \ldots, x_{q_n} - 1)$. For each j satisfying $1 \le j \le p_i$ there exists a bijection $\bar{b}_i : \mathbb{B}_{x_i-1} \mapsto A_{i,j}$ for which the $(x_i - 1)$-blocks of lengths $x_i - 1$ and x_i in $\mathcal{T}\backslash\mathcal{U}$ get mapped to colored squares and colored dominoes, respectively. This gives p_i possible choices for the function $\overline{b}_i$ for each of the a_i appearances of a $(x_i - 1)$-block in $\mathcal{T}\backslash\mathcal{U}$, and thus

$$\prod_{i=1}^{r} p_i^{a_i}$$

possible choices in total.

Since A_0 is not isomorphic to $\mathbb{B}_{-1}$ when $\beta_0 > 1$, we may in such cases define $\overline{b}_0$ to be a function from $\mathbb{B}_{-1}$ to A_0, mapping t_0, the tiling of zero length, to one of the β_0 elements of A_0. This gives β_0 possible choices for the function $\overline{b}_0$ for each of the a_0 appearances of a (-1)-block in $\mathcal{T}\backslash\mathcal{U}$.

The pair $(\mathcal{A}, \mathcal{T})$ therefore induces a total of

$$\left(\sum_{i=1}^{r} p_i F_{x_i-2}\right)^{a_0} \left(\prod_{i=1}^{r} p_i^{a_i}\right)$$

distinct mappings from $\mathcal{T}\backslash\mathcal{U}$ to the set of colored tilings of length n using elements from the set $\{s_1, \ldots, s_c, d_1, \ldots, d_c\}$. The remainder of the proof is identical to the final two paragraphs of that of Theorem 1. $\square$

5. An Algebraic Proof

For the sake of completeness, we now provide an algebraic proof of Theorem 2, and begin by stating two simple lemmas.

Lemma 1. *$\phi^m = \phi F_m + F_{m-1}$, where the golden ratio ϕ is given by*

$$\phi = \frac{1+\sqrt{5}}{2}.$$

Lemma 2. *Let α be irrational. Then $a\alpha + b = c\alpha + d$ for some $a, b, c, d \in \mathbb{Q}$ if, and only if, $a = c$ and $b = d$.*

Lemma 1 may be found in [9, 10], and is straightforward to prove by induction using the Fibonacci recurrence relation and the fact that $\phi^2 = \phi + 1$. Lemma 2 is applicable to any irrational number α, although we specialize it to ϕ here.

First, in order to prove the theorem, we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{r} p_i\phi^{x_i}+\sum_{i=1}^{r} p_iF_{x_i-2}\right)^n\\
&\qquad=\left(p_1\phi^{x_1}+p_2\phi^{x_2}+\cdots+p_r\phi^{x_r}+\left(\sum_{i=1}^{r} p_iF_{x_i-2}\right)\right)^n\\
&\qquad=\sum_{\Sigma_{i=0}^{r}a_i=n}\binom{n}{a_0,\ldots,a_r}\left(\prod_{i=1}^{r}\left(p_i\phi^{x_i}\right)^{a_i}\right)\left(\sum_{i=1}^{r} p_iF_{x_i-2}\right)^{a_0}\\
&\qquad=\sum_{\Sigma_{i=0}^{r}a_i=n}\binom{n}{a_0,\ldots,a_r}\left(\prod_{i=1}^{r}{p_i}^{a_i}\right)\left(\sum_{i=1}^{r} p_iF_{x_i-2}\right)^{a_0}\phi^s\\
&\qquad=\sum_{\Sigma_{i=0}^{r}a_i=n}\binom{n}{a_0,\ldots,a_r}\left(\prod_{i=1}^{r}{p_i}^{a_i}\right)\left(\sum_{i=1}^{r} p_iF_{x_i-2}\right)^{a_0}\left(\phi F_s+F_{s-1}\right),
\end{aligned}$$

where $s=\sum_{i=1}^{r}x_ia_i$ and use has been made of Lemma 1 in the final step. It is also the case that

$$\begin{aligned}
\left(\sum_{i=1}^{r} p_i\phi^{x_i}+\sum_{i=1}^{r} p_iF_{x_i-2}\right)^n &=\left(\sum_{i=1}^{r} p_i\left(\phi F_{x_i}+F_{x_i-1}\right)+\sum_{i=1}^{r} p_iF_{x_i-2}\right)^n\\
&=\left(\phi\sum_{i=1}^{r} p_iF_{x_i}+\sum_{i=1}^{r} p_i\left(F_{x_i-1}+F_{x_i-2}\right)\right)^n\\
&=\left(\phi\sum_{i=1}^{r} p_iF_{x_i}+\sum_{i=1}^{r} p_iF_{x_i}\right)^n\\
&=\left(\sum_{i=1}^{r} p_iF_{x_i}\right)^n(\phi+1)^n\\
&=\left(\sum_{i=1}^{r} p_iF_{x_i}\right)^n\phi^{2n}\\
&=\left(\sum_{i=1}^{r} p_iF_{x_i}\right)^n\left(\phi F_{2n}+F_{2n-1}\right),
\end{aligned}$$

where Lemma 1 has been utilized once more. An application of Lemma 2 then completes the proof of the theorem.

6. The Zeckendorf Form

The expression

$$\sum_{i=1}^{r} p_i F_{x_i}$$

appearing on the right-hand side of (8) is a sum of Fibonacci numbers in which the summands are not necessarily distinct. At the beginning of Section 4 we gave as an example the sum

$$2F_3 + F_4 + 5F_7 + F_9 + F_{12},$$

which is equal to 250. This may alternatively be written as

$$4F_3 + 3F_4 + F_7 + 4F_{10} \quad \text{or} \quad F_2 + F_4 + F_7 + F_{11} + F_{12} \quad \text{or} \quad F_2 + F_4 + F_7 + F_{13},$$

and there are clearly many other ways of expressing 250 as a sum of Fibonacci numbers.

Indeed, as can be seen from the middle and right-most sums above, not only is it possible to express 250 as a sum of Fibonacci numbers in more than one way, but it is also possible to find several ways of representing it as a sum of distinct Fibonacci numbers. This is the case in general. As will be demonstrated in the current section, this leads to a natural application of a result known as Zeckendorf's theorem, which is concerned with the possibility of writing, subject to certain restrictions, each positive integer as a unique sum of distinct Fibonacci numbers.

Zeckendorf's theorem states that every $m \in \mathbb{N}$ can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. Somewhat more formally, for any $m \in \mathbb{N}$ there exists a unique strictly-increasing finite sequence of positive integers of length $k \in \mathbb{N}$, $(y_1, y_2, \ldots, y_k)$ say, such that $y_1 \geq 2$, $y_i \geq y_{i-1} + 2$ for $i = 2, 3, \ldots, k$, and

$$m = \sum_{i=1}^{k} F_{y_i}.$$

Proofs of this theorem are given in [4, 13, 14]. Note that no Zeckendorf representation requires the use of F_1. Clearly, if a Zeckendorf representation of m contains F_2 then, on replacing F_2 with F_1, we would still have a representation for m that does not include any consecutive Fibonacci numbers. This would, however, violate the uniqueness of these representations. It is thus necessary to stipulate that F_1 does not appear in any Zeckendorf representation.

Continuing with our example, we see that $F_2 + F_4 + F_7 + F_{13}$ is the Zeckendorf representation for $2F_3 + F_4 + 5F_7 + F_9 + F_{12}$. In particular, therefore, we have

$$\sum_{i=1}^{r} p_i F_{x_i} = \sum_{i=1}^{k} F_{y_i},$$

where $r = 5, x_1 = 3, x_2 = 4, x_3 = 7, x_4 = 9, x_5 = 12, p_1 = 2, p_2 = 1, p_3 = 5, p_4 = 1, p_5 = 1$; and $k = 4, y_1 = 2, y_2 = 4, y_3 = 7, y_4 = 13$.

More generally, with

$$\sum_{i=1}^{r} p_i F_{x_i} = \sum_{i=1}^{k} F_{y_i},$$

for some $k \in \mathbb{N}$ and $(y_1, y_2, \ldots, y_k)$ such that $y_1 \geq 2$, $y_i \geq y_{i-1}+2$ for $i = 2, 3, \ldots, k$, we may transform (8) to what might be regarded as the canonical form of the identity, namely

$$\left(\sum_{i=1}^{k} F_{y_i}\right)^n F_{2n} = \sum_{\Sigma_{i=0}^{k} a_i = n} \binom{n}{a_0, \ldots, a_k} \left(\sum_{i=1}^{k} F_{y_i - 2}\right)^{a_0} F_{\Sigma_{i=1}^{k} y_i a_i}.$$

7. 'Powerless' Forms

It is sometimes possible to express the left-hand sides of (7) and (8) in the particularly simple form

$$\sum_{\Sigma_{i=0}^{r} a_i = n} \binom{n}{a_0, \ldots, a_r} F_{\Sigma_{i=1}^{r} a_i x_i},$$

that we term a 'powerless identity'. The trivial case mentioned in Section 3 is in fact a special case of this. It arises when $x_1 = x_2 = \cdots = x_r = 2$, and the corresponding identity is given by

$$\sum_{\Sigma_{i=1}^{r} a_i = n} \binom{n}{a_1, \ldots, a_r} F_{2n} = r^n F_{2n}.$$

We now consider non-trivial cases. To take an example,

$$\sum_{\Sigma_{i=0}^{3} a_i = n} \binom{n}{a_0, a_1, a_2, a_3} F_{2a_0 + 2a_1 + 3a_3} = (F_2 + F_2 + F_3)^n F_{2n} = 4^n F_n.$$

In order to obtain a powerless identity in general, we need $p_1 = p_2 = \cdots = p_r = 1$ and $\sum_{i=1}^{r} F_{x_i - 2} = 1$. Therefore, disregarding the order of the suffices, the number of non-trivial identities for some fixed value of r is equal to the number of solutions of the Diophantine equation

$$\sum_{i=1}^{r} F_{x_i - 2} = 1, \tag{9}$$

where $x_i \geq 0$ for $1 \leq i \leq r$.

The number of non-trivial powerless binomial identities of the type we are considering is equal to the number of solutions to (9) when $r = 1$. It is easily checked that there are exactly 3 such identities. They are given by

$$\sum_{a_0+a_1=n} \binom{n}{a_0, a_1} F_{a_1 x_1} = (F_{x_1})^n F_{2n},$$

on setting x_1 equal to 1, 3 and 4. Note that these correspond to the identities (1), (2) and (3), respectively, given in Section 1.

Similarly, the 4 solutions to (9) when $r = 2$ correspond to the number of non-trivial powerless trinomial identities. These are given by

$$\sum_{a_0+a_1+a_2=n} \binom{n}{a_0, a_1, a_2} F_{a_1 x_1 + a_2 x_2} = (F_{x_1} + F_{x_2})^n F_{2n},$$

on setting the pair (x_1, x_2) equal to (0,5), (1,2), (2,3) and (2,4). Then, with $r = 3$, we find that there are 11 non-trivial powerless quadrinomial identities, and so on. It is clear that for any particular value of r, there exist only finitely many powerless identities.

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