EXPANSIONS IN NON-INTEGER BASES: LOWER ORDER REVISITED

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Abstract

Let $q \in (1,2)$ and $x \in [0, \frac{1}{q-1}]$. We say that a sequence $(\varepsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ is an expansion of x in base q (or a q-expansion) if

$$x = \sum_{i=1}^{\infty} \varepsilon_i q^{-i}.$$

For any $k \in \mathbb{N}$, let $\mathcal{B}_k$ denote the set of q such that there exists x with exactly k expansions in base q. In 2009, the second-named author showed $\min \mathcal{B}_2 = q_2 \approx 1.71064$, the appropriate root of $x^4 = 2x^2 + x + 1$. In this paper we show that for any $k \geq 3$, $\min \mathcal{B}_k = q_f \approx 1.75488$, the appropriate root of $x^3 = 2x^2 - x + 1$.

1. Introduction

Let $q \in (1,2)$ and $I_q = [0, \frac{1}{q-1}]$. Given $x \in \mathbb{R}$, we say that a sequence $(\varepsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ is a *q-expansion* for x if

$$x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i}. \tag{1}$$

Expansions in non-integer bases were pioneered in the papers of Rényi [10] and Parry [9].

It is a simple exercise to show that x has a q-expansion if and only if $x \in I_q$. When (1) holds, we will adopt the notation $x = (\varepsilon_1, \varepsilon_2, \ldots)_q$. Given $x \in I_q$, we denote the set of q-expansions for x by $\Sigma_q(x)$, i.e.,

$$\Sigma_q(x) = \Big\{(\varepsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = x\Big\}.$$

In [5] it is shown that for $q \in (1, \frac{1+\sqrt{5}}{2})$ the set $\Sigma_q(x)$ is uncountable for all $x \in (0, \frac{1}{q-1})$. The endpoints of I_q trivially have a unique q-expansion for all $q \in (1,2)$. In [14] it is shown that for $q = \frac{1+\sqrt{5}}{2}$ every $x \in (0, \frac{1}{q-1})$ has uncountably many q-expansions unless $x = \frac{(1+\sqrt{5})n}{2} \bmod 1$, for some $n \in \mathbb{Z}$, in which case $\Sigma_q(x)$ is infinite countable. Moreover, in [3] it is shown that for all $q \in (\frac{1+\sqrt{5}}{2}, 2)$ there exists $x \in (0, \frac{1}{q-1})$ with a unique q-expansion. In this paper we will be interested in the set of $q \in (1,2)$ for which there exists $x \in I_q$ with precisely k q-expansions. More specifically, we will be interested in the set

$$\mathcal{B}_k := \Big\{q \in (1,2) |\text{ there exists } x \in \Big(0, \frac{1}{q-1}\Big) \text{ satisfying } \#\Sigma_q(x) = k\Big\}.$$

It was shown in [4] that $\mathcal{B}_k \neq \varnothing$ for any $k \geq 2$. Similarly we can define $\mathcal{B}_{\aleph_0}$ and $\mathcal{B}_{2^{\aleph_0}}$. The reader should bear in mind the possibility that the number of expansions could lie strictly between countable infinite and the continuum. By the above remarks it is clear that $\mathcal{B}_1 = (\frac{1+\sqrt{5}}{2}, 2)$. In [12] the following theorem was shown to hold.

Theorem 1.1.

- *The smallest element of $\mathcal{B}_2$ is*

$$q_2 \approx 1.71064,$$

  the appropriate root of $x^4 = 2x^2 + x + 1$.

- *The next smallest element of $\mathcal{B}_2$ is*

$$q_f \approx 1.75488,$$

  the appropriate root of $x^3 = 2x^2 - x + 1$.

- *For each $k \in \mathbb{N}$ there exists $\gamma_k > 0$ such that $(2-\gamma_k, 2) \subset \mathcal{B}_j$ for all $1 \leq j \leq k$.*

The following theorem is the central result of the present paper. It answers a question posed by V. Komornik [7] (see also [12, Section 5]).

Theorem 1.2. *For $k \geq 3$ the smallest element of $\mathcal{B}_k$ is q_f.*

The range of $q > \frac{1+\sqrt{5}}{2}$ which are "sufficiently close" to the golden ratio is referred to in [12] as the *lower order*, which explains the title of the present paper.

In the course of our proof of Theorem 1.2 we will also show that $q_f \in \mathcal{B}_{\aleph_0}$. Combined with our earlier remarks, Theorem 1.1, Theorem 1.2, and a result in [11] which states that for $q \in [\frac{1+\sqrt{5}}{2}, 2)$ almost every $x \in I_q$ has a continuum of q-expansions, we can conclude the following.

Theorem 1.3. *In base q_f all situations occur: there exist $x \in I_q$ having exactly k q-expansions for each $k = 1, 2, \ldots$, $k = \aleph_0$ or $k = 2^{\aleph_0}$. Moreover, q_f is the smallest $q \in (1, 2)$ satisfying this property.*

Before proving Theorem 1.2 it is necessary to recall some theory. In what follows we fix $T_{q,0}(x) = qx$ and $T_{q,1}(x) = qx - 1$. We will typically denote an element of $\bigcup_{n=0}^{\infty}\{T_{q,0}, T_{q,1}\}^n$ by a; here $\{T_{q,0}, T_{q,1}\}^0$ denotes the set consisting of the identity map. Moreover, if $a = (a_1, \ldots, a_n)$ we shall use $a(x)$ to denote $(a_n \circ \cdots \circ a_1)(x)$ and $|a|$ to denote the length of a.

We let

$$\Omega_q(x) = \Big\{(a_i)_{i=1}^{\infty} \in \{T_{q,0}, T_{q,1}\}^{\mathbb{N}} : (a_n \circ \ldots \circ a_1)(x) \in I_q \text{ for all } n \in \mathbb{N}\Big\}.$$

The significance of $\Omega_q(x)$ is made clear by the following lemma.

Lemma 1.4. *$\#\Sigma_q(x) = \#\Omega_q(x)$ where our bijection identifies $(\varepsilon_i)_{i=1}^{\infty}$ with $(T_{q,\varepsilon_i})_{i=1}^{\infty}$.*

The proof of Lemma 1.4 is contained within [2]. It is an immediate consequence of Lemma 1.4 that we can interpret Theorem 1.2 in terms of $\Omega_q(x)$ rather than $\Sigma_q(x)$.

An element $x \in I_q$ satisfies $T_{q,0}(x) \in I_q$ and $T_{q,1}(x) \in I_q$ if and only if $x \in [\frac{1}{q}, \frac{1}{q(q-1)}]$. Moreover, if $\#\Sigma_q(x) > 1$ or equivalently $\#\Omega_q(x) > 1$, then there exists a unique minimal sequence of transformations a such that $a(x) \in [\frac{1}{q}, \frac{1}{q(q-1)}]$. In what follows we let $S_q := [\frac{1}{q}, \frac{1}{q(q-1)}]$. The set S_q is usually referred to as the *switch region*. We will also make regular use of the fact that if $x \in I_q$ and a is a sequence of transformations such that $a(x) \in I_q$, then

$$\#\Omega_q(x) \geq \#\Omega_q(a(x)) \text{ or equivalently } \#\Sigma_q(x) \geq \#\Sigma_q(a(x)). \tag{2}$$

This is immediate from the definition of $\Omega_q(x)$ and Lemma 1.4.

In the course of our proof of Theorem 1.2 we will frequently switch between $\Sigma_q(x)$ and the dynamical interpretation of $\Sigma_q(x)$ provided by Lemma 1.4. Often considering $\Omega_q(x)$ will help our exposition.

The following lemma is a consequence of [6, Theorem 2].

Lemma 1.5. *Let $q \in (\frac{1+\sqrt{5}}{2}, q_f]$, if $x \in I_q$ has a unique q-expansion $(\varepsilon_i)_{i=1}^{\infty}$, then*

$$(\varepsilon_i)_{i=1}^{\infty} \in \Big\{0^k(10)^{\infty}, 1^k(10)^{\infty}, 0^{\infty}, 1^{\infty}\Big\},$$

where $k \geq 0$. Similarly, if $(\varepsilon_i)_{i=1}^{\infty} \in \{0^k(10)^{\infty}, 1^k(10)^{\infty}, 0^{\infty}, 1^{\infty}\}$, then for $q \in (\frac{1+\sqrt{5}}{2}, 2)$ $x = ((\varepsilon_i)_{i=1}^{\infty})_q$ has a unique q-expansion given by $(\varepsilon_i)_{i=1}^{\infty}$.

In Lemma 1.5 we have adopted the notation $(\varepsilon_1 \ldots \varepsilon_n)^k$ to denote the concatenation of $(\varepsilon_1 \ldots \varepsilon_n) \in \{0,1\}^n$ by itself k times and $(\varepsilon_1 \ldots \varepsilon_n)^{\infty}$ to denote the infinite sequence obtained by concatenating $\varepsilon_1 \dots \varepsilon_n$ by itself infinitely many times. We will use this notation throughout.

The following lemma follows from the branching argument first introduced in [13].

Lemma 1.6. *Let $k \geq 2$, $x \in I_q$, and suppose $\#\Sigma_q(x) = k$ or equivalently $\#\Omega_q(x) = k$. If a is the unique minimal sequence of transformations such that $a(x) \in S_q$, then*

$$\#\Omega_q(T_{q,1}(a(x))) + \#\Omega_q(T_{q,0}(a(x))) = k.$$

Moreover, $1 \leq \#\Omega_q(T_{q,1}(a(x))) < k$ and $1 \leq \#\Omega_q(T_{q,0}(a(x))) < k$.

The following result is an immediate consequence of Lemma 1.4 and Lemma 1.6.

Corollary 1.7. *$\mathcal{B}_k \subset \mathcal{B}_2$ for all $k \geq 3$.*

An outline of our proof of Theorem 1.2 is as follows: first of all we will show that $q_f \in \mathcal{B}_k$ for all $k \geq 1$. Then by Theorem 1.1 and Corollary 1.7, to prove Theorem 1.2, it suffices to show that $q_2 \notin \mathcal{B}_k$ for all $k \geq 3$. But by an application of Lemma 1.6, to show that $q_2 \notin \mathcal{B}_k$ for all $k \geq 3$ it suffices to show that $q_2 \notin \mathcal{B}_3$ and $q_2 \notin \mathcal{B}_4$. This will yield the claim of Theorem 1.2.

2. Proof that $q_f \in \mathcal{B}_k$ for all $k \geq 1$

To show that $q_f \in \mathcal{B}_k$ for all $k \geq 1$, we construct an $x \in I_{q_f}$ satisfying $\#\Sigma_{q_f}(x) = k$ explicitly.

Proposition 2.1. *For each $k \geq 1$ the number $x_k = (1(0000)^{k-1}0(10)^\infty)_{q_f}$ satisfies $\#\Sigma_{q_f}(x_k) = k$. Moreover, $x_{\aleph_0} = (10^\infty)_{q_f}$ satisfies $\operatorname{card} \Sigma_{q_f}(x) = \aleph_0$.*

Proof. We proceed by induction. For $k = 1$ we have $x_1 = ((10)^\infty)_{q_f}$, and therefore $\#\Sigma_{q_f}(x_1) = 1$ by Lemma 1.5. Let us assume $x_k = (1(0000)^{k-1}0(10)^\infty)_{q_f}$ satisfies $\#\Sigma_{q_f}(x_k) = k$. To prove our result, it suffices to show that $x_{k+1} = (1(0000)^k 0(10)^\infty)_{q_f}$ satisfies $\#\Sigma_{q_f}(x_{k+1}) = k + 1$.

We begin by remarking that by Lemma 1.5 $((0000)^k 0(10)^\infty))_{q_f}$ has a unique q_f-expansion. Therefore there is a unique q_f-expansion of x_{k+1} beginning with 1. Furthermore, it is a simple exercise to show that q_f satisfies the equation $x^4 = x^3 + x^2 + 1$, which implies that $(0(1101)(0000)^{k-1}0(10)^\infty)$ is also a q_f-expansion for x_{k+1}.

To prove the claim, we will show that if $(\varepsilon_i)_{i=1}^\infty$ is a q-expansion for x_{k+1} and $\varepsilon_1 = 0$, then $\varepsilon_2 = 1$, $\varepsilon_3 = 1$ and $\varepsilon_4 = 0$. Which combined with our inductive hypothesis implies that the set of q-expansions for x_{k+1} satisfying $\varepsilon_1 = 0$ consists of k distinct elements. Combining these q-expansions with the unique q-expansion of x_{k+1} satisfying $\varepsilon_1 = 1$ we may conclude $\#\Sigma_{q_f}(x_{k+1}) = k + 1$.

Let us suppose $\varepsilon_1 = 0$; if $\varepsilon_2 = 0$, then we would require

$$x_{k+1} = (1(0000)^k 0(10)^\infty)_{q_f} \leq (00(1)^\infty)_{q_f};$$

however, $x_{k+1} > \frac{1}{q_f}$ and $\sum_{i=3}^{\infty} \frac{1}{q^i} < \frac{1}{q}$ for all $q > \frac{1+\sqrt{5}}{2}$, and therefore $\varepsilon_2 = 1$. If $\varepsilon_3 = 0$, then we would require

$$x_{k+1} = (1(0000)^k 0(10)^\infty)_{q_f} \leq (010(1)^\infty)_{q_f}, \tag{3}$$

which is equivalent to

$$x_{k+1} = \frac{1}{q_f} + \frac{1}{q_f^{4k+3}} \sum_{i=0}^{\infty} \frac{1}{q_f^{2i}} \leq \frac{1}{q_f^2} + \frac{1}{q_f^4} \sum_{i=0}^{\infty} \frac{1}{q_f^i};$$

however,

$$\frac{1}{q_f} = \frac{1}{q_f^2} + \frac{1}{q_f^4} \sum_{i=0}^{\infty} \frac{1}{q_f^i},$$

whence (3) cannot occur and $\varepsilon_3 = 1$. Now let us suppose $\varepsilon_4 = 1$. Then we must have

$$x_{k+1} = (1(0000)^k 0(10)^\infty)_{q_f} \geq (01110^\infty)_{q_f}, \tag{4}$$

which is equivalent to

$$x_{k+1} = \frac{1}{q_f} + \frac{1}{q_f^{4k+3}} \sum_{i=0}^{\infty} \frac{1}{q_f^{2i}} \geq \frac{1}{q_f^2} + \frac{1}{q_f^3} + \frac{1}{q_f^4}. \tag{5}$$

The left-hand side of (5) is maximized when $k = 1$, and therefore to show that $\varepsilon_4 = 0$ it suffices to show that

$$\frac{1}{q_f} + \frac{1}{q_f^7} \sum_{i=0}^{\infty} \frac{1}{q_f^{2i}} \geq \frac{1}{q_f^2} + \frac{1}{q_f^3} + \frac{1}{q_f^4} \tag{6}$$

does not hold. By a simple manipulation (6) is equivalent to

$$q_f^6 - q_f^5 - 2q_f^4 + q_f^2 + q_f + 1 \geq 0, \tag{7}$$

but by an explicit calculation we can show that the left-hand side of (7) is strictly negative; therefore (4) does not hold and $\varepsilon_4 = 0$.

Now we consider $x_{\aleph_0}$. Replicating our analysis for x_k, we can show that if $(\varepsilon_i)_{i=1}^{\infty}$ is a q-expansion for $x_{\aleph_0}$ and $\varepsilon_1 = 0$, then $\varepsilon_2 = 1$. Unlike our previous case it is possible for ε_3 be equal to 0; however, in this case $\varepsilon_i = 1$ for all $i \geq 4$. If $\varepsilon_3 = 1$, then as in our previous case we must have $\varepsilon_4 = 0$. We observe that

$$x_{\aleph_0} = (10^\infty)_{q_f} = (010(1)^\infty)_{q_f} = (011010^\infty)_{q_f}.$$

Clearly, there exists a unique q-expansion for $x_{\aleph_0}$ satisfying $\varepsilon_1 = 1$ and a unique q-expansion for $x_{\aleph_0}$ satisfying $\varepsilon_1 = 0$, $\varepsilon_2 = 1$ and $\varepsilon_3 = 0$. Therefore all other q-expansions of $x_{\aleph_0}$ have (0110) as a prefix. Repeating the above argument arbitrarily many times we can determine that all the q_f-expansions of $x_{\aleph_0}$ are of the form:

$$\begin{aligned}
x_{\aleph_0} &= (10^{\infty})_{q_f}\\
&= (010(1)^{\infty})_{q_f}\\
&= (011010^{\infty})_{q_f}\\
&= (0110010(1)^{\infty})_{q_f}\\
&= (0110011010^{\infty})_{q_f}\\
&= (01100110010(1)^{\infty})_{q_f}\\
&= (01100110011010^{\infty})_{q_f},\\
&\vdots
\end{aligned}$$

which is clearly infinite countable. □

Thus, to prove Theorem 1.2, it suffices to show that $q_2 \notin \mathcal{B}_3 \cup \mathcal{B}_4$. This may look like a fairly innocuous exercise, but in reality it requires a substantial effort.

3. Proof that $q_2 \notin \mathcal{B}_3$

By Lemma 1.6, to show that $q_2 \notin \mathcal{B}_k$ for all $k \geq 3$, it suffices to show $q_2 \notin \mathcal{B}_3$ and $q_2 \notin \mathcal{B}_4$. To prove this, we begin by characterizing those $x \in S_{q_2}$ that satisfy $\#\Sigma_{q_2}(x) = 2$. To simplify our notation, we denote for the rest of the paper $\beta := q_2$ and $T_i := T_{q_2,i}$ for $i = 0, 1$.

Proposition 3.1. *The only $x \in S_\beta$ which satisfy $\#\Sigma_\beta(x) = 2$ are*

$$x = (01(10)^{\infty})_\beta = (10000(10)^{\infty})_\beta \text{ and } x = (0111(10)^{\infty})_\beta = (100(10)^{\infty})_\beta.$$

Proof. It was shown in the proof of [12, Proposition 2.4] that if $\frac{1+\sqrt{5}}{2} < q < q_f$ and $y, y+1$ have unique q-expansions, then necessarily $q = \beta$ and either $y = (0000(10)^{\infty})_\beta$ and $y + 1 = (1(10)^{\infty})_\beta$ or $y = (00(10)^{\infty})_\beta$ and $y + 1 = (111(10)^{\infty})_\beta$ respectively. Since for either case there exists a unique $x \in S_\beta$ such that $\beta x - 1 = y$, Lemma 1.6 yields the claim. □

In what follows we will let $(\varepsilon_i^1)_{i=1}^{\infty} = 01(10)^{\infty}$, $(\varepsilon_i^2)_{i=1}^{\infty} = 10000(10)^{\infty}$, $(\varepsilon_i^3)_{i=1}^{\infty} = 0111(10)^{\infty}$ and $(\varepsilon_i^4)_{i=1}^{\infty} = 100(10)^{\infty}$.

Remark 3.2. Let $(\bar{\varepsilon}_i)_{i=1}^{\infty} = (1 - \varepsilon_i)_{i=1}^{\infty}$, we refer to $(\bar{\varepsilon}_i)_{i=1}^{\infty}$ as the *reflection* of $(\varepsilon_i)_{i=1}^{\infty}$. Clearly $(\bar{\varepsilon}_i^1)_{i=1}^{\infty} = (\varepsilon_i^4)_{i=1}^{\infty}$ and $(\bar{\varepsilon}_i^2)_{i=1}^{\infty} = (\varepsilon_i^3)_{i=1}^{\infty}$. This is to be expected as every $x \in I_q$ satisfies $\#\Sigma_q(x) = \#\Sigma_q(\frac{1}{q-1} - x)$ and mapping $(\varepsilon_i)_{i=1}^{\infty}$ to $(\bar{\varepsilon}_i)_{i=1}^{\infty}$ is a bijection between $\Sigma_q(x)$ and $\Sigma_q(\frac{1}{q-1} - x)$. If $(\varepsilon_i^1)_{i=1}^{\infty}$ and $(\varepsilon_i^2)_{i=1}^{\infty}$ were not the reflections of $(\varepsilon_i^4)_{i=1}^{\infty}$ and $(\varepsilon_i^3)_{i=1}^{\infty}$ respectively, then there would exist other $x \in S_\beta$ satisfying $\#\Sigma_\beta(x) = 2$, contradicting Proposition 3.1.

In this section we show that no $x \in I_\beta$ can satisfy $\#\Sigma_\beta(x) = 3$. To show that $\beta \notin \mathcal{B}_3$ and $\beta \notin \mathcal{B}_4$ we will make use of the following proposition.

Proposition 3.3. *Suppose $x \in I_\beta$ satisfies $\#\Sigma_\beta(x) = 2$ or equivalently $\#\Omega_\beta(x) = 2$. Then there exists a unique sequence of transformations a such that $a(x) \in S_\beta$. Moreover, $a(x) = ((\varepsilon_i^1)_{i=1}^{\infty})_\beta$ or $a(x) = ((\varepsilon_i^3)_{i=1}^{\infty})_\beta$.*

Proof. Since $\#\Omega_\beta(x) = 2$, there must exist a satisfying $a(x) \in S_\beta$; otherwise $\#\Omega_\beta(x) = 1$. We begin by showing uniqueness; suppose a' satisfies $a'(x) \in S_\beta$ and $a' \neq a$. If $|a'| < |a|$, then we have two cases. If a' is a prefix of a, then by (2) and Lemma 1.6,

$$\#\Omega_\beta(x) \geq \#\Omega_\beta(a'(x)) = \#\Omega_\beta(T_0(a'(x))) + \#\Omega_\beta(T_1(a'(x))) \geq 3,$$

which contradicts $\#\Omega_\beta(x) = 2$. If a' is not a prefix of a, then there exists $b \in \bigcup_{n=0}^{\infty}\{T_0, T_1\}^n$ such that $b(x) \in S_\beta$ and either $b0$ is a prefix for a' and $b1$ is a prefix for a, or $b0$ is a prefix for a and $b1$ is a prefix for a'. In either case it follows from (2) and Lemma 1.6 that

$$\#\Omega_\beta(x) \geq \#\Omega_\beta(b(x)) = \#\Omega_\beta(T_0(b(x))) + \#\Omega_\beta(T_1(b(x))) \geq 4,$$

a contradiction. By analogous arguments we can show that if $|a'| = |a|$ or $|a'| > |a|$, then this implies $\#\Omega_\beta(x) > 2$. Therefore, a must be unique.

Now let a be the unique sequence of transformations such that $a(x) \in S_\beta$. By Lemma 1.6,

$$\#\Omega_\beta(T_0(a(x))) = \#\Omega_\beta(T_1(a(x))) = 1.$$

However, it follows from Proposition 3.1 that this can only happen when $a(x) = ((\varepsilon_i^1)_{i=1}^{\infty})_\beta$ or $a(x) = ((\varepsilon_i^3)_{i=1}^{\infty})_\beta$. □

Remark 3.4. By Proposition 3.3, to show that $x \in I_\beta$ satisfies $\operatorname{card} \Sigma_\beta(x) > 2$ (or equivalently, $\operatorname{card} \Omega_\beta(x) > 2$), it suffices to construct a sequence of transformations a such that $a(x) \in S_\beta$ with $a(x) \neq ((\varepsilon_i^1)_{i=1}^{\infty})_\beta$ and $a(x) \neq ((\varepsilon_i^3)_{i=1}^{\infty})_\beta$. We will make regular use of this strategy in our later proofs.

Before proving $\beta \notin \mathcal{B}_3$ it is appropriate to state numerical estimates[1] for S_β, $((\varepsilon_i^1)_{i=1}^{\infty})_\beta$ and $((\varepsilon_i^3)_{i=1}^{\infty})_\beta$. Our calculations yield

$$S_\beta = [0.584575\ldots, 0.822599\ldots],$$

[1]The explicit calculations performed in this paper were done using MATLAB. In our calculations we approximated β by 1.710644095045033, which is correct to the first fifteen decimal places.

$(0^k(01)^\infty)_\beta + 1$	Iterates of $(0^k(01)^\infty)_\beta + 1$ (To 6 decimal places)
$(0(01)^\infty)_\beta + 1$	Unique q-expansion by Proposition 3.1
$(00(01)^\infty)_\beta + 1$	$1.177400, 1.014114, 0.734788$
$(000(01)^\infty)_\beta + 1$	Unique q-expansion by Proposition 3.1
$(0000(01)^\infty)_\beta + 1$	$1.060622, 0.8143482$
$(00000(01)^\infty)_\beta + 1$	$1.035438, 0.771266$
$(000000(01)^\infty)_\beta + 1$	$1.020716, 0.746082$
1	$1, 0.710644$

Table 1: Successive iterates of $(0^k(01)^\infty)_\beta + 1$ falling into $S_\beta \setminus \{(\varepsilon^1)_\beta, (\varepsilon^3)_\beta\}$

$$((\varepsilon_i^1)_{i=1}^\infty)_\beta = 0.645198\ldots \text{ and } ((\varepsilon_i^3)_{i=1}^\infty)_\beta = 0.761976\ldots.$$

These estimates will make clear when $a(x) \in S_\beta$ and whether $a(x) = ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ or $a(x) = ((\varepsilon_i^3)_{i=1}^\infty)_\beta$.

Theorem 3.5. *We have $\beta \notin \mathcal{B}_3$.*

Proof. Suppose $x' \in I_\beta$ satisfies $\#\Sigma_\beta(x') = 3$ or equivalently $\#\Omega_\beta(x') = 3$. Let a denote the unique minimal sequence of transformations such that $a(x') \in S_\beta$. By considering reflections, we may assume without loss of generality that

$$\#\Omega_\beta(T_1(a(x'))) = 1 \text{ and } \#\Omega_\beta(T_0(a(x'))) = 2.$$

Put $x = T_1(a(x'))$; by a simple argument it can be shown that $x \neq 0$, so we may assume that $x = (0^k(01)^\infty)_\beta$ for some $k \geq 1$. To show that $\beta \notin \mathcal{B}_3$ we consider $T_0(a(x')) = x + 1 = (0^k(01)^\infty)_\beta + 1$. We will show that for each $k \geq 1$ there exists a finite sequence of transformations a such that $a(x+1) \in S_\beta$, $a(x+1) \neq ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $a(x+1) \neq ((\varepsilon_i^3)_{i=1}^\infty)_\beta$. By Proposition 3.3 and Remark 3.4, this implies $\#\Omega_\beta(x+1) > 2$, which is a contradiction. Hence $\beta \notin \mathcal{B}_3$.

Table 1 states the orbits of $(0^k(01)^\infty)_\beta + 1$ under T_0 and T_1 until eventually $(0^k(01)^\infty)_\beta + 1$ is mapped into S_β. Table 1 also includes the orbit of 1 under T_0 and T_1 until 1 is mapped into S_β. The reason we have included the orbit of 1 is because $(0^k(01)^\infty)_\beta + 1 \to 1$ as $k \to \infty$. Therefore understanding the orbit of 1 allows us to understand the orbit of $(0^k(01)^\infty)_\beta + 1$ for large values of k.

By inspection of Table 1, we conclude that for $1 \leq k \leq 6$ either $(0^k(01)^\infty)_\beta + 1$ has a unique q-expansion which contradicts $\#\Omega_\beta(T_0(a(x'))) = 2$, or there exists a such that $a((0^k(01)^\infty)_\beta + 1) \in S_\beta$ with $a((0^k(01)^\infty)_\beta + 1) \neq ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $a((0^k(01)^\infty)_\beta + 1) \neq ((\varepsilon_i^3)_{i=1}^\infty)_\beta$. By Proposition 3.3, this contradicts $\#\Omega_\beta(x+1) = 2$.

To conclude our proof, it suffices to show that for each $k \geq 7$ there exists a such that $a((0^k(01)^\infty)_\beta + 1) \in S_\beta$, $a((0^k(01)^\infty)_\beta + 1) \neq ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $a((0^k(01)^\infty)_\beta + 1) \neq ((\varepsilon_i^3)_{i=1}^\infty)_\beta$. For all $k \geq 7$, we have $(0^k(01)^\infty)_\beta + 1 \in (1, (000000(01)^\infty)_\beta + 1)$; however, by inspection of Table 1, it is clear that $T_1(x) \in (0.710644\ldots, 0.746082\ldots)$

for all $x \in (1, (000000(01)^\infty)_\beta + 1)$. Therefore, we can infer that such an a exists for all $k \geq 7$, which concludes our proof. $\square$

4. Proof that $q_2 \notin \mathcal{B}_4$

To prove $\beta \notin \mathcal{B}_4$, we will use a similar method to that used in the previous section, the primary difference being there are more cases to consider. Before giving our proof we give details of these cases.

Suppose $x' \in I_\beta$ satisfies $\#\Sigma_\beta(x') = 4$ or equivalently $\#\Omega_\beta(x') = 4$. Let a' denote the unique minimal sequence of transformations such that $a'(x) \in S_\beta$. By Lemma 1.6,

$$\#\Omega_\beta(T_0(a'(x'))) + \#\Omega_\beta(T_1(a'(x'))) = 4.$$

By Theorem 3.5, $\#\Omega_\beta(T_0(a'(x'))) \neq 3$ and $\#\Omega_\beta(T_1(a'(x'))) \neq 3$, whence

$$\#\Omega_\beta(T_0(a'(x'))) = \#\Omega_\beta(T_1(a'(x'))) = 2. \tag{8}$$

Letting $x = T_1(a'(x'))$, we observe that (8) is equivalent to

$$\#\Omega_\beta(x) = \#\Omega_\beta(x+1) = 2. \tag{9}$$

By Proposition 3.3, there exists a unique sequence of transformations a such that $a(x) \in S_\beta$ and $a(x) = ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ or $a(x) = ((\varepsilon_i^3)_{i=1}^\infty)_\beta$. We now determine the possible unique sequences of transformations a that satisfy $a(x) \in S_\beta$.

To determine the unique a such that $a(x) \in S_\beta$, it is useful to consider the interval $[\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}]$. The significance of this interval is that $T_0(\frac{1}{\beta^2-1}) = \frac{\beta}{\beta^2-1}$ and $T_1(\frac{\beta}{\beta^2-1}) = \frac{1}{\beta^2-1}$. The monotonicity of the maps T_0 and T_1 implies that if $x \in (0, \frac{1}{\beta-1})$ and $x \notin [\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}]$, then there exists $i \in \{0,1\}$ and a minimal $k \geq 1$ such that $T_i^k(x) \in [\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}]$. Furthermore, $S_\beta \subset [\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}]$, in view of $\beta > \frac{1+\sqrt{5}}{2}$.

In particular, if $x \in (0, \frac{1}{\beta^2-1})$, then there exists a minimal $k \geq 1$ such that $T_0^k(x) \in (\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1})$; $T_0^k(x)$ cannot equal $\frac{1}{\beta^2-1}$ or $\frac{\beta}{\beta^2-1}$ as this would imply $\#\Omega_\beta(x) = 1$. There are three cases to consider: either $T_0^k(x) \in S_\beta$, in which case $T_0^k(x) = ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ or $T_0^k(x) = ((\varepsilon_i^3)_{i=1}^\infty)_\beta$ by Proposition 3.3, or alternatively $T_0^k(x) \in (\frac{1}{\beta^2-1}, \frac{1}{\beta})$ or $T_0^k(x) \in (\frac{1}{\beta(\beta-1)}, \frac{\beta}{\beta^2-1})$. It is a simple exercise to show that if $T_0^k(x) = ((\varepsilon_i^3)_{i=1}^\infty)_\beta$ or $T_0^k(x) \in (\frac{1}{\beta(\beta-1)}, \frac{\beta}{\beta^2-1})$, then $k \geq 2$. By Lemma 1.4 and Proposition 3.3, if $T_0^k(x) \in S_\beta$, then

$$x = (0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta \text{ for some } k \geq 1 \text{ or } x = (0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta \text{ for some } k \geq 2.$$

For any $q \in (\frac{1+\sqrt{5}}{2}, q_f)$ and $y \in (\frac{1}{q^2-1}, \frac{1}{q})$ there exists a unique minimal sequence a'' such that $a''(y) \in S_q$. Moreover, $a''(y) = (T_{q,1} \circ T_{q,0})^j(y)$ for some $j \geq 1$

and $(T_{q,1} \circ T_{q,0})^i(y) \in (\frac{1}{q^2-1}, \frac{1}{q})$ for all $i < k$. For all $y \in (\frac{1}{q^2-1}, \frac{1}{q})$ we have that $(T_{q,1} \circ T_{q,0})(y) = q^2y - 1 < q - 1$. Furthermore, it can be checked directly that $\beta - 1 < ((\varepsilon_i^3)_{i=1}^{\infty})_\beta$. Hence if $T_0^k(x) \in (\frac{1}{\beta^2-1}, \frac{1}{\beta})$, then

$$x = (0^k(01)^j(\varepsilon_i^1)_{i=1}^{\infty})_\beta,$$

for some $k \geq 1$ and $j \geq 1$. By a similar argument it can be shown that if $T_0^k(x) \in (\frac{1}{\beta(\beta-1)}, \frac{\beta}{\beta^2-1})$, then

$$x = (0^k(10)^j(\varepsilon_i^3)_{i=1}^{\infty})_\beta,$$

for some $k \geq 2$ and $j \geq 1$. The above arguments are summarized in the following proposition.

Proposition 4.1. *Let x be as in (9); then one of the following four cases holds:*

$$x = (0^k(\varepsilon_i^1)_{i=1}^{\infty})_\beta \text{ for some } k \geq 1, \tag{10}$$

$$x = (0^k(\varepsilon_i^3)_{i=1}^{\infty})_\beta \text{ for some } k \geq 2, \tag{11}$$

$$x = (0^k(01)^j(\varepsilon_i^1)_{i=1}^{\infty})_\beta \text{ for some } k \geq 1 \text{ and } j \geq 1 \tag{12}$$

or

$$x = (0^k(10)^j(\varepsilon_i^3)_{i=1}^{\infty})_\beta \text{ for some } k \geq 2 \text{ and } j \geq 1. \tag{13}$$

To prove that $\beta \notin \mathcal{B}_4$ we will show that for each of the four cases described in Proposition 4.1 there exists a such that

$$a(x+1) \in S_\beta \setminus \{((\varepsilon_i^1)_{i=1}^{\infty})_\beta, ((\varepsilon_i^3)_{i=1}^{\infty})_\beta\}, \tag{14}$$

which contradicts $\#\Omega_\beta(x+1) = 2$ by Proposition 3.3 and Remark 3.4.

For the majority of our cases an argument analogous to that used in Section 3 will suffice. However, in the case where $k = 1, 3$ in (12) and $k = 2, 4$ in (13) a different argument is required. We refer to these cases as the *exceptional cases*. For the exceptional cases we will also show (14); however, the approach used in slightly more technical and as such we will treat these cases separately.

Proposition 4.2. *For each of the cases described by Proposition 4.1 there exists a such that (14) holds.*

Proof of Proposition 4.2 for the non-exceptional cases. In the cases where we have $x = (0^k(\varepsilon_i^1)_{i=1}^{\infty})_\beta$ for some $k \geq 1$ or $x = (0^k(\varepsilon_i^3)_{i=1}^{\infty})_\beta$ for some $k \geq 1$ it is clear that $x \to 0$ as $k \to \infty$. Therefore, to understand the orbit of $(0^k(\varepsilon_i^1)_{i=1}^{\infty})_\beta + 1$ or $(0^k(\varepsilon_i^3)_{i=1}^{\infty})_\beta + 1$ for large values of k, it suffices to consider the orbit of 1. Similarly, in the cases described by (12) and (13), if we fix $k \geq 1$, then, as $j \to \infty$, both $(0^k(01)^j(\varepsilon_i^1)_{i=1}^{\infty})_\beta$ and $(0^k(10)^j(\varepsilon_i^3)_{i=1}^{\infty})_\beta$ converge to $(0^l(10)^\infty)$ for some $l \geq 1$. Consequently, in order to understand the orbits of $(0^k(01)^j(\varepsilon_i^1)_{i=1}^{\infty})_\beta + 1$

$(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$	Iterates of $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$ (to 6 decimal places)
$(0(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$	$1.377166, 1.355842, 1.319363, 1.256961, 1.150213, 0.967605, 0.655228$
$(00(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$	$1.220482, 1.087810, 0.860857, 0.472620, 0.808484$
$(000(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$	$1.128888, 0.931126, 0.592825$
$(0000(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$	$1.075344, 0.839532, 0.436141, 0.746082$
$(00000(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$	$1.044044, 0.785989$
$(000000(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$	$1.025747, 0.754688$
1	$1, 0.710644$

Table 2: Successive iterates of $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$

and $(0^k(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta+1$ for large values of j, it suffices to consider the orbit of $(0^l(10)^\infty)_\beta+1$, for some $l\geq 1$. By considering these limits it will be clear when a sequence of transformations a exists that satisfies (14) for large values of k and j.

We begin by considering the case $x=(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta$. Table 2 plots successive (unique) iterates of $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$ until $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$ is mapped into S_β for $1\leq k\leq 6$. It is clear from inspection of Table 2 that for $1\leq k\leq 6$ there exists a such that $a(x+1)\in S_\beta$, $a(x+1)\neq((\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $a(x+1)\neq((\varepsilon_i^3)_{i=1}^\infty)_\beta$. The case $k\geq 7$ follows from the fact that $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta+1\in(1,(000000(\varepsilon_i^1)_{i=1}^\infty)_\beta+1)$ for all $k\geq 7$ and $T_1(y)\in(0.710644\ldots,0.754688\ldots)$ for all $y\in(1,(000000(\varepsilon_i^1)_{i=1}^\infty)_\beta+1)$. The case described by (11) follows by an analogous argument, so we omit the details and just include the relevant orbits in Table 3.

For the non-exceptional cases described by (12) and (13) an analogous argument works for the first few values of k by considering the limit of $x+1$ as $j\to\infty$, so, as above, we just include the relevant orbits in Table 3. It is clear by inspection of Table 3 that $(0^k(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta+1\in(1,(00000001(\varepsilon_i^1)_{i=1}^\infty)_\beta+1)$ for all $k\geq 7$ and $j\geq 1$. However, $T_1(y)\in(0.710644\ldots,0.749023\ldots)$ for all $y\in(1,(00000001(\varepsilon_i^1)_{i=1}^\infty)_\beta+1)$; by inspection of Table 3, we can conclude the case described by (12) in the non-exceptional cases. Similarly, it is clear from inspection of Table 3 that $(0^k(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta+1\in(1,(0000000(10)^\infty)_\beta+1)$ for all $k\geq 8$ and $j\geq 1$. However, $T_1(y)\in(0.710644\ldots,0.7460826\ldots)$ for all $y\in(1,(0000000(10)^\infty)_\beta+1)$, therefore by inspection of Table 3 we can conclude the case described by (13) in the non-exceptional cases. $\square$

Proof of Proposition 4.2 for the exceptional cases. The reason we cannot use the same method as used for the non-exceptional cases is because as $j\to\infty$ the limits of $(0(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta+1$, $(000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta+1, (00(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta+1$ and $(0000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta+1$ all have unique β-expansions, which follows from Proposition 3.1. As a consequence of the uniqueness of the β-expansion of the relevant limit, the number of transformations required to map $x+1$ into S_β becomes arbitrarily large as $j\to\infty$. However, the following proposition shows that we can still

$(0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	Iterates of $(0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$ (to 6 decimal places)
$(00(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	$1.260388, 1.156076, 0.977635, 0.672385$
$(000(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	$1.152216, 0.971032, 0.661091$
$(0000(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	$1.088982, 0.862860, 0.476047, 0.814348$
$(00000(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	$1.052016, 0.799626$
$(000000(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	$1.030407, 0.762660$
$(0000000(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	$1.017775, 0.741051$
1	$1, 0.710644$
$(00(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	Iterates of $(00(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$
$(0001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	$1.192123, 1.039298, 0.777869$
$(000101(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	$1.182431, 1.022720, 0.749510$
$(00(01)^\infty)_\beta + 1$	$1.177400, 1.014114, 0.734788$
$(0000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	Iterates of $(0000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$
$(000001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	$1.065653, 0.822954, 0.407782, 0.697570$
$(00000101(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	$1.062342, 0.817289$
$(0000(01)^\infty)_\beta + 1$	$1.060622, 0.814348$
$(00000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	Iterates of $(00000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$
$(0000001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	$1.038379, 0.776297$
$(00000(01)^\infty)_\beta + 1$	$1.035438, 0.771266$
$(000000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	Iterates of $(000000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$
$(00000001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	$1.022435, 0.749023$
$(000000(01)^\infty)_\beta + 1$	$1.020716, 0.746082$
$(000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	Iterates of $(000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$
$(00010(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	$1.168794, 0.999391, 0.709603$
$(000(10)^\infty)_\beta + 1$	$1.177400, 1.014114, 0.734788$
$(00000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	Iterates of $(00000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$
$(0000010(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	$1.057681, 0.809317$
$(00000(10)^\infty)_\beta + 1$	$1.060622, 0.814348$
$(000000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	Iterates of $(000000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$
$(00000010(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	$1.033719, 0.768326$
$(000000(10)^\infty)_\beta + 1$	$1.035438, 0.771266$
$(0000000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	Iterates of $(0000000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$
$(000000010(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	$1.019711, 0.744363$
$(0000000(10)^\infty)_\beta + 1$	$1.020716, 0.746082$

Table 3: Successive iterates of $(0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$

construct an a satisfying (14) for all but three of the exceptional cases.

Proposition 4.3. *The following identities hold:*

$$((T_1\circ T_0)^{j-2}\circ(T_1)^4)((0(01)^j(\varepsilon_i^1)_{i=1}^{\infty})_\beta+1)=\frac{\beta-1}{\beta^3(\beta^2-1)}+\frac{1}{\beta^2-1}\approx 0.59282 \text{ for } j\geq 3, \tag{15}$$

$$((T_1\circ T_0)^{j}\circ(T_1)^2)((000(01)^j(\varepsilon_i^1)_{i=1}^{\infty})_\beta+1)=\frac{\beta-1}{\beta^3(\beta^2-1)}+\frac{1}{\beta^2-1}\approx 0.59282 \text{ for } j\geq 1, \tag{16}$$

$$((T_0\circ T_1)^{j-1}\circ(T_1)^3)((00(10)^j(\varepsilon_i^3)_{i=1}^{\infty})_\beta+1)=\frac{\beta}{\beta^2-1}+\frac{1-\beta}{\beta^3(\beta^2-1)}\approx 0.81434 \text{ for } j\geq 2 \tag{17}$$

and

$$((T_0\circ T_1)^{j+1}\circ(T_1))((0000(10)^j(\varepsilon_i^3)_{i=1}^{\infty})_\beta+1)=\frac{\beta}{\beta^2-1}+\frac{1-\beta}{\beta^3(\beta^2-1)}\approx 0.81434 \text{ for } j\geq 1. \tag{18}$$

Proof. Each of the identities (15), (16), (17) and (18) is proved by similar arguments, so we will just show that (15) holds. Note that

$$(0(01)^j(\varepsilon_i^1)_{i=1}^{\infty})_\beta+1=\frac{\beta^{2j+2}+\beta-1}{\beta^{2j+3}(\beta^2-1)}+1,$$

for all $j\geq 1$. We observe the following:

$$\begin{aligned}
&((T_1\circ T_0)^{j-2}\circ(T_1)^4)\Big(\frac{\beta^{2j+2}+\beta-1}{\beta^{2j+3}(\beta^2-1)}+1\Big)\\
=&(T_1\circ T_0)^{j-2}\Big(\frac{\beta^{2j+2}+\beta-1}{\beta^{2j-1}(\beta^2-1)}+\beta^4-\beta^3-\beta^2-\beta-1\Big)\\
=&\beta^{2j-4}\Big(\frac{\beta^{2j+2}+\beta-1}{\beta^{2j-1}(\beta^2-1)}+\beta^4-\beta^3-\beta^2-\beta-1\Big)-\sum_{i=0}^{j-3}\beta^{2i}\\
=&\frac{\beta^{2j+2}+\beta-1}{\beta^3(\beta^2-1)}+\beta^{2j}-\beta^{2j-1}-\beta^{2j-2}-\beta^{2j-3}-\beta^{2j-4}-\frac{\beta^{2j-4}-1}{\beta^2-1}\\
=&\frac{\beta^{2j+2}}{\beta^3(\beta^2-1)}+\beta^{2j}-\beta^{2j-1}-\beta^{2j-2}-\beta^{2j-3}-\beta^{2j-4}-\frac{\beta^{2j-4}}{\beta^2-1}+\frac{\beta-1}{\beta^3(\beta^2-1)}+\frac{1}{\beta^2-1}.
\end{aligned}$$

Therefore, to conclude our proof, it suffices to show that

$$\frac{\beta^{2j+2}}{\beta^3(\beta^2-1)}+\beta^{2j}-\beta^{2j-1}-\beta^{2j-2}-\beta^{2j-3}-\beta^{2j-4}-\frac{\beta^{2j-4}}{\beta^2-1}=0. \tag{19}$$

Exceptional cases	Iterates (to 6 decimal places)
$(001(\varepsilon_i^1)_{i=1}^{\infty})_\beta + 1$	$1.328654, 1.272854, 1.177400, 1.014114, 0.734788$
$(00101(\varepsilon_i^1)_{i=1}^{\infty})_\beta + 1$	$1.312076, 1.244495, 1.128888, 0.931126, 0.592825$
$(0010(\varepsilon_i^3)_{i=1}^{\infty})_\beta + 1$	$1.288747, 1.204588, 1.060622, 0.814348$

Table 4: Remaining exceptional cases: $k = 1, j \in \{1, 2\}$ in (12) and $k = 2, j = 1$ in (13)

By manipulating the left-hand side of (19), we conclude that satisfying (19) is equivalent to

$$\frac{\beta^{2j-1} - \beta^{2j-4} + (\beta^{2j} - \beta^{2j-1} - \beta^{2j-2} - \beta^{2j-3} - \beta^{2j-4})(\beta^2 - 1)}{\beta^2 - 1} = 0$$

or

$$\frac{\beta^{2j-3}(\beta - 1)(\beta^4 - 2\beta^2 - \beta - 1)}{\beta^2 - 1} = 0.$$

This is true in view of $\beta^4 - 2\beta^2 - \beta - 1 = 0$. □

Proposition 4.3 and Table 4 (which displays the orbits of the exceptional cases that are not covered by Proposition 4.3) conclude our proof of Proposition 4.2 for all the exceptional cases. Therefore, $\beta \notin \mathcal{B}_4$, and Theorem 1.2 holds.

□

5. Open Questions

To conclude the paper, we pose a few open questions:

- What is the topology of $\mathcal{B}_k$ for $k \geq 2$? In particular, what is the smallest limit point of $\mathcal{B}_k$? Is it below or above the Komornik-Loreti constant introduced in [8]?

- What is the smallest q such that $x = 1$ has k q-expansions? (For $k = 1$ this is precisely the Komornik-Loreti constant.)

- What is the structure of $\mathcal{B}_{\aleph_0} \cap \left(\frac{1+\sqrt{5}}{2}, q_f\right)$? In view of the results of the present paper, knowing this would lead to a complete understanding of card $\Sigma_q(x)$ for all $q \leq q_f$ and all $x \in I_q$.

- Let, as above,
$$\mathcal{B}_\infty = \bigcap_{k=1}^{\infty} \mathcal{B}_k \cap \mathcal{B}_{\aleph_0} \cap \mathcal{B}_{2^{\aleph_0}}.$$

By Theorem 1.3, q_f is the smallest element of $\mathcal{B}_\infty$. What is the second smallest element of $\mathcal{B}_\infty$? What is the topology of $\mathcal{B}_\infty$?

- In [1] the authors study the order in which periodic orbits appear in the set of points with unique q-expansion; they show that as $q \uparrow 2$, the order in which periodic orbits appear in the set of uniqueness is intimately related to the classical Sharkovskiĭ ordering. Does a similar result hold in our case? That is, if $k > k'$ with respect to the usual Sharkovskiĭ ordering, does this imply $\mathcal{B}_k \subset \mathcal{B}_{k'}$?

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