



THE LOCATION OF THE FIRST ASCENT IN A 123-AVOIDING PERMUTATION

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Abstract

It is natural to ask, given a permutation with no three-term ascending subsequence, at what index the first ascent occurs. We shall show, using both a recursion and a bijection, that the number of 123-avoiding permutations at which the first ascent occurs at positions $k, k + 1$ is given by the k -fold Catalan convolution $C_{n,k}$. For $1 \leq k \leq n$, $C_{n,k}$ is also seen to enumerate the number of 123-avoiding permutations with n being in the k th position. Two interesting discrete probability distributions, related obliquely to the Poisson and geometric random variables, are derived as a result.

1. Introduction

For $n \geq 0$, the Catalan numbers C_n are given by

$$C_n = \frac{1}{n+1} \binom{2n}{n};$$

generalizing this fact, Catalan [3] proved the k -fold Catalan convolution formula

$$C_{n,k} := \sum_{i_1 + \dots + i_k = n} \prod_{r=1}^k C_{i_r-1} = \frac{k}{2n-k} \binom{2n-k}{n}.$$

The theory of pattern avoidance in permutations is now well-established and thriving, and a survey of the many results in that area may be found in the text by

Kitaev[5]. One of the earliest and most fundamental results in the field is that the number of permutations of $[n] := \{1, 2, \dots, n\}$ in which the longest increasing sequence is of length ≤ 2 , the so-called 123-avoiding permutations, is given by the Catalan numbers, and classical bijective techniques give that each of the ijk -avoiding permutations with $\{i, j, k\} = \{1, 2, 3\}$ are equinumerous. In this paper, we ask a very natural question, namely in how many permutations in which the longest increasing subsequence is of length at most 2, does the first ascent occur in positions $k, k + 1$. Actually, we were web-searching for the answer to this question to use in a different context, and were rather surprised to find that the solution appeared to be not known explicitly: Bousquet-Mélou [2] addressed this question indirectly when she used the location of the first ascent as a “catalytical variable” in the “laziest proof, combinatorially speaking,” of the fact that there are C_n 123-avoiding permutations. In Sections 2 and 3 of this paper, we will give recursive and bijective proofs respectively of the fact that there are $C_{n,k}$ 123-avoiding permutations on $[n]$ for which the first ascent occurs at positions $k, k + 1$. Critical to the bijective proof are the various combinatorial interpretations of the Catalan convolutions due to Tedford [9], and the bijections between Dyck paths and avoiding permutations due to Krattenthaler[7]. In Section 2, additionally, we show that $C_{n,k}$ also enumerates 123-avoiding permutations with the position of “ n ” being $k \in [1, n]$. Finally, in Section 4, we produce two interesting probability distributions on \mathbf{Z}^+ related to these issues. These are reminiscent of the geometric and Poisson random variables, and are studied systematically in [4].

2. Recursive Proof

Throughout, we refer to the first ascent as being in position k if the ascent is at the k th and $k + 1$ st positions of the permutation. If a permutation of $[n]$ letters has no ascents at all (i.e., it is the descending permutation π given by $\pi(j) = n + 1 - j; 1 \leq j \leq n$), we define it as having first ascent in position n , as though it had a first ascent “after” the last letter in the permutation – perhaps using a number such as 1.5 in the $(n + 1)$ st spot.

Any 123-avoiding permutation of $[n]$ becomes a 123-avoiding permutation of $[n - 1]$ when the letter n is removed from it, so we can view each 123-avoiding permutation of $[n]$ as being grown uniquely by taking a 123-avoiding permutation of $[n - 1]$ and inserting the letter n into certain positions.

Suppose we have a 123-avoiding permutation of $[n - 1]$ with first ascent in positions $k, k + 1$. How can this be grown into a longer 123-avoiding permutation by inserting n ? If n is inserted at the very front of this permutation, the resulting permutation is still 123-avoiding with the original first ascent being “pushed forward” one position due to the presence of n . If n is inserted anywhere between the very

front of this permutation and the peak of the original first ascent, i.e., as anything from the 2nd letter to the $k + 1$ st letter of the permutation, then n becomes the peak of the new first ascent, which therefore has position 1 less than the position of n . Furthermore, as there are no ascents before n , and as n cannot be involved in any subsequent ascents, the new permutation is still 123-avoiding. Finally, if n is inserted after the original first ascent, then the resulting permutation is no longer 123-avoiding. Note that all of the above hold exactly even if the original permutation is the descending permutation π given by $\pi(j) = n - j; 1 \leq j \leq n - 1$, with first ascent in positions $n - 1, n$. To summarize, a 123-avoiding permutation of length $n - 1$ with first ascent in positions $k, k + 1$ gives rise to exactly one 123-avoiding permutation of length n with first ascent in position $i, i + 1$ for each $1 \leq i \leq k + 1$, and each 123-avoiding permutation of length n must be grown in such a way.

Turning this around, the number $A_{n,k}$ of 123-avoiding permutations of length n with first ascent in position k is equal to the number of 123-avoiding permutations of length $n - 1$ with first ascent in position $k - 1$ or later, as it is these permutations that give rise to them. Thus

$$A_{n,k} = \sum_{i=k-1}^n A_{n-1,i}$$

In particular, we note that

$$A_{n,k} = \sum_{i=k-1}^n A_{n-1,i} = A_{n-1,k-1} + \sum_{i=k}^n A_{n-1,i} = A_{n-1,k-1} + A_{n,k+1}.$$

Since $A_{n,0} = 0$ for all n , the above recursion indicates that $A_{n,1} = A_{n,2}$ for all n , with both being equal to the total number of 123-avoiding permutations of length $n - 1$. We can see that this is true: any such $n - 1$ permutation can be uniquely grown into a 123-avoiding permutation with first ascent in position 2 by inserting n either as the third letter (if the original permutation did not have first ascent in position 1) or as the first letter (otherwise); or it can be uniquely grown into a 123-avoiding permutation with first ascent in position 1 by inserting n as the second letter. Combined with the base cases $A_{n,0} = 0$ and $A_{n,n} = 1$ for all $n \geq 1$, this recurrence relation is sufficient to fully characterize $A_{n,k}$ for any $1 \leq k \leq n$.

Note that $C_{n,0} = 0$ and $C_{n,n} = 1$ for all $n \geq 1$. Moreover, we find that the Catalan convolutions $C_{n,k}$ obey the same recurrence relation as above:

$$\begin{aligned}
 & C_{n-1,k-1} + C_{n,k+1} \\
 &= \frac{k-1}{2n-k-1} \binom{2n-k-1}{n-1} + \frac{k+1}{2n-k-1} \binom{2n-k-1}{n} \\
 &= \frac{k-1}{2n-k-1} \binom{n}{2n-k} \binom{2n-k}{n} + \frac{k+1}{2n-k-1} \binom{n-k}{2n-k} \binom{2n-k}{n} \\
 &= \frac{k}{2n-k} \binom{2n-k}{n} = C_{n,k}.
 \end{aligned}$$

Since $C_{n,k}$ obey the same recurrence relation as the $A_{n,k}$, and they have the same base cases (which generate their values for all $1 \leq k \leq n$), we find that $C_{n,k} = A_{n,k}$ everywhere.

Corollary 1. *The number of 123-avoiding permutations where n is in the k th spot are also given by the Catalan convolutions $C_{n,k}$.*

Proof. The result is obvious for $k = 1$ where the answer equals $C_{n-1} = C_{n,1}$. Let $k = 2$. We have that n is in position 2 in a 123-avoiding permutation iff the first ascent is at positions (1,2), necessarily to n . Thus again there are $C_{n,1} = C_{n-1}$ such possibilities. For $k \geq 3$ let $\alpha_{n,k}$ be the number of 123-avoiding permutations on $[n]$ with n in the k th spot. Since, as will be emphasized in Section 3, for the first ascent to be at spots $(k-1, k)$, n must either be in position 1, or the first ascent must be to n , we have that $\alpha_{n,k} = C_{n,k-1} - C_{n-1,k-2} = C_{n,k}$, by the above recursion.

To give an alternate bijective proof for $k \geq 2$, we proceed as follows. Consider a 123-avoiding permutation π with first ascent at spots $(k, k+1)$, and move n , originally in position 1 or $k+1$, into position k , while keeping the relative order of the other numbers unchanged. Regardless of whether n was in position 1 or position $(k+1)$, the new permutation has first ascent at position $k-1$ and is still 123-free. Since just one of these original configurations is valid for a given relative ordering of $[n-1]$, we see that this map φ from the set of 123-avoiders with first ascent at $(k, k+1)$ to the set of 123-avoiding permutations with n in position k is one-to-one. Moreover the map has an inverse: If $\varphi(\pi)(k-1) < \varphi(\pi)(k+1)$, n must have been at the beginning of π , and n must have been in position $k+1$ if we find that $\varphi(\pi)(k-1) > \varphi(\pi)(k+1)$. \square

3. Bijective Proof

We have, from Tedford [9] that $C_{n,k}$ is given (adjusting for his different indexing) by the number of lattice paths from $(k-1, 0)$ to $(n-1, n-1)$ consisting of steps of $(0, 1)$ and $(1, 0)$ and never crossing above the line $x = y$. Note that these paths,

and hence, $C_{n,k}$ are in bijection with these same types of paths between $(k, 1)$ and (n, n) . We will show that these paths are in bijection with the paths corresponding, by the Krattenthaler bijection, to 123-avoiding n -permutations with first ascent at k .

Krattenthaler’s bijection between 123-avoiding permutations and Dyck paths can be described as follows [7]: Given a permutation π of n integers, denote the right to left maxima (RLM) of π , reading from left to right, by $\{m_s, m_{s-1}, \dots, m_2, m_1\}$ and denote the (necessarily descending) word between m_{i+1} and m_i by w_i . π will now read left to right as $w_s m_s \dots w_2 m_2 w_1 m_1$. Read π from left to right, and draw as follows, beginning at $(0, 0)$: upon encountering w_i add $|w_i| + 1$ steps in the x direction, where $|w_i|$ is the length of w_i . Upon encountering m_i , add $m_i - m_{i-1}$ steps in the y direction (m_0 is taken to be 0 by convention). This will give a lattice path between $(0, 0)$ and (n, n) consisting of steps of $(1, 0)$ and $(0, 1)$ and never crossing above the line $x = y$.

Example 1. The permutation 76584213 corresponds with the path encoded by

EEEENNNNENEENN

where E, N represent steps to the east and north directions respectively (since, e.g., a positive step in the x direction takes a path to the east.) In this example, the descending words w_i are of length 3, 0, and 2; and the RLM’s are 3, 4, and 8.

Lemma 1. *If π is a 123-avoiding n -permutation with first ascent at k , then μ , the leftmost right to left maximum preceded by a non-empty word, is at position $k + 1$. Also, μ is either n , or μ is one less than the nearest right to left maximum to its left.*

Proof. The first ascent of π must be to a RLM, as follows: If the first ascent is either at positions $n - 1$ or n , we easily or vacuously have the next symbol being a RLM. If the first ascent is at position $k \in [1, n - 2]$ then the $k + 1$ st symbol must be a RLM, since otherwise the next RLM to the right would enable the formation of a 123. If the permutation does not begin with n , then n must be at the top of the first ascent. This is because if n is not the first number, it must be the top of an ascent, but if an ascent precedes that with the n , that ascent, along with n , would form a 123. This is the case in which $\mu = n$. If n is the first term in π , then π begins with the integers $n(n - 1) \dots (n - a)$ for some $0 \leq a \leq k - 2$ (we choose the maximum such a , and from now on we will say that “ π begins with a regular descent from n of length $a + 1$ ”). The upper bound on a comes from the fact that if π began with regular descent from n of length k (i.e., if we had $a = k - 1$) then all integers greater than the one appearing at index k have already appeared and so the first ascent could not be at k . After the end of the regular descent from n , we can think of the rest of the permutation as a 123-avoiding $(n - a - 1)$ -permutation;

so, by the same reasoning as above, along with the fact that $n - a - 1$ cannot be the first term, or else it would lengthen the regular descent from n , $n - a - 1$ must be at the top of the first ascent, and must be preceded by a non-empty word. \square

Theorem 1. *The Dyck paths that correspond, by the Krattenthaler bijection, with 123-avoiding n -perms with first ascent at k are in bijection with the lattice paths given by Tedford [9], as counted by $C_{n,k}$.*

Proof. Let π be a 123-avoiding n -permutation with first ascent at k . By Lemma 1, the following two cases are exhaustive: either (i) the first ascent is to n , and so $|w_s| = k$ meaning that the path begins with $k + 1$ x steps for a word of length k , followed by a y step for an RLM, or (ii) π begins with a regular descent from n of length j where $1 \leq j \leq k - 1$ and subsequently contains a word of length $k - j$, and then a RLM with value less than the last term in the regular descent from n . In the latter case, the path will begin with j iterations of the pattern (x step, y step) each representing an empty word followed by a RLM one greater than the following RLM, and then will have $k - j + 1$ x steps, corresponding to the word of length $k - j$, and then a y step corresponding to a RLM. In the first case, we have a specific path from $(0, 0)$ to $(k + 1, 1)$. In the second case, we have one specific path for each $1 \leq j \leq k - 1$ from $(0, 0)$ to $(k + 1, j + 1)$. This is to say that every Dyck path that gets to one of these points via the path associated with it represents a 123-avoiding permutation with first ascent at k , and vice versa. Therefore the number of 123-avoiding n -permutations with first ascent at k is given by the number of unique ways to finish a Dyck path from each of these endpoints, i.e., denoting by “good” paths the ones that do not cross the line $x = y$,

$$C_{n,k} = \sum_{i=1}^k |\text{good lattice paths with } E/N \text{ steps from } (k + 1, i) \text{ to } (n, n)|.$$

A bijection between these paths and the paths from $(k, 1)$ to (n, n) is given by taking a path from $(k, 1)$ to (n, n) , and disregarding every step up through the first x step, so that a path from $(k + 1, i); 1 \leq i \leq k$ to (n, n) remains. \square

4. Limit Distributions

The probability that a random permutation on $[n]$ has its first ascent at position k is given, for $1 \leq k \leq n - 1$, by $\frac{k}{(k+1)!}$. To see this, choose any one of the $k + 1$ elements in positions 1 through $k + 1$, except for the smallest, to occupy the $k + 1$ st position, and then arrange the other elements in a monotone decreasing fashion. The chance that the first ascent is at position n is, of course, $\frac{1}{n!}$. We will find it more convenient in this section to consider infinite analogs of the finite distributions we derive. An

infinite permutation may be realized, e.g., by considering a sequence $X_1, X_2 \dots$ of independent and identically distributed (i.i.d.) random variables with say a uniform distribution on $[0,1]$. Under this scheme we get the first ascent distribution as being

$$f(x) = \frac{x}{(x+1)!}, x = 1, 2, \dots,$$

which is similar in form to the unit Poisson distribution with parameter $\lambda = 1$ – and mass function $g(y) = e^{-1}/y!; y = 0, 1, \dots$, mean and variance equal to 1, and generating function $\mathbb{E}(s^Y) = \exp\{s - 1\}$. By contrast, it is shown in [4] that the first ascent distribution above satisfies

$$\mathbb{E}(X) = e - 1; \mathbb{V}(X) = e(3 - e); \mathbb{E}(s^X) = \frac{(1 - e^s + se^s)}{s}.$$

What, on the other hand, can be said about the location distribution of the first ascent in a random 123-avoiding permutation? We see from our earlier results that for a randomly chosen 123-avoiding permutation on $[n]$ the distribution of the location of first ascent is given by

$$f(k) = \frac{C_{n,k}}{C_n} = k \frac{(2n - k - 1)!(n + 1)!}{(2n)!(n - k)!}, \quad k = 1, 2, \dots, n,$$

which, for small k and large n , may be approximated by $f(k) = \frac{k}{2^{k+1}}$. Accordingly, in [4] the authors studied the geometric-like distribution on $\mathbf{Z}^+ = 1, 2, \dots$ defined by

$$f(w) = \frac{w}{2^{w+1}}, w = 1, 2, \dots,$$

showing that

$$\mathbb{E}(W) = 3; \mathbb{E}(s^W) = \frac{s}{s^2 - 4s + 4}.$$

Roughly speaking, the above facts indicate that for a random permutation on a large $[n]$, we expect the first ascent to be at position $e - 1 \approx 1.718$, whereas this value increases to 3 for a random 123-avoiding permutation.

5. Open Questions

A whole series of questions would relate to enumeration of permutations, free of a certain pattern, in which the first occurrence of another pattern occurs at a certain spot. Another direction to pursue might be to consider a specific partial order on $[n]$ and answer the same question as that studied in this paper. Finally, can we use the notion of first ascents in the context of 123-avoiding permutations to give another combinatorial proof of Shapiro’s Catalan Convolution identity, as in [1], [6] (both papers were written in response to a query of R. M. Stanley)?

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