



ON SPARSELY SCHEMMEL TOTIENT NUMBERS

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Abstract

For each positive integer r , let S_r denote the r^{th} Schemmel totient function, a multiplicative arithmetic function defined by

$$S_r(p^\alpha) = \begin{cases} 0, & \text{if } p \leq r; \\ p^{\alpha-1}(p-r), & \text{if } p > r \end{cases}$$

for all primes p and positive integers α . The function S_1 is simply Euler's totient function ϕ . Masser and Shiu have established several fascinating results concerning sparsely totient numbers, positive integers n satisfying $\phi(n) < \phi(m)$ for all integers $m > n$. We define a sparsely Schemmel totient number of order r to be a positive integer n such that $S_r(n) > 0$ and $S_r(n) < S_r(m)$ for all $m > n$ with $S_r(m) > 0$. We then generalize some of the results of Masser and Shiu.

1. Introduction

Throughout this paper, we will let \mathbb{N} and \mathbb{P} denote the set of positive integers and the set of prime numbers, respectively. For any prime p and positive integer n , we will let $\omega(n)$ denote the number of distinct prime factors of n , and we will let $v_p(n)$ denote the exponent of p in the prime factorization of n . Furthermore, we will let $n\#$ denote the product of all the prime numbers less than or equal to n (with the convention $1\# = 1$), and we will let p_i denote the i^{th} prime number.

The Euler totient function $\phi(n)$ counts the number of positive integers less than or equal to n that are relatively prime to n . In 1869, V. Schemmel introduced a class of functions S_r , now known as Schemmel totient functions, that generalize Euler's totient function. $S_r(n)$ counts the number of positive integers $k \leq n$ such that $\gcd(k+j, n) = 1$ for all $j \in \{0, 1, \dots, r-1\}$. Clearly, $S_1 = \phi$. It has been

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shown [6] that S_r is a multiplicative function that satisfies

$$S_r(p^\alpha) = \begin{cases} 0, & \text{if } p \leq r \\ p^{\alpha-1}(p-r), & \text{if } p > r \end{cases}$$

for all primes p and positive integers α . It follows easily from this formula that, for any positive integers r, u, v , we have

$$S_r(u)S_r(v) \leq S_r(uv) \leq \min(uS_r(v), vS_r(u)). \tag{1}$$

For any positive integer r , we will let B_r denote the set of positive integers whose smallest prime factor is greater than r , and we will convene to let $1 \in B_r$. Equivalently,

$$B_r = \{n \in \mathbb{N} : S_r(n) > 0\}.$$

Masser and Shiu have studied the set F of positive integers n that satisfy $\phi(n) < \phi(m)$ for all $m > n$ [5]. These integers are known as sparsely totient numbers, and they motivate the following definition.

Definition 1.1. Let r be a positive integer. A positive integer n is a *sparsely Schemmel totient number of order r* if $n \in B_r$ and $S_r(n) < S_r(m)$ for all $m \in B_r$ with $m > n$. We will let F_r be the set of all sparsely Schemmel totient numbers of order r .

Remark 1.1. Lee-Wah Yip has shown that if r is a positive integer, then there exists a positive constant $c_1(r)$ such that $S_r(n) \geq \frac{c_1(r)n}{(\log \log 3n)^r}$ for all $n \in B_r$ [8]. In fact, this result follows quite easily from Mertens's estimates. Therefore, each set F_r is infinite.

The aim of this paper is to modify some of the proofs that Masser and Shiu used to establish results concerning sparsely totient numbers in order to illustrate how those results generalize to results concerning sparsely Schemmel totient numbers.

2. A Fundamental Construction

The fundamental result in Masser and Shiu's paper, upon which all subsequent theorems rely, is a construction of a certain subset of F , so we will give a similar construction of subsets of the sets F_r .

Lemma 1. Fix some positive integer r , and suppose $x_1, x_2, \dots, x_s, y_1, y_2, \dots, y_s, X, Y$ are real numbers such that $r < x_i \leq y_i$ for all $i \in \{1, 2, \dots, s\}$. If $Y \geq$

$\max(x_1, x_2, \dots, x_s)$ and $X \prod_{i=1}^s x_i < Y \prod_{i=1}^s y_i$, then

$$(X - r) \prod_{i=1}^s (x_i - r) < (Y - r) \prod_{i=1}^s (y_i - r).$$

Proof. The proof is by induction on s , so we will assume that $s \geq 2$ and that the lemma is true if we replace s with $s - 1$. Note that

$$\prod_{i=1}^s (x_i - r) \leq (Y - r) \prod_{i=1}^{s-1} (y_i - r),$$

so the proof is simple if $X < y_s$. Therefore, we will assume that $X \geq y_s$. If we write the inequality $X \prod_{i=1}^s x_i < Y \prod_{i=1}^s y_i$ as $\frac{Xx_s}{y_s} \prod_{i=1}^{s-1} x_i < Y \prod_{i=1}^{s-1} y_i$, then the induction hypothesis tells us that

$$\left(\frac{Xx_s}{y_s} - r\right) \prod_{i=1}^{s-1} (x_i - r) < (Y - r) \prod_{i=1}^{s-1} (y_i - r). \tag{2}$$

Multiplying each side of (2) by $y_s - r$, we see that it suffices to show, in order to complete the induction step, that

$$\left(\frac{Xx_s}{y_s} - r\right) (y_s - r) \geq (X - r)(x_s - r). \tag{3}$$

We may rewrite (3) as $r \left(\frac{Xx_s}{y_s} + y_s\right) \leq r(X + x_s)$, or, equivalently, $y_s - x_s \leq X \left(1 - \frac{x_s}{y_s}\right)$. This inequality holds because $X \geq y_s$, so we have completed the induction step of the proof.

For the case $s = 1$, we note again that the proof is trivial if $X < y_1$, so we will assume that $X \geq y_1$. This implies that $y_1 - x_1 \leq X \left(1 - \frac{x_1}{y_1}\right)$, which we may rewrite as $y_1 + \frac{Xx_1}{y_1} \leq X + x_1$. Multiplying this last inequality by $-r$ and adding $Xx_1 + r^2$ to each side, we get

$$Xx_1 - r \left(y_1 + \frac{Xx_1}{y_1}\right) + r^2 \geq Xx_1 - r(X + x_1) + r^2,$$

so $(y_1 - r) \left(\frac{Xx_1}{y_1} - r\right) \geq (x_1 - r)(X - r)$. As $x_1X < y_1Y$ by hypothesis, we find that $(y_1 - r)(Y - r) > (x_1 - r)(X - r)$. □

In what follows, we will let $\pi(r)$ denote the number of primes less than or equal to r .

Theorem 1. *Let r be a positive integer, and let ℓ and k be nonnegative integers such that $k \geq \pi(r) + 2$. Suppose d is an element of B_r such that $d < p_{k+1} - r$ and $d(p_{k+\ell} - r) < (d + 1)(p_k - r)$. If we set $n = dp_{k+\ell} \prod_{i=\pi(r)+1}^{k-1} p_i$, then $n \in F_r$.*

Proof. First, note that $n \in B_r$. Setting $u = d$ and $v = p_{k+\ell} \prod_{i=\pi(r)+1}^{k-1} p_i$ in (1) yields

$$S_r(n) \leq d(p_{k+\ell} - r) \prod_{i=\pi(r)+1}^{k-1} (p_i - r). \tag{4}$$

Using the hypothesis $d(p_{k+\ell} - r) < (d + 1)(p_k - r)$, we get

$$S_r(n) < (d + 1) \prod_{i=\pi(r)+1}^k (p_i - r), \tag{5}$$

from which the hypothesis $d < p_{k+1} - r$ yields

$$S_r(n) < \prod_{i=\pi(r)+1}^{k+1} (p_i - r). \tag{6}$$

Now, choose some arbitrary $m \in B_r$ with $m > n$. We will show that $S_r(m) > S_r(n)$. There is a unique integer $t > \pi(r)$ such that

$$\prod_{i=\pi(r)+1}^t p_i \leq m < \prod_{i=\pi(r)+1}^{t+1} p_i.$$

Clearly, $\omega(m) \leq t - \pi(r)$, so $\frac{S_r(m)}{m} \geq \prod_{i=\pi(r)+1}^t \left(1 - \frac{r}{p_i}\right)$. This implies that $S_r(m) \geq$

$\prod_{i=\pi(r)+1}^t (p_i - r)$. If $t \geq k + 1$, then we may use (6) to conclude that $S_r(n) < S_r(m)$. Therefore, let us assume that $t \leq k$. Then $\omega(m) \leq k - \pi(r)$. Suppose $\omega(m) \leq k - 1 - \pi(r)$ so that $\frac{S_r(m)}{m} \geq \prod_{i=\pi(r)+1}^{k-1} \left(1 - \frac{r}{p_i}\right)$. From (4), we

have $\frac{S_r(n)}{n} < \prod_{i=\pi(r)+1}^{k-1} \left(1 - \frac{r}{p_i}\right)$, so $\frac{S_r(n)}{n} < \frac{S_r(m)}{m}$. Because $m > n$, we see that $S_r(n) < S_r(m)$.

Now, assume $\omega(m) = k - \pi(r)$. Then we may write $m = \mu \prod_{i=1}^{k-\pi(r)} q_i$, where μ is a positive integer whose prime factors are all in the set $\{q_1, q_2, \dots, q_{k-\pi(r)}\}$ and, for all $i, j \in \{1, 2, \dots, k - \pi(r)\}$ with $i < j$, q_i is a prime and $p_{\pi(r)+i} \leq q_i < q_j$. This means that $S_r(m) = \mu \prod_{i=1}^{k-\pi(r)} (q_i - r)$. If $\mu \geq d + 1$, then we may use (5) to find that

$S_r(n) < \mu \prod_{i=\pi(r)+1}^k (p_i - r) = \mu \prod_{i=1}^{k-\pi(r)} (p_{\pi(r)+i} - r) \leq S_r(m)$. Hence, we may assume that $\mu \leq d$. Because $m > n$, we have

$$\prod_{i=1}^{k-\pi(r)} q_i > \frac{d}{\mu} p_{k+\ell} \prod_{i=\pi(r)+1}^{k-1} p_i. \tag{7}$$

For each $i \in \{1, 2, \dots, k - 1 - \pi(r)\}$, let $x_i = p_{\pi(r)+i}$, and let $y_i = q_i$. If we set $s = k - 1 - \pi(r)$, $X = \frac{d}{\mu} p_{k+\ell}$, and $Y = q_{k-\pi(r)}$, then we may use Lemma 1 and (7) to conclude that

$$\prod_{i=1}^{k-\pi(r)} (q_i - r) > \left(\frac{d}{\mu} p_{k+\ell} - r \right) \prod_{i=\pi(r)+1}^{k-1} (p_i - r).$$

Thus, because $\mu \leq d$, we have

$$\begin{aligned} S_r(m) &= \mu \prod_{i=1}^{k-\pi(r)} (q_i - r) > (d p_{k+\ell} - r \mu) \prod_{i=\pi(r)+1}^{k-1} (p_i - r) \\ &\geq d(p_{k+\ell} - r) \prod_{i=\pi(r)+1}^{k-1} (p_i - r). \end{aligned}$$

Recalling (4), we have $S_r(m) > S_r(n)$, so the proof is complete. □

3. Prime Divisors of Sparsely Schemmel Totient Numbers

In their paper, Masser and Shiu casually mention that 2 is the only sparsely totient prime power [5], but their brief proof utilizes the fact that, for $r = 1$, $r + 1$ is prime. We will see that if $r + 1$ is prime, then $r + 1$ is indeed the only sparsely Schemmel totient number of order r that is a prime power. However, if $r + 1$ is composite, there could easily be multiple sparsely Schemmel totient numbers of order r that are prime powers. The following results will provide an upper bound (in terms of r) for the values of sparsely Schemmel totient prime powers of order r .

r	Sparsely Schemmel Totient Numbers of Order r
2	3, 15*, 21*, 45*, 105*, 165*, 195, 315*, 345, 525*, 585, 735*, 1155*, 1365*, 1785*, 1995*, 2145, 2415, 2625, 3465*, 4095*, 4305, 4515, 5775*, 5985, 6825, 8085, 8925, 9555, 10395*, 15015*, 19635*, 21945*, 23205, 25935, 26565*, 28875, 31395, 33495, 33915, 35805, 45045*, 47355, 49665, 50505, 58905, 65835, 75075*, 77805, 79695, 82005, 84315, 98175
3	1, 5, 7, 35*, 55, 65, 85, 95, 175*, 245*, 385*, 455*, 595*, 665, 805, 875, 1015, 1085, 1295, 1435, 1925*, 2275, 2695*, 3185, 5005*, 6545*, 7315*, 7735, 8855, 10465, 11165, 11935, 13195, 14245, 15785, 16555, 16835, 18095, 18655, 18865, 25025*, 25795, 27335, 35035*, 36575, 38675, 45815, 55055*, 65065*, 85085*, 95095*
4	5, 35*, 55, 65, 85, 175*, 385*, 455*, 595*, 665, 805, 1015, 1085, 1295, 1925*, 2275, 2695*, 5005*, 6545*, 7315*, 7735, 8855, 10465, 11165, 11935, 13195, 14245, 15785, 16555, 16835, 18095, 25025*, 35035*, 36575, 38675, 45815, 55055*, 85085*, 95095*
5	1, 7, 11, 13, 77*, 91*, 119, 133, 161, 203, 217, 259, 287, 301, 539*, 1001*, 1309*, 1463*, 1547, 1771, 2233, 2387, 2849, 3157, 3311, 3619, 4081, 4543, 7007*, 9163, 11011*, 17017*, 19019*, 23023*, 24871, 29029, 31031, 33649, 37037, 41041, 43043, 47047, 53053, 59059, 61061, 67067, 71071, 73073, 79079, 83083
6	7, 77*, 91*, 119, 133, 161, 203, 217, 259, 1001*, 1309*, 1463*, 1547, 1771, 2233, 2387, 2849, 3157, 3311, 3619, 4081, 7007*, 17017*, 19019*, 23023*, 24871, 29029, 31031, 37037, 41041, 43043, 47047, 53053, 59059, 61061, 67067, 71071, 73073, 79079

Table 1: This table lists all Sparsely Schemmel totient numbers of order r that are less than 10^5 for each $r \in \{2, 3, 4, 5\}$. A table of Sparsely totient numbers ($r = 1$) is given in [5]. Those numbers constructed by Theorem 1 are marked with asterisks.

Lemma 2. *If $j \in \mathbb{N} \setminus \{1, 2, 4\}$, then $\frac{p_{j+1}}{p_j} \leq \frac{7}{5}$.*

Proof. Pierre Dusart [2] has shown that, for $x \geq 396\,738$, there must be at least one prime in the interval $\left[x, x + \frac{x}{25 \log^2 x}\right]$. Therefore, whenever $p_j > 396\,738$, we may set $x = p_j + 1$ to get $p_{j+1} \leq (p_j + 1) + \frac{p_j + 1}{25 \log^2(p_j + 1)} < \frac{7}{5}p_j$. Using Mathematica 9.0 [7], we may quickly search through all the primes less than 396 738 to conclude the desired result. \square

Lemma 3. *Let p be a prime, and let r, α , and γ be positive integers such that $\alpha > 1$ and $p \nmid \gamma$. If $p^{\alpha\gamma} \in F_r$, then $p^{\alpha-1}\gamma \in F_r$.*

Proof. Suppose, for the sake of finding a contradiction, that $p^{\alpha-1}\gamma \notin F_r$ and $p^{\alpha\gamma} \in$

F_r . Because $p^\alpha\gamma \in F_r \subseteq B_r$, we know that $p^{\alpha-1}\gamma \in B_r$. Then, because $p^{\alpha-1}\gamma \notin F_r$, there must exist some $m \in B_r$ such that $m > p^{\alpha-1}\gamma$ and $S_r(m) \leq S_r(p^{\alpha-1}\gamma) = p^{\alpha-2}(p-r)S_r(\gamma)$. However, this implies that $pm > p^\alpha\gamma$ and $S_r(pm) \leq pS_r(m) \leq p^{\alpha-1}(p-r)S_r(\gamma) = S_r(p^\alpha\gamma)$, which contradicts the fact that $p^\alpha\gamma \in F_r$. \square

Theorem 2. *If p is a prime and r is a positive integer, then $p \in F_r$ if and only if $r < p < (p_{\pi(r)+1} - r)(p_{\pi(r)+2} - r) + r$.*

Proof. First, suppose $r < p < (p_{\pi(r)+1} - r)(p_{\pi(r)+2} - r) + r$, and let m be an arbitrary element of B_r that is greater than p . We will show that $S_r(m) > p - r$.

If $\omega(m) \geq 2$, then $S_r(m) \geq \prod_{i=1}^{\omega(m)} (p_{\pi(r)+i} - r) \geq (p_{\pi(r)+1} - r)(p_{\pi(r)+2} - r) > p - r$. Therefore, we may assume that $\omega(m) = 1$ so that we may write $m = q^\beta$ for some prime $q > r$ and positive integer β . Furthermore, we may assume $\beta > 1$ because if $\beta = 1$, then $S_r(m) = q - r = m - r > p - r$. If $r \notin \{1, 2, 3, 5\}$, then it is easy to see, with the help of Lemma 2, that $p_{\pi(r)+2} < 2r$. Thus, if $r \notin \{1, 2, 3, 5\}$, then we have $S_r(m) = q^{\beta-1}(q-r) \geq q(q-r) \geq p_{\pi(r)+1}(p_{\pi(r)+1} - r) > r(p_{\pi(r)+1} - r) > (p_{\pi(r)+1} - r)(p_{\pi(r)+2} - r) > p - r$. If $r = 1$, then the inequality $p < (p_{\pi(r)+1} - r)(p_{\pi(r)+2} - r) + r$ forces $p = 2$, so $S_r(m) = q^{\beta-1}(q-r) > 1 = p - r$. If $r = 2$, then the inequality $r < p < (p_{\pi(r)+1} - r)(p_{\pi(r)+2} - r) + r$ forces $p = 3$, so $S_r(m) = q^{\beta-1}(q-r) > 1 = p - r$. If $r = 3$, then $q \geq 5$ and either $p = 5$ or $p = 7$. Therefore, $S_r(m) = q^{\beta-1}(q-r) \geq 5(5-3) > p - r$. Finally, if $r = 5$, then $p \in \{7, 11, 13\}$ and $q \geq 7$. Thus, $S_r(m) = q^{\beta-1}(q-r) \geq 7(7-5) > p - r$.

To prove the converse, suppose $p \geq (p_{\pi(r)+1} - r)(p_{\pi(r)+2} - r) + r$. We wish to find some $m \in B_r$ such that $m > p$ and $S_r(m) \leq p - r$. We may assume that $p > p_{\pi(r)+1}p_{\pi(r)+2}$ because, otherwise, we may simply set $m = p_{\pi(r)+1}p_{\pi(r)+2}$. We know that there exists a unique integer $t \geq \pi(r)+2$ such that $p_{\pi(r)+1}p_t < p < p_{\pi(r)+1}p_{t+1}$. Suppose $r > 3$ so that, with the help of Lemma 2 and some very short casework, we may conclude that $p_{\pi(r)+1} \leq \frac{11}{7}r$ and $p_{t+1} \leq \frac{11}{7}p_t$. Then, setting $m = p_{\pi(r)+1}p_{t+1}$, we have

$$\begin{aligned} S_r(m) &= (p_{\pi(r)+1} - r)(p_{t+1} - r) \leq \frac{4}{7}r \left(\frac{11}{7}p_t - r \right) \\ &< \frac{44}{49}p_{\pi(r)+1}p_t - \frac{4}{7}r^2 < p_{\pi(r)+1}p_t - r < p - r. \end{aligned}$$

We now handle the cases in which $r \leq 3$. If $r = 1$, then p is odd, so we may set $m = 2p$ to get $S_1(m) = S_1(2)S_1(p) = p - 1 = p - r$. If $r = 2$, then $3 \nmid p$, so we may set $m = 3p$ to find $S_2(m) = S_2(3)S_2(p) = p - 2 = p - r$. Finally, if $r = 3$, then we have $5p_t < p < 5p_{t+1}$. Set $m = 5p_{t+1}$. As $p_{t+1} < \frac{5}{2}p_t$, we have $2p_{t+1} - 3 < p$, so $S_3(m) = 2(p_{t+1} - 3) < p - 3 = p - r$. \square

Theorem 3. *Let r be a positive integer, and let p be a prime. Then $p_{\pi(r)+1}^2 \notin F_r$ and $p^3 \notin F_r$.*

Proof. Suppose $p_{\pi(r)+1}^2 \in F_r$. Then, as $p_{\pi(r)+1}p_{\pi(r)+2} > p_{\pi(r)+1}^2$, we must have $(p_{\pi(r)+1} - r)(p_{\pi(r)+2} - r) > p_{\pi(r)+1}(p_{\pi(r)+1} - r)$. Therefore, $r < p_{\pi(r)+2} - p_{\pi(r)+1}$. It is easy to see that this inequality fails to hold for all $r \leq 10$. For $r \geq 11$, we may use Lemma 2 to write $p_{\pi(r)+1} < \sqrt{2}r$ and $p_{\pi(r)+2} < \sqrt{2}p_{\pi(r)+1}$. Hence, $p_{\pi(r)+2} - p_{\pi(r)+1} < (\sqrt{2} - 1)p_{\pi(r)+1} < (2 - \sqrt{2})r < r$, which is a contradiction.

Now, suppose $p^3 \in F_r$. Then, by Lemma 3, we know that $p^2 \in F_r$, so $p > p_{\pi(r)+1}$. Let t be the unique integer such that $p_{\pi(r)+1}p_t < p^2 < p_{\pi(r)+1}p_{t+1}$. Then $p^3 < p_{\pi(r)+1}p_{t+1}p$ and $p_{\pi(r)+1} < p < p_{t+1}$. Therefore, as $p^3 \in F_r$, we see that $S_r(p^3) = p^2(p - r) < (p_{\pi(r)+1} - r)(p_{t+1} - r)(p - r)$, implying that $p^2 < (p_{t+1} - r)(p_{\pi(r)+1} - r) < p_{t+1}(p_{\pi(r)+1} - r)$. Using Bertrand's Postulate, we see that $p_{t+1} < 2p_t$ and $p_{\pi(r)+1} \leq 2r$. Therefore, $p_{\pi(r)+1}p_t < p^2 < p_{t+1}(p_{\pi(r)+1} - r) < 2p_t(p_{\pi(r)+1} - r)$, so $2r < p_{\pi(r)+1}$. This is our desired contradiction. \square

Combining Lemma 3, Theorem 2, and Theorem 3, we see that any $n \in F_r$ satisfying $n \geq ((p_{\pi(r)+1} - r)(p_{\pi(r)+2} - r) + r)^2$ must have at least two prime factors. Furthermore, we record the following conjecture about the nonexistence of sparsely Schemmel totient numbers that are squares of primes. This conjecture has been checked for all $r \leq 16$ and $p \leq 317$.

Conjecture 1. *For any prime p and positive integer r , $p^2 \notin F_r$.*

We now proceed to establish asymptotic results concerning the primes that divide and do not divide sparsely Schemmel totient numbers. For a given $r \in \mathbb{N}$ and $n \in F_r$, we will define $P_k(n)$ to be the k^{th} largest prime divisor of n (provided $\omega(n) \geq k$), and we will let $Q_k(n)$ denote the k^{th} smallest prime that is larger than r and does not divide n (the functions Q_k depends on r , but this should not lead to confusion because we will work with fixed values of r). We will let $R(n) = n \prod_{\substack{p \in \mathbb{P} \\ p|n}} p^{-1}$. We will also make use of the Jacobsthal function J . For a positive

integer n , $J(n)$ is defined to be the smallest positive integer a such that every set of a consecutive integers contains an element that is relatively prime to n [4]. In particular, for any positive integer r , $J(r\#)$ is the largest possible difference between consecutive elements of B_r . This means that, for any positive real x , the smallest element of B_r that is greater than x is at most $x + J(r\#)$. For convenience, we will write $J_r = J(r\#)$. The first sixteen values of J_r (starting with $r = 1$) are 1, 2, 4, 6, 10, 14, 22, 26, 34, 40, 46, 58, 66, 74, 90, 100.

Finally, we will let $\lambda_k(r)$ be the unique positive real root of the polynomial $\frac{J_r}{r}x^k + kx - (k - 1)$.

Lemma 4. *If r , n , and k are positive integers such that $k \geq 2$, $n \in F_r$, and $\omega(n) \geq k$, then $Q_{k-1}(n) > \lambda_k(r)(P_k(n) - r)$.*

Proof. Write $M = \prod_{i=1}^k P_i(n)$ and $N = \prod_{i=1}^{k-1} Q_i(n)$. Let μ be the smallest element of B_r that is greater than $\frac{M}{N}$. As noted above, we must have $\mu < \frac{M}{N} + J_r$. Let us put $m = \frac{\mu N}{M}n$ so that $m \in B_r$ and

$$1 < \frac{m}{n} < 1 + \frac{J_r N}{M} < 1 + J_r \frac{Q_{k-1}(n)^{k-1}}{P_k(n)^k}.$$

Since, $\frac{n}{M}$ is an integer that is relatively prime to N , we may use the multiplicativity of S_r along with (1) to write

$$S_r(m) = S_r\left(\mu N \frac{n}{M}\right) \leq \mu S_r\left(N \frac{n}{M}\right) = \mu S_r(N) S_r\left(\frac{n}{M}\right) \leq \mu \frac{S_r(N) S_r(n)}{S_r(M)}.$$

This implies that

$$\begin{aligned} \frac{S_r(m)}{m} &\leq \frac{S_r(N)}{N} \frac{M}{S_r(M)} \frac{S_r(n)}{n} = \prod_{i=1}^{k-1} \left(1 - \frac{r}{Q_i(n)}\right) \prod_{j=1}^k \left(1 - \frac{r}{P_j(n)}\right)^{-1} \frac{S_r(n)}{n} \\ &< \left(1 - \frac{r}{Q_{k-1}(n)}\right)^{k-1} \left(1 - \frac{r}{P_k(n)}\right)^{-k} \frac{S_r(n)}{n}. \end{aligned}$$

Therefore,

$$S_r(m) < \left(1 + J_r \frac{Q_{k-1}(n)^{k-1}}{P_k(n)^k}\right) \left(1 - \frac{r}{Q_{k-1}(n)}\right)^{k-1} \left(1 - \frac{r}{P_k(n)}\right)^{-k} S_r(n),$$

so the fact that $n \in F_r$ implies that

$$\left(1 + J_r \frac{Q_{k-1}(n)^{k-1}}{P_k(n)^k}\right) \left(1 - \frac{r}{Q_{k-1}(n)}\right)^{k-1} \left(1 - \frac{r}{P_k(n)}\right)^{-k} > 1. \tag{8}$$

Write $x_1 = J_r \frac{Q_{k-1}(n)^{k-1}}{P_k(n)^k}$, $x_2 = \frac{r}{Q_{k-1}(n)}$, and $x_3 = \frac{r}{P_k(n)}$ so that (8) becomes $(1+x_1)(1-x_2)^{k-1}(1-x_3)^{-k} > 1$. Because x_1 and x_2 are positive and $0 < x_3 < 1$, we may invoke the inequalities $1+x_1 < e^{x_1}$, $1-x_2 < e^{-x_2}$, and $(1-x_3)^{-1} < e^{x_3/(1-x_3)}$ to write

$$e^{x_1 - (k-1)x_2 + kx_3/(1-x_3)} > 1. \tag{9}$$

After a little algebraic manipulation, (9) becomes

$$\frac{J_r}{r} \left(\frac{Q_{k-1}(n)}{P_k(n)}\right)^k + k \frac{Q_{k-1}(n)}{P_k(n) - r} - (k-1) > 0.$$

Thus, if we write $A(x) = \frac{J_r}{r}x^k + kx - (k - 1)$, then $A\left(\frac{Q_{k-1}(n)}{P_k(n) - r}\right) > 0$. This means that $\frac{Q_{k-1}(n)}{P_k(n) - r} > \lambda_k(r)$, so we are done. \square

Lemma 5. *For any positive integers r and n with $n \in F_r$,*

$$P_1(n) < Q_1(n) \left(1 - J_r + \frac{J_r}{r}Q_1(n)\right).$$

Proof. Fix r and n , and write $P = P_1(n)$ and $Q = Q_1(n)$. Suppose, for the sake of finding a contradiction, that

$$P \geq Q \left(1 - J_r + \frac{J_r}{r}Q\right).$$

Let μ be the smallest element of B_r that is greater than $\frac{P}{Q}$. Then $\mu < \frac{P}{Q} + J_r$. Write $m = \frac{Q\mu}{P}n$ so that $m \in B_r$ and $1 < \frac{m}{n} < 1 + \frac{J_rQ}{P} \leq 1 + \frac{J_r}{1 - J_r + \frac{J_r}{r}Q}$. Because m is divisible by Q and all the prime divisors of n except possibly P , we have

$$\frac{S_r(m)}{m} \leq \left(1 - \frac{r}{Q}\right) \left(1 - \frac{r}{P}\right)^{-1} \frac{S_r(n)}{n}.$$

Therefore,

$$\begin{aligned} S_r(m) &< \left(1 - \frac{r}{Q}\right) \left(1 - \frac{r}{P}\right)^{-1} \left(1 + \frac{J_r}{1 - J_r + \frac{J_r}{r}Q}\right) S_r(n) \\ &\leq \left(1 - \frac{r}{Q}\right) \left(1 - \frac{r}{Q(1 - J_r + \frac{J_r}{r}Q)}\right)^{-1} \left(1 + \frac{J_r}{1 - J_r + \frac{J_r}{r}Q}\right) S_r(n) \\ &= \left(1 - \frac{r}{Q}\right) \left(1 - J_r + \frac{J_r}{r}Q - \frac{r}{Q}\right)^{-1} \left(1 + \frac{J_r}{r}Q\right) S_r(n) = S_r(n). \end{aligned}$$

This is our desired contradiction, so the proof is complete. \square

For the following lemma, recall that we defined $R(n) = n \prod_{\substack{p \in \mathbb{P} \\ p|n}} p^{-1}$.

Lemma 6. *Let r be a positive integer, and let $n \in F_r$. Then*

$$R(n) < \frac{J_r}{r}Q_1(n)(Q_1(n) - r).$$

Proof. Fix r , and n , and write $Q = Q_1(n)$ and $R = R(n)$. Suppose $R \geq \frac{J_r}{r}Q(Q - r)$. Let μ be the smallest element of B_r greater than $\frac{R}{Q}$. Then $\mu < \frac{R}{Q} + J_r$. If we put $m = \frac{Q\mu}{R}n$, then $m \in B_r$ and

$$1 < \frac{m}{n} < 1 + \frac{J_r Q}{R} \leq 1 + \frac{r}{Q - r}.$$

Because m is divisible by Q and all the prime divisors of n , we have

$$\frac{S_r(m)}{m} \leq \left(1 - \frac{r}{Q}\right) \frac{S_r(n)}{n}.$$

This implies that

$$S_r(m) < \left(1 - \frac{r}{Q}\right) \left(1 + \frac{r}{Q - r}\right) S_r(n) = S_r(n),$$

which is a contradiction. □

Corollary 1. *Let r and Ω be positive integers, and let*

$$V_r(\Omega) = \frac{J_r}{r} p_{\pi(r)+\Omega} (p_{\pi(r)+\Omega} - r).$$

If $n \in F_r$ and $n \geq V_r(\Omega) \frac{(V_r(\Omega) + p_{\pi(r)+\Omega})\#}{p_{\pi(r)}\#}$, then $\omega(n) \geq \Omega$. In particular, for $n \in F_r$,

$$\lim_{n \rightarrow \infty} \omega(n) = \infty.$$

Proof. We prove the contrapositive. Suppose $n \in F_r$ and $\omega(n) < \Omega$. Then $Q_1(n) \leq p_{\pi(r)+\Omega}$, so $P_1(n) < p_{\pi(r)+\Omega} \left(1 - J_r + \frac{J_r}{r} p_{\pi(r)+\Omega}\right) = V_r(\Omega) + p_{\pi(r)+\Omega}$ by Lemma

5. This implies that $R(n) = n \prod_{\substack{p \in \mathbb{P} \\ p|n}} p^{-1} \geq n \frac{p_{\pi(r)}\#}{(V_r(\Omega) + p_{\pi(r)+\Omega})\#}$. Lemma 6 tells us

that $R(n) < V_r(\Omega)$, so

$$n \leq R(n) \frac{(V_r(\Omega) + p_{\pi(r)+\Omega})\#}{p_{\pi(r)}\#} < V_r(\Omega) \frac{(V_r(\Omega) + p_{\pi(r)+\Omega})\#}{p_{\pi(r)}\#}.$$

□

Corollary 2. *Let r be a positive integer. For sufficiently large $n \in F_r$, $P_1(n)^4 \nmid n$.*

Proof. For any integer $n > 1$, write $v_{P_1(n)}(n) = \eta(n)$. Using Lemma 6, we see that, for any $n \in F_r$ satisfying $n > 1$,

$$P_1(n)^{\eta(n)-1} \leq R(n) < \frac{J_r}{r} Q_1(n)(Q_1(n) - r).$$

Because $Q_1(n)$ is at most the smallest prime exceeding $P_1(n)$, we may use Bertrand's Postulate to write

$$\frac{J_r}{r} Q_1(n)(Q_1(n) - r) \leq 2 \frac{J_r}{r} P_1(n)(2P_1(n) - r) < 4 \frac{J_r}{r} P_1(n)^2.$$

If $P_1(n)^4 | n$, then $\eta(n) - 1 \geq 3$, so $P_1(n) < 4 \frac{J_r}{r}$. By Corollary 1, we see that this is impossible for sufficiently large n . \square

Masser and Shiu show that $P_1(n)^3 \nmid n$ for all sparsely totient numbers n , but their methods are not obviously generalizable [5]. Thus, we make the following conjecture, which has been checked for all $r \leq 16$ and $n \leq 100000$.

Conjecture 2. *For any positive integers r and n with $n \in F_r$ and $n > 1$, $P_1(n)^3 \nmid n$.*

For small values of r , we may effortlessly make small amounts of progress toward Conjecture 2. For example, it is easy to use Lemma 6 to show that $P_1(n)^4 \nmid n$ for all $n \in F_2$. Indeed, if $P_1(n)^4 | n$ for some $n \in F_2$, then $P_1(n) < 4 \cdot \frac{J_2}{2} = 4$. This forces n to be a power of 3, but Theorem 3 tells us that there are no powers of 3 in F_2 except 3 itself.

We are finally ready to establish our promised asymptotic results.

Theorem 4. *Let r, K , and L be positive integers with $K \geq 2$. For $n \in F_r$, we have*

- (a) $\limsup_{n \rightarrow \infty} \frac{P_1(n)}{\log n} \geq 2,$
- (b) $\limsup_{n \rightarrow \infty} \frac{Q_L(n)}{\log n} = 1,$
- (c) $\limsup_{n \rightarrow \infty} \frac{P_K(n)}{\log n} \leq \lambda_K(r)^{-1},$
- (d) $\limsup_{n \rightarrow \infty} \frac{P_1(n)}{\log^2 n} \leq \frac{J_r}{r}.$

Proof. To prove (a), let us begin by choosing some integer $k \geq \pi(r) + 2$. Let $\ell(k)$ be the largest integer such that $p_{k+\ell(k)} < 2p_k - r$. Setting $n(k) = p_{k+\ell(k)} \prod_{i=\pi(r)+1}^{k-1} p_i$, we

see, by Theorem 1, that $n(k) \in F_r$. Furthermore, as $\prod_{i=\pi(r)+1}^k p_i \leq n(k) < \prod_{i=\pi(r)+1}^{k+1} p_i$, the Prime Number Theorem tells us that $p_k \sim \log n(k)$ as $k \rightarrow \infty$. Thus, as $k \rightarrow \infty$, $P_1(n(k)) = p_{k+\ell(k)} \sim 2p_k \sim 2 \log n(k)$.

To prove (b), choose any $n \in F_r$ with $n > 1$, and let $k(n)$ be the unique integer satisfying $\prod_{i=\pi(r)+1}^{k(n)} p_i \leq n < \prod_{i=\pi(r)+1}^{k(n)+1} p_i$. Using the Prime Number Theorem again, we have $Q_L(n) \leq p_{k(n)+L} \sim \log n$ as $n \rightarrow \infty$. In addition, for those $n \in F_r$ (guaranteed by Theorem 1) of the form $n = \prod_{i=\pi(r)+1}^{k(n)} p_i$, we see that $Q_L(n) = p_{k(n)+L} \sim \log n$.

Corollary 1 guarantees that the limit in (c) is well-defined. To prove the limit, we use Lemma 4 to find that if $n \in F_r$ and $\omega(n) \geq K$, then

$$\frac{P_K(n)}{\log n} < \lambda_K(r)^{-1} \frac{Q_{K-1}(n)}{\log n} + \frac{r}{\log n}.$$

Then the desired result follows from setting $L = K - 1$ in (b).

Finally, (d) follows immediately from Lemma 5 and from setting $L = 1$ in (b). \square

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