REPRESENTATIONS BY CERTAIN OCTONARY QUADRATIC FORMS WITH COEFFICIENTS 1, 2, 3, AND 6

Şaban Alaca[1]
School of Mathematics and Statistics, Carleton University, Ottawa, Canada
salaca@math.carleton.ca

Yavuz Kesicioğlu[2]
Department of Mathematics, Recep Tayyip Erdoğan University, Rize, Turkey
yavuz.kesicioglu@erdogan.edu.tr

Received: 1/2/14, Accepted: 1/20/15, Published: 2/18/15

Abstract

We determine formulae for the representation numbers for certain octonary quadratic forms with coefficients 1, 2, 3 and 6. We use a modular form approach.

1. Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, rational numbers and complex numbers respectively. For $a_1, \ldots, a_8 \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$N(a_1, \ldots, a_8; n) := \operatorname{card}\{(x_1, \ldots, x_8) \in \mathbb{Z}^8 \mid n = a_1x_1^2 + \cdots + a_8x_8^2\}.$$

Clearly $N(a_1, \ldots, a_8; 0) = 1$. Without loss of generality we may suppose that

$$a_1 \leq \cdots \leq a_8 \ \text{ and } \ \gcd(a_1, \ldots, a_8) = 1.$$

Formulae for $N(a_1, \ldots, a_n; n)$ for the octonary quadratic forms

$$(x_1^2 + \cdots + x_i^2) + 2(x_{i+1}^2 + \cdots + x_{i+j}^2) \\ +3(x_{i+j+1}^2 + \cdots + x_{i+j+k}^2) + 6(x_{i+j+k+1}^2 + \cdots + x_{i+j+k+l}^2) \quad (1.1)$$

under the conditions $i + j + k + l = 8$ and $i \equiv j \equiv k \equiv l \equiv 0 \pmod 2$ appeared in literature. See [1], [2], [3], [4], [7] and [8]. In this paper we determine formulae for $N(a_1, \ldots, a_n; n)$ for the octonary quadratic forms (1.1) under the conditions

$$i + j + k + l = 8 \ \text{ and } \ i \equiv j \equiv k \equiv l \equiv 1 \pmod 2. \quad (1.2)$$

[1]The research of the first author was supported by a research grant from the Natural Sciences and Engineering Research Council of Canada.

[2]The research of the second author was supported by Recep Tayyip Erdoğan University and the Council of Higher Education of Turkey.

There are ten such octonary quadratic forms namely

$$(i,j,k,l) = \begin{array}{l}(1,1,3,3),\ (1,3,1,3),\ (1,3,3,1),\ (3,3,1,1),\ (3,1,3,1),\\ (3,1,1,3)\ (1,1,1,5),\ (1,1,5,1),\ (1,5,1,1),\ (5,1,1,1).\end{array} \quad (1.3)$$

We write $N(1^i, 2^j, 3^k, 6^l; n)$ to denote the number of representations of n by an octonary quadratic form (i, j, k, l) listed in (1.3). We use a modular form approach. For $q \in \mathbb{C}$ with $|q| < 1$ we define

$$F(q) := \prod_{n=1}^{\infty}(1-q^n). \quad (1.4)$$

The following eight infinite products are defined in [1]. The extra q factors in the products are due to the slightly different definition of $F(q)$ in this paper.

$$A_1(q) := \sum_{n=1}^{\infty} a_1(n)q^n = qF^4(q^2)F^4(q^4), \quad (1.5)$$

$$A_2(q) := \sum_{n=1}^{\infty} a_2(n)q^n = qF^2(q)F^2(q^2)F^2(q^3)F^2(q^6), \quad (1.6)$$

$$A_3(q) := \sum_{n=1}^{\infty} a_3(n)q^n = \frac{qF^5(q^2)F(q^3)F^2(q^4)F^5(q^6)}{F^3(q)F^2(q^{12})}, \quad (1.7)$$

$$A_4(q) := \sum_{n=1}^{\infty} a_4(n)q^n = \frac{q^2F(q)F(q^2)F^9(q^6)}{F^3(q^3)}, \quad (1.8)$$

$$A_5(q) := \sum_{n=1}^{\infty} a_5(n)q^n = q^2F^2(q^2)F^2(q^4)F^2(q^6)F^2(q^{12}), \quad (1.9)$$

$$A_6(q) := \sum_{n=1}^{\infty} a_6(n)q^n = q^3F^4(q^6)F^4(q^{12}), \quad (1.10)$$

$$A_7(q) := \sum_{n=1}^{\infty} a_7(n)q^n = \frac{q^4F(q^2)F(q^4)F^9(q^{12})}{F^3(q^6)}, \quad (1.11)$$

$$A_8(q) := \sum_{n=1}^{\infty} a_8(n)q^n = q^4F^2(q^4)F^2(q^8)F^2(q^{12})F^2(q^{24}). \quad (1.12)$$

For $q \in \mathbb{C}$ with $|q| < 1$ Ramanujan's theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

We have

$$\sum_{n=0}^{\infty} N(a_1, \ldots, a_8; n)q^n = \varphi(q^{a_1}) \cdots \varphi(q^{a_8}). \quad (1.13)$$

The infinite product representation of $\varphi(q)$ is due to Jacobi [4] namely

$$\varphi(q) = \frac{F^5(q^2)}{F^2(q)F^2(q^4)}. \tag{1.14}$$

For $k \in \mathbb{N}$ the sum of divisors function $\sigma_k(n)$ is defined by $\sigma_k(n) = \sum_{d|n} d^k$, where the sum is over the positive divisors of n. If $n \notin \mathbb{N}$, we set $\sigma_k(n) = 0$. For $q \in \mathbb{C}$ with $|q| < 1$ the Eisenstein series $E_4(q)$ is defined by

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n. \tag{1.15}$$

We now state our main result. We prove it in Section 3.

Theorem 1.1. *Let $n \in \mathbb{N}$. Then*

(i) $$N(1^3, 2^3, 3, 6; n) = \frac{13}{10}\sigma_3(n) - \frac{13}{10}\sigma_3(n/2) + \frac{27}{10}\sigma_3(n/3) - \frac{26}{5}\sigma_3(n/4) - \frac{27}{10}\sigma_3(n/6) + \frac{416}{5}\sigma_3(n/8) - \frac{54}{5}\sigma_3(n/12) + \frac{864}{5}\sigma_3(n/24) + 3a_1(n) + 27a_1(n/3) + \frac{17}{10}a_2(n) + 8a_2(n/2) + \frac{256}{5}a_2(n/4) + 3a_4(n) - 12a_4(n/2)$$

(ii) $$N(1^3, 2, 3^3, 6; n) = \sigma_3(n) - \sigma_3(n/2) - 9\sigma_3(n/3) + 2\sigma_3(n/4) + 9\sigma_3(n/6) - 32\sigma_3(n/8) - 18\sigma_3(n/12) + 288\sigma_3(n/24) - 2a_1(n) - 27a_1(n/3) + 2a_2(n) + 7a_2(n/2) - 72a_2(n/4) + 5a_3(n) - 12a_4(n) + 24a_4(n/2)$$

(iii) $$N(1^3, 2, 3, 6^3; n) = \frac{1}{2}\sigma_3(n) - \frac{1}{2}\sigma_3(n/2) - \frac{9}{2}\sigma_3(n/3) + 2\sigma_3(n/4) + \frac{9}{2}\sigma_3(n/6) - 32\sigma_3(n/8) - 18\sigma_3(n/12) + 288\sigma_3(n/24) - a_1(n) - 9a_1(n/3) + \frac{5}{2}a_2(n) + 12a_2(n/2) - 32a_2(n/4) + 4a_3(n) - 9a_4(n) + 12a_4(n/2)$$

(iv) $$N(1, 2^3, 3^3, 6; n) = \frac{1}{2}\sigma_3(n) - \frac{1}{2}\sigma_3(n/2) - \frac{9}{2}\sigma_3(n/3) + 2\sigma_3(n/4) + \frac{9}{2}\sigma_3(n/6) - 32\sigma_3(n/8) - 18\sigma_3(n/12) + 288\sigma_3(n/24) - 3a_1(n) - 27a_1(n/3) + \frac{1}{2}a_2(n) - 64a_2(n/4) + 4a_3(n) - 9a_4(n) + 12a_4(n/2)$$

(v) $$N(1, 2^3, 3, 6^3; n) = \frac{1}{4}\sigma_3(n) - \frac{1}{4}\sigma_3(n/2) - \frac{9}{4}\sigma_3(n/3) + 2\sigma_3(n/4) + \frac{9}{4}\sigma_3(n/6) - 32\sigma_3(n/8) - 18\sigma_3(n/12) + 288\sigma_3(n/24) - \frac{3}{2}a_1(n) - 9a_1(n/3) + \frac{3}{4}a_2(n) + \frac{5}{2}a_2(n/2) - 28a_2(n/4) + \frac{5}{2}a_3(n) - \frac{9}{2}a_4(n) + 6a_4(n/2)$$

(vi) $N(1,2,3^3,6^3;n) = \frac{1}{10}\sigma_3(n) - \frac{1}{10}\sigma_3(n/2) + \frac{39}{10}\sigma_3(n/3) - \frac{2}{5}\sigma_3(n/4)$
$-\frac{39}{10}\sigma_3(n/6) + \frac{32}{5}\sigma_3(n/8) - \frac{78}{5}\sigma_3(n/12) + \frac{1248}{5}\sigma_3(n/24) + a_1(n)$
$+9a_1(n/3) + \frac{9}{10}a_2(n) + 4a_2(n/2) + \frac{32}{5}a_2(n/4) - a_4(n) + 4a_4(n/2)$

(vii) $N(1^5,2,3,6;n) = \frac{13}{5}\sigma_3(n) - \frac{13}{5}\sigma_3(n/2) + \frac{27}{5}\sigma_3(n/3) - \frac{26}{5}\sigma_3(n/4)$
$-\frac{27}{5}\sigma_3(n/6) + \frac{416}{5}\sigma_3(n/8) - \frac{54}{5}\sigma_3(n/12) + \frac{864}{5}\sigma_3(n/24) + 2a_1(n)$
$+27a_1(n/3) + \frac{12}{5}a_2(n) + 17a_2(n/2) + \frac{296}{5}a_2(n/4) + 3a_3(n) - 24a_4(n/2)$

(viii) $N(1,2^5,3,6;n) = \frac{13}{20}\sigma_3(n) - \frac{13}{20}\sigma_3(n/2) + \frac{27}{20}\sigma_3(n/3) - \frac{26}{5}\sigma_3(n/4)$
$-\frac{27}{20}\sigma_3(n/6) + \frac{416}{5}\sigma_3(n/8) - \frac{54}{5}\sigma_3(n/12) + \frac{864}{5}\sigma_3(n/24) + \frac{5}{2}a_1(n)$
$+27a_1(n/3) + \frac{7}{20}a_2(n) + \frac{5}{2}a_2(n/2) + \frac{276}{5}a_2(n/4) - \frac{3}{2}a_3(n) + \frac{15}{2}a_4(n)$
$-18a_4(n/2)$

(ix) $N(1,2,3^5,6;n) = \frac{1}{5}\sigma_3(n) - \frac{1}{5}\sigma_3(n/2) + \frac{39}{5}\sigma_3(n/3) - \frac{2}{5}\sigma_3(n/4)$
$-\frac{39}{5}\sigma_3(n/6) + \frac{32}{5}\sigma_3(n/8) - \frac{78}{5}\sigma_3(n/12) + \frac{1248}{5}\sigma_3(n/24) + 2a_1(n)$
$+15a_1(n/3) + \frac{4}{5}a_2(n) + 5a_2(n/2) + \frac{72}{5}a_2(n/4) - a_3(n) + 8a_4(n/2)$

(x) $N(1,2,3,6^5;n) = \frac{1}{20}\sigma_3(n) - \frac{1}{20}\sigma_3(n/2) + \frac{39}{20}\sigma_3(n/3) - \frac{2}{5}\sigma_3(n/4)$
$-\frac{39}{20}\sigma_3(n/6) + \frac{32}{5}\sigma_3(n/8) - \frac{78}{5}\sigma_3(n/12) + \frac{1248}{5}\sigma_3(n/24) + \frac{1}{2}a_1(n)$
$+3a_1(n/3) + \frac{19}{20}a_2(n) + \frac{9}{2}a_2(n/2) - \frac{28}{5}a_2(n/4) + \frac{1}{2}a_3(n) - \frac{5}{2}a_4(n)$
$+6a_4(n/2)$

2. Preliminary Results

Let N be a positive integer and k an integer. We write $M_k(\Gamma_0(N))$ to denote the space of modular forms of weight k with trivial multiplier system for the modular subgroup $\Gamma_0(N)$ defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad-bc=1,\ c \equiv 0 \pmod{N} \right\}.$$

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by the product formula

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}). \tag{2.1}$$

Throughout the remainder of the paper we take $q = q(z) := e^{2\pi i z}$ with $z \in \mathbb{H}$ so that $|q| < 1$. By (1.4) and (2.1) we have

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = q^{1/24} F(q). \tag{2.2}$$

An eta quotient is defined to be a finite product of the form

$$f(z) = \prod_{\delta} \eta^{r_\delta}(\delta z), \tag{2.3}$$

where δ runs through a finite set of positive integers and the exponents r_δ are any integers. By taking N to be the least common multiple of the δ's we can write the eta quotient (2.3) as

$$f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z),$$

where δ runs through positive divisors of N. When all of the exponents r_δ are nonnegative, $f(z)$ is said to be an eta product.

The following lemma follows from [5, Theorem 5.7, p.99] and[6, Corollary 2.3, p.37]. We use it to determine if a given eta quotient $f(z)$ is in $M_k(\Gamma_0(N))$.

Lemma 2.1. *Let N be a positive integer and let $f(z) = \prod_{1 \le \delta \mid N} \eta^{r_\delta}(\delta z)$ be an eta quotient which satisfies the following conditions:*

(i) $\sum_{1 \le \delta \mid N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,

(ii) $\sum_{1 \le \delta \mid N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,

(iii) $\prod_{1 \le \delta \mid N} \delta^{r_\delta}$ *is the square of a rational number,*

(iv) *For each $d \mid N$,* $\sum_{1 \le \delta \mid N} \frac{\gcd(d, \delta)^2 \cdot r_\delta}{\delta} \ge 0$,

(v) *The weight $k = \frac{1}{2} \sum_{1 \le \delta \mid N} r_\delta$ is an even integer.*

Then $f(z)$ is in $M_k(\Gamma_0(N))$.

It is shown in [1] that

$$\left\{E_4(q^k)\ (k=1,2,3,4,6,8,12,24), A_k(q)\ (1\le k\le 8)\right\} \tag{2.4}$$

is a basis for $M_4(\Gamma_0(24))$. We use Lemma 2.1 to prove the following theorem.

Theorem 2.1. *Let i,j,k,l be nonnegative odd integers such that $i+j+k+l=8$. Then $\varphi^i(q)\varphi^j(q^2)\varphi^k(q^3)\varphi^l(q^6)$ are in $M_4(\Gamma_0(24))$.*

Proof. We just present the proof for $\varphi^3(q)\varphi(q^2)\varphi^3(q^3)\varphi(q^6)$ as all the other cases can be proved similarly. From (1.14) and (2.2) we obtain

$$\varphi^3(q)\varphi(q^2)\varphi^3(q^3)\varphi(q^6) = \frac{\eta^{13}(q^2)\eta^{13}(q^6)}{\eta^6(q)\eta^6(q^3)\eta(q^4)\eta^2(q^8)\varphi(q^{12})\varphi^2(q^{24})}.$$

We take $N=24$. We have

$\delta_1=1, r_1=-6,\quad \delta_3=3, r_3=-6,\quad \delta_6=6, r_6=13,\quad \delta_{12}=12, r_{12}=-1,$
$\delta_2=2, r_2=13,\quad \delta_4=4, r_4=-1,\quad \delta_8=8, r_8=-2,\quad \delta_{24}=24, r_{24}=-2.$

It can be easily seen that conditions (i)–(iii) and (v) of Lemma 2.1 are satisfied. We show that (iv) is also satisfied for each positive divisor d of 24. We just show it for $d=3$ as the remaining cases can be done similarly. Note that

$$\sum_{0<\delta|N}\frac{\gcd(d,\delta)^2\cdot r_\delta}{\delta} = \frac{1^2\cdot(-6)}{1}+\frac{1^2\cdot 13}{2}+\frac{3^2\cdot(-6)}{3}+\frac{1^2\cdot(-1)}{4}$$
$$+\frac{3^2\cdot 13}{6}+\frac{1^2\cdot(-2)}{8}+\frac{3^2\cdot(-1)}{12}+\frac{3^2\cdot(-2)}{24}=0.$$

Thus by Lemma 2.1 we have $\varphi^3(q)\varphi(q^2)\varphi^3(q^3)\varphi(q^6)\in M_4(\Gamma_0(24))$. □

3. Proof of Theorem 1.1

We first prove the following theorem from which we deduce the proof of Theorem 1.1.

Theorem 3.1.

(i) $$\varphi^3(q)\varphi^3(q^2)\varphi(q^3)\varphi(q^6) = \frac{13}{2400}E_4(q)-\frac{13}{2400}E_4(q^2)+\frac{9}{800}E_4(q^3)$$
$$-\frac{13}{600}E_4(q^4)-\frac{9}{800}E_4(q^6)+\frac{26}{75}E_4(q^8)-\frac{9}{200}E_4(q^{12})+\frac{18}{25}E_4(q^{24})$$
$$+3A_1(q)+\frac{17}{10}A_2(q)+3A_4(q)+8A_5(q)+27A_6(q)-12A_7(q)+\frac{256}{5}A_8(q)$$

(ii) $$\varphi^3(q)\varphi(q^2)\varphi^3(q^3)\varphi(q^6) = \frac{1}{240}E_4(q)-\frac{1}{240}E_4(q^2)-\frac{3}{80}E_4(q^3)+\frac{1}{120}E_4(q^4)$$

$$+\frac{3}{80}E_4(q^6)-\frac{2}{15}E_4(q^8)-\frac{3}{40}E_4(q^{12})+\frac{6}{5}E_4(q^{24})-2A_1(q)+2A_2(q)$$
$$+5A_3(q)-12A_4(q)+7A_5(q)-27A_6(q)+24A_7(q)-72A_8(q)$$

(iii) $$\varphi^3(q)\varphi(q^2)\varphi(q^3)\varphi^3(q^6)=\frac{1}{480}E_4(q)-\frac{1}{480}E_4(q^2)-\frac{3}{160}E_4(q^3)+\frac{1}{120}E_4(q^4)$$
$$+\frac{3}{160}E_4(q^6)-\frac{2}{15}E_4(q^8)-\frac{3}{40}E_4(q^{12})+\frac{6}{5}E_4(q^{24})-A_1(q)+\frac{5}{2}A_2(q)$$
$$+4A_3(q)-9A_4(q)+12A_5(q)-9A_6(q)+12A_7(q)-32A_8(q)$$

(iv) $$\varphi(q)\varphi^3(q^2)\varphi^3(q^3)\varphi(q^6)=\frac{1}{480}E_4(q)-\frac{1}{480}E_4(q^2)-\frac{3}{160}E_4(q^3)$$
$$+\frac{1}{120}E_4(q^4)+\frac{3}{160}E_4(q^6)-\frac{2}{15}E_4(q^8)-\frac{3}{40}E_4(q^{12})+\frac{6}{5}E_4(q^{24})$$
$$-3A_1(q)+\frac{1}{2}A_2(q)+4A_3(q)-9A_4(q)-27A_6(q)+12A_7(q)-64A_8(q)$$

(v) $$\varphi(q)\varphi^3(q^2)\varphi(q^3)\varphi^3(q^6)=\frac{1}{960}E_4(q)-\frac{1}{960}E_4(q^2)-\frac{3}{320}E_4(q^3)+\frac{1}{120}E_4(q^4)$$
$$+\frac{3}{320}E_4(q^6)-\frac{2}{15}E_4(q^8)-\frac{3}{40}E_4(q^{12})+\frac{6}{5}E_4(q^{24})-\frac{3}{2}A_1(q)+\frac{3}{4}A_2(q)$$
$$+\frac{5}{2}A_3(q)-\frac{9}{2}A_4(q)+\frac{5}{2}A_5(q)-9A_6(q)+6A_7(q)-28A_8(q)$$

(vi) $$\varphi(q)\varphi(q^2)\varphi^3(q^3)\varphi^3(q^6)=\frac{1}{2400}E_4(q)-\frac{1}{2400}E_4(q^2)+\frac{13}{800}E_4(q^3)$$
$$-\frac{1}{600}E_4(q^4)-\frac{13}{800}E_4(q^6)+\frac{2}{75}E_4(q^8)-\frac{13}{200}E_4(q^{12})+\frac{26}{25}E_4(q^{24})$$
$$+A_1(q)+\frac{9}{10}A_2(q)-A_4(q)+4A_5(q)+9A_6(q)+4A_7(q)+\frac{32}{5}A_8(q)$$

(vii) $$\varphi^5(q)\varphi(q^2)\varphi(q^3)\varphi(q^6)=\frac{13}{1200}E_4(q)-\frac{13}{1200}E_4(q^2)+\frac{9}{400}E_4(q^3)$$
$$-\frac{13}{600}E_4(q^4)-\frac{9}{400}E_4(q^6)+\frac{26}{75}E_4(q^8)-\frac{9}{200}E_4(q^{12})+\frac{18}{25}E_4(q^{24})$$
$$+2A_1(q)+\frac{12}{5}A_2(q)+3A_3(q)+17A_5(q)+27A_6(q)-24A_7(q)+\frac{296}{5}A_8(q)$$

(viii) $$\varphi(q)\varphi^5(q^2)\varphi(q^3)\varphi(q^6)=\frac{13}{4800}E_4(q)-\frac{13}{4800}E_4(q^2)+\frac{9}{1600}E_4(q^3)$$
$$-\frac{13}{600}E_4(q^4)-\frac{9}{1600}E_4(q^6)+\frac{26}{75}E_4(q^8)-\frac{9}{200}E_4(q^{12})+\frac{18}{25}E_4(q^{24})+\frac{5}{2}A_1(q)$$
$$+\frac{7}{20}A_2(q)-\frac{3}{2}A_3(q)+\frac{15}{2}A_4(q)+\frac{5}{2}A_5(q)+27A_6(q)-18A_7(q)+\frac{276}{5}A_8(q)$$

(ix) $$\varphi(q)\varphi(q^2)\varphi^5(q^3)\varphi(q^6)=\frac{1}{1200}E_4(q)-\frac{1}{1200}E_4(q^2)+\frac{13}{400}E_4(q^3)-\frac{1}{600}E_4(q^4)$$
$$-\frac{13}{400}E_4(q^6)+\frac{2}{75}E_4(q^8)-\frac{13}{200}E_4(q^{12})+\frac{26}{25}E_4(q^{24})$$

$$+2A_1(q)+\frac{4}{5}A_2(q)-A_3(q)+5A_5(q)+15A_6(q)+8A_7(q)+\frac{72}{5}A_8(q)$$

(x) $$\varphi(q)\varphi(q^2)\varphi(q^3)\varphi^5(q^6)=\frac{1}{4800}E_4(q)-\frac{1}{4800}E_4(q^2)+\frac{13}{1600}E_4(q^3)-\frac{1}{600}E_4(q^4)$$
$$-\frac{13}{1600}E_4(q^6)+\frac{2}{75}E_4(q^8)-\frac{13}{200}E_4(q^{12})+\frac{26}{25}E_4(q^{24})+\frac{1}{2}A_1(q)+\frac{19}{20}A_2(q)$$
$$+\frac{1}{2}A_3(q)-\frac{5}{2}A_4(q)+\frac{9}{2}A_5(q)+3A_6(q)+6A_7(q)-\frac{28}{5}A_8(q)$$

Proof. Let i,j,k,l be nonnegative odd integers such that $i+j+k+l=8$. By Theorem 2.1, $\varphi^i(q)\varphi^j(q^2)\varphi^k(q^3)\varphi^l(q^6)\in M_4(\Gamma_0(24))$. By (2.4) we have that $\left\{E_4(q^k)\ (k=1,2,3,4,6,8,12,24), A_k(q)\ (1\le k\le 8)\right\}$ is a basis of $M_4(\Gamma_0(24))$. Thus $\varphi^i(q)\varphi^j(q^2)\varphi^k(q^3)\varphi^l(q^6)$ can be expressed as a linear combination of $E_4(q^k)$ $(k=1,2,3,4,6,8,12,24)$ and $A_k(q)$ $(1\le k\le 8)$. Using MAPLE we obtain the asserted coefficients. □

We now use Theorem 3.1 to prove Theorem 1.1.

Proof of Theorem 1.1. We just present the proof of (ii) as all the other cases can be proved similarly. By (1.13), Theorem 3.1(ii), (1.5)–(1.12) and (1.15) we have

$$\sum_{n=0}^{\infty}N(1^3,2,3^3,6;n)q^n=\varphi^3(q)\varphi(q^2)\varphi^3(q^3)\varphi(q^6)$$
$$=\frac{1}{240}E_4(q)-\frac{1}{240}E_4(q^2)-\frac{3}{80}E_4(q^3)+\frac{1}{120}E_4(q^4)+\frac{3}{80}E_4(q^6)-\frac{2}{15}E_4(q^8)$$
$$-\frac{3}{40}E_4(q^{12})+\frac{6}{5}E_4(q^{24})-2A_1(q)+2A_2(q)+5A_3(q)-12A_4(q)$$
$$+7A_5(q)-27A_6(q)+24A_7(q)-72A_8(q)$$
$$=1+\sum_{n=1}^{\infty}\Big(\sigma_3(n)-\sigma_3(n/2)-9\sigma_3(n/3)+2\sigma_3(n/4)+9\sigma_3(n/6)-32\sigma_3(n/8)$$
$$-18\sigma_3(n/12)+288\sigma_3(n/24)-2a_1(n)-27a_1(n/3)+2a_2(n)+7a_2(n/2)$$
$$-72a_2(n/4)+5a_3(n)-12a_4(n)+24a_4(n/2)\Big)q^n.$$

Equating coefficients of q^n yields the result. □

Acknowledgments. The research was carried out during the second author's visit to Carleton University, Ottawa, Canada. He would like to thank Carleton University for the hospitality during his stay.

References

[1] Ş. Alaca and Y. Kesicioğlu, *Representations by certain octonary quadratic forms whose coefficients are* 1, 2, 3 *and* 6, Int. J. Number Theory **10** (2014), 133-150.

[2] A. Alaca, Ş. Alaca and K. S. Williams, *Fourteen octonary quadratic forms,* Int. J. Number Theory **6** (2010), 37-50.

[3] Ş. Alaca and K. S. Williams, *The number of representations of a positive integer by certain octonary quadratic forms,* Funct. Approx. Comment. Math. **43** (2010), 45-54.

[4] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum,* Bornträger, Regiomonti, 1829. (Gesammelte Werke, Erster Band, Chelsea Publishing Co., New York, 1969, pp. 49-239.)

[5] L. J. P. Kilford, *Modular Forms: A classical and computational introduction,* Imperial College Press, London, 2008.

[6] G. Köhler, *Eta Products and Theta Series Identities,* Springer Monographs in Mathematics, Springer, 2011.

[7] B. Köklüce, *The representation numbers of three octonary quadratic forms,* Int. J. Number Theory **9** (2013), 505-516.

[8] B. Ramakrishnan and B. Sahu, *Evaluation of the convolution sums* $\sum_{l+15m=n} \sigma(l)\sigma(m)$ *and* $\sum_{3l+5m=n} \sigma(l)\sigma(m)$ *and an application,* Int. J. Number Theory **9** (2013), 799-809.