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THE STRUCTURE OF RAINBOW-FREE COLORINGS FOR LINEAR EQUATIONS ON THREE VARIABLES IN $\mathbb{Z}_p$

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Abstract

Let p be a prime number and $\mathbb{Z}_p$ be the cyclic group of order p. A coloring of $\mathbb{Z}_p$ is called rainbow–free with respect to a certain equation, if it contains no rainbow solution of the equation, that is, a solution whose elements have pairwise distinct colors. In this paper we describe the structure of rainbow–free 3–colorings of $\mathbb{Z}_p$ with respect to all linear equations on three variables. Consequently, we determine those linear equations on three variables for which every 3–coloring (with nonempty color classes) of $\mathbb{Z}_p$ contains a rainbow solution of it.

1. Introduction

A *k–coloring* of a set X is a surjective mapping $c : X \to \{1, 2, ...k\}$, or equivalently a partition $X = C_1 \cup C_2 \cup ... \cup C_k$, where each nonempty set C_i is called a *color class*. A subset $Y \subseteq X$ is *rainbow* under c, if the coloring pairwise assigns distinct colors to the elements of Y. The study of the existence of rainbow structures falls into the anti–Ramsey Theory. Canonical versions of this theory prove the existence of either a monochromatic structure or a rainbow structure. In contrast, in the recent so–called *Rainbow Ramsey Theory* the existence of rainbow structures is guaranteed under some density conditions on the color classes (see [1, 3, 5, 4] and references therein). Beyond this approach, recent works [6, 7] have addressed the problem of describing the shape of colorings containing no rainbow structures, called *rainbow–free* colorings.

Let p be a prime number and $\mathbb{Z}_p$ be the cyclic group of order p. Among other results, Jungić et al. [3] proved that every 3–coloring of $\mathbb{Z}_p$ with the cardinality of the smallest color class greater than or equal to four has a rainbow solution of all linear equations in three variables with the only possible exception of $x+y+z=d$. In other words, the authors proved that rainbow–free colorings of $\mathbb{Z}_p$ concerning the equation $a_1x+a_2y+a_3z=b$, where some $a_i \neq a_j$, are such that the smallest color class has less than four elements. In this work we analyze the "small cases" (cases when the smallest color class has one, two or three elements) in order to fully characterize the structure of rainbow–free colorings.

Our main result, Theorem 6, implies that, actually, rainbow–free colorings of $\mathbb{Z}_p$ concerning equation $a_1x+a_2y+a_3z=b$, with some $a_i \neq a_j$, are such that the cardinality of the smallest color class is one. Moreover, Theorem 6 characterizes the structure of such colorings. Therefore, we provide a criterion to decide whether or not, for a given equation and a given prime number, there exists a rainbow–free coloring. In other words, we classify equations (depending on a_1, a_2, a_3, b and p) for which every 3–coloring contains a rainbow solution (Corollary 8).

The paper is organized as follows. In Section 2 we establish the notation and give some preliminary results. In Section 3 we present our results: first we give the structure characterization of rainbow free colorings of $\mathbb{Z}_p$ concerning equation $x+y+z=b$ (Theorem 5), which is the only one admitting rainbow–free colorings with large color classes. We point out that Theorem 5 is deduced from known results with relatively little effort. In Theorem 6 we give the structure characterization of rainbow–free colorings concerning equation $a_1x+a_2y+a_3z=b$ with some $a_i \neq a_j$. The proof of Theorem 6 is divided into three parts. In Section 4 we handle the case when there is a color class of cardinality one, which is the only case where rainbow–free colorings exist. In Sections 6 and 7 we discard the cases when the smallest color class has cardinality two or three respectively. As usual in the area, we will use as tools to solve those later cases some inverse results in additive number theory presented in Section 5.

2. Notation and Preliminaries

Let p be a prime number and $\mathbb{Z}_p$ be the cyclic group of order p. Given a set $S \subseteq \mathbb{Z}_p$ and elements $t, d \in \mathbb{Z}_p$, the sets: $S+t := \{x+t : x \in S\}$ and $dS := \{dx : x \in S\}$ are called the *t–translation* and the *d–dilation* of S respectively. Concerning the multiplicative group $\mathbb{Z}_p^* := \mathbb{Z}_p \setminus \{0\}$, for every $d \in \mathbb{Z}_p^*$ we denote by d^{-1} the multiplicative inverse of d, and by $\langle d \rangle$ the subgroup of $\mathbb{Z}_p^*$ generated by d. We say that a subset $S \subseteq \mathbb{Z}_p^*$ is *$\langle d \rangle$–periodic* if it is invariant under d–dilation, that is $S = dS$. Note that a set which is $\langle d \rangle$–periodic is a union of cosets of $\langle d \rangle$. A set S which is $\langle -1 \rangle$–periodic is also called *symmetric*. An arithmetic progression with common difference 1 is called an *interval*.

Let $d, t \in \mathbb{Z}_p$, $d \neq 0$, and $S \subseteq \mathbb{Z}_p$; throughout the paper we will work with the transformation $T_{d,t} : \mathbb{Z}_p \to \mathbb{Z}_p$ defined as:

$$T_{d,t}(S) = dS + t = \{dx + t : x \in S\}.$$

Naturally, a set $S \subseteq \mathbb{Z}_p$ is called *invariant* under $T_{d,t}$, if $T_{d,t}(S) = S$. The following observation is not difficult to prove:

Observation 1.

- *If $d = 1$, $T_{d,t}$ is a t–translation.*
- *If $d \neq 1$, the transformation $T_{d,t}$ has a unique fixed point which is $t(1-d)^{-1}$.*
- *For $d \neq 1$, a set $S \subseteq \mathbb{Z}_p$ is invariant under $T_{d,t}$ if and only if $S + t(d-1)^{-1}$ is a $\langle d \rangle$–periodic set.*

We will work with the most general linear equation on three variables written as:

$$a_1x + a_2y + a_3z = b \tag{1}$$

which has $x, y, z \in \mathbb{Z}_p$ variables, and $a_1, a_2, a_3, b \in \mathbb{Z}_p$ constants such that $a_1a_2a_3 \neq 0$. Naturally, a set $\{s_1, s_2, s_3\}$ of elements in $\mathbb{Z}_p$ is a solution of Equation (1), if for some choice of $\{i, j, k\} = \{1, 2, 3\}$, $a_1s_i + a_2s_j + a_3s_k = b$. An observation that will be important later is the following: a solution $\{s_1, s_2, s_3\}$ of Equation (1) with $s_i = s_j := s$ for some $i \neq j$ is such that $s_1 = s_2 = s_3$ if and only if $s(a_1+a_2+a_3) = b$.

A 3–*coloring* of $\mathbb{Z}_p$ is a partition $\mathbb{Z}_p = A \cup B \cup C$ with nonempty color classes. A solution of Equation (1) is *rainbow*, if the elements belong pairwise to distinct color classes. A 3–coloring of $\mathbb{Z}_p$ is said to be *rainbow–free for Equation (1)* if it contains no rainbow solution of Equation (1).

In [3] it was proved that every 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $4 \leq |A| \leq |B| \leq |C|$ has a rainbow solution of Equation (1) except when $a_1 = a_2 = a_3$.

Theorem 2 (Jungić et al. Theorem 4.1 of [3])**.** *Let $a_1, a_2, a_3, b \in \mathbb{Z}_p$ with $a_1a_2a_3 \neq 0$. Then every partition of $\mathbb{Z}_p = A \cup B \cup C$ with $|A|, |B|, |C| \geq 4$ contains a rainbow solution of $a_1x + a_2y + a_3z = b$ with the only exception being when $a_1 = a_2 = a_3 =: a$, and every color class is an arithmetic progression with the same common difference d, such that $d^{-1}A = \{i\}_{i=t_1}^{t_2-1}$, $d^{-1}B = \{i\}_{i=t_2}^{t_3-1}$, and $d^{-1}C = \{i\}_{i=t_3}^{t_1-1}$, where $t_1 + t_2 + t_3 = a^{-1}b + 1$, or $a^{-1}b + 2$.*

We shall note that this description has an error. It works well when $d = 1$, but do not for other values of d. Take for instance $\mathbb{Z}_{13} = A \cup B \cup C$ with $A = \{2, 4, 6, 8\}$, $B = \{10, 12, 1, 3\}$, and $C = \{5, 7, 9, 11, 0\}$ (three arithmetic progressions with difference $d = 2$), which is a rainbow–free coloring for $x + y + z = 2$, and does not satisfy the condition mentioned above. In Theorem 5 we correct the statement.

In [6] the case when some $a_i = a_j$ and $b = 0$ in Equation (1) was considered. The authors provided the description of rainbow–free colorings in this particular case with no restrictions on the size of the color classes.

Theorem 3 (Llano, Montejano. Theorem 2 of [6])**.** *A 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $1 \le |A| \le |B| \le |C|$ is rainbow–free for $x+y = cz$ if and only if, under dilation, one of the following holds true:*

1. $A = \{0\}$, *with both* B *and* C *symmetric* $\langle c \rangle$*–periodic subsets.*
2. $A = \{1\}$ *for*
   i) $c = 2$, *with* $(B-1)$ *and* $(C-1)$ *symmetric* $\langle 2 \rangle$*–periodic subsets;*
   ii) $c = -1$, *with* $(B \setminus \{-2\}) + 2^{-1}$ *and* $(C \setminus \{-2\}) + 2^{-1}$ *symmetric subsets.*
3. $|A| \ge 2$, *for* $c = -1$, *with* A, B, *and* C *arithmetic progressions with difference* 1, *such that* $A = \{i\}_{i=t_1}^{t_2-1}$, $B = \{i\}_{i=t_2}^{t_3-1}$, *and* $C = \{i\}_{i=t_3}^{t_1-1}$, *where* $(t_1+t_2+t_3) = 1$ *or* 2.

We will use both Theorems 2 and 3 in order to fully characterize the structure of rainbow–free colorings concerning Equation (1).

Next we prove that, besides the case when $a_1 + a_2 + a_3 = 0$, the structure of rainbow–free colorings of $\mathbb{Z}_p$ concerning Equation (1) with $b \neq 0$ is the same as the structure of rainbow–free colorings for Equation (1) with $b = 0$ under a suitable translation.

Lemma 4. *Let* $a := a_1+a_2+a_3 \neq 0$, *and let* $T := T_{1,-ba^{-1}}$. *Then,* $\mathbb{Z}_p = A \cup B \cup C$ *is rainbow–free for Equation (1) if and only if* $\mathbb{Z}_p = T(A) \cup T(B) \cup T(C)$ *is rainbow–free for equation* $a_1x + a_2y + a_3z = 0$.

Proof. The set $\{s_1, s_2, s_3\}$ is a solution of Equation (1) if and only if the set $\{T(s_1), T(s_2), T(s_3)\}$ is a solution of $a_1x + a_2y + a_3z = 0$. □

Lemma 4 indicates that, if $a_1 + a_2 + a_3 \neq 0$, it is sufficient to study rainbow–free colorings of Equation (1) for $b = 0$.

3. Results

First we consider Equation (1) with $a_1 = a_2 = a_3$. That is, we describe all rainbow–free colorings of $\mathbb{Z}_p$ concerning equation $x + y + z = b$. This equation is the only one admitting rainbow–free colorings with large color classes. The next theorem is a direct consequence of Theorem 3, and Lemma 4.

Theorem 5. *For* $p > 3$, *a 3–coloring* $\mathbb{Z}_p = A \cup B \cup C$ *with* $1 \le |A| \le |B| \le |C|$ *is rainbow–free with respect to equation* $x + y + z = b$, *if and only if one of the following holds true:*

i) $A = \{s\}$ *with both* $(B \setminus \{b-2s\}) + (s-b)2^{-1}$ *and* $(C \setminus \{b-2s\}) + (s-b)2^{-1}$ *symmetric sets.*

ii) $|A| \geq 2$ *and all* A*,* B *and* C *are arithmetic progressions with the same common difference* d *so that* $d^{-1}A = \{i\}_{i=t_1}^{t_2-1}$*,* $d^{-1}B = \{i\}_{i=t_2}^{t_3-1}$*, and* $d^{-1}C = \{i\}_{i=t_3}^{t_1-1}$ *satisfy* $t_1 + t_2 + t_3 \in \{1 + d^{-1}b, 2 + d^{-1}b\}$*.*

Proof. For $b = 0$ we deduce the structure of rainbow–free colorings from Theorem 3. Since $a_1 + a_2 + a_3 \neq 0$, we use Lemma 4 to complete the structure characterization. □

Consider now Equation (1) with some $a_i \neq a_j$. In contrast with Theorem 5, we find that all rainbow–free colorings are such that there is one color class of cardinality one. We will let this color class be $A = \{s\}$. Before stating our main result, we define for every $i \in \{1, 2, \ldots, 6\}$ the transformation $T_i : \mathbb{Z}_p \to \mathbb{Z}_p$ as $T_i(x) := T_{d_i,t_i}(x) = d_i x + t_i$ where:

$$\begin{array}{ll} d_1 = -a_3 a_1^{-1} & t_1 = (b - a_2 s)a_1^{-1} \\ d_2 = -a_2 a_1^{-1} & t_2 = (b - a_3 s)a_1^{-1} \\ d_3 = -a_1 a_2^{-1} & t_3 = (b - a_3 s)a_2^{-1} \\ d_4 = -a_3 a_2^{-1} & t_4 = (b - a_1 s)a_2^{-1} \\ d_5 = -a_1 a_3^{-1} & t_5 = (b - a_2 s)a_3^{-1} \\ d_6 = -a_2 a_3^{-1} & t_6 = (b - a_1 s)a_3^{-1} \end{array}$$

Theorem 6. *A* 3*–coloring* $\mathbb{Z}_p = A \cup B \cup C$ *with* $1 \leq |A| \leq |B| \leq |C|$ *is rainbow–free for equation:*

$$a_1 x + a_2 y + a_3 z = b, \quad \text{with some } a_i \neq a_j \tag{2}$$

if and only if $A = \{s\}$ *with* $s(a_1 + a_2 + a_3) = b$*, and both* B *and* C *are sets invariant under* T_i *for every* $i \in \{1, 2, \ldots, 6\}$*.*

Proof. The proof is deduced from Theorem 2, and the lemmas in Sections 6 and 7. Let $\mathbb{Z}_p = A \cup B \cup C$ with $1 \leq |A| \leq |B| \leq |C|$ be a rainbow–free coloring of Equation (2). Then Theorem 2 implies that $|A| \in \{1, 2, 3\}$. From Lemmas 20 and 21 (concerning the case $|A| = 2$), and Lemmas 26 and 30 (concerning the case $|A| = 3$) we deduce that actually $|A| = 1$. The rest of the proof follows by Lemma 12 (concerning the case $|A| = 1$). □

Given a prime number p, an equation will be called *rainbow* with respect to p, if every 3–coloring of $\mathbb{Z}_p$ contains a rainbow solution of it. Consequently, a *non–rainbow* equation with respect to a prime number p is an equation such that there are rainbow–free colorings of $\mathbb{Z}_p$ for the equation. For instance, $x+y+z = b$ is a non–rainbow equation with respect to all primes, and $x + y = 2z$ is a rainbow equation with respect to p, if and only if p satisfies either $|\langle 2\rangle| = p-1$ or $|\langle 2\rangle| = (p-1)/2$ where $(p-1)/2$ is an odd number (see Theorem 3.5 of [3], or Corollary 1 of [6]).

We can deduce from Theorem 6 which equations are rainbow (hence, which ones are non–rainbow). We generalize the above result about 3–term arithmetic progressions in the next corollary. Before continuing we highlight an important consequence of Observation 1.

Observation 7. *If $s(a_1 + a_2 + a_3) = b$, then s is the fixed point of each T_i for $i \in \{1, ..., 6\}$. Moreover, when $s(a_1 + a_2 + a_3) = b$, a set X is invariant under T_i for all $i \in \{1, ..., 6\}$ if and only if $X - s$ is a $\langle d_1, d_2, ..., d_6\rangle$-periodic set.*

Theorem 6 can also be stated in the opposite manner: a coloring $\mathbb{Z}_p = A \cup B \cup C$ has a rainbow solution of Equation (2) if and only if some of the following holds true:

i) $2 \leq \min\{|A|, |B|, |C|\}$.

ii) $a_i + a_2 + a_3 = 0 \neq b$.

iii) $T_i(X) \neq X$ for some $X \in \{B, C\}$ and some $i \in \{1, ..., 6\}$.

Corollary 8. *Every 3–coloring of $\mathbb{Z}_p$ contains a rainbow solution of Equation (2) if and only if one of the following holds true:*

i) $a_1 + a_2 + a_3 = 0 \neq b$

ii) $|\langle d_1, d_2, ..., d_6\rangle| = p - 1$.

Proof. If $a_1 + a_2 + a_3 = 0 \neq b$ then the second point in the previous paragraph implies that every 3–coloring of $\mathbb{Z}_p$ contains a rainbow solution of Equation (2). If $|\langle d_1, d_2, ..., d_6\rangle| = p - 1$, then, by Observation 7, it is impossible to simultaneously satisfy $T_i(B) = B$ and $T_i(C) = C$ for every $i \in \{1, ..., 6\}$. Thus, by the third point in the previous paragraph we conclude the desired implication.

On the other hand, suppose that $a_1 + a_2 + a_3 = 0 = b$ or $a_1 + a_2 + a_3 \neq 0$, and $|\langle d_1, d_2, ..., d_6\rangle| < p-1$. Then, according to Theorem 6 there exist rainbow-free colorings of $\mathbb{Z}_p$. □

Corollary 8 gives a criterion to determine which equations are rainbow; see Figure 1. To finish this section we describe with two particular examples how to construct rainbow–free colorings of $\mathbb{Z}_p$ for equations provided with the following conditions: $a_1 + a_2 + a_3 = 0 = b$ or $a_1 + a_2 + a_3 \neq 0$, and $|\langle d_1, d_2, ..., d_6\rangle| < p - 1$.

Example 9. Consider $\mathbb{Z}_{13}$ and the equation $x - 4y + 3z = 0$. Since $1 - 4 + 3 = 0 = b$, in order to construct a rainbow–free coloring, we can let A be any point of $\mathbb{Z}_{13}$. Let $A = \{0\}$. Since $\langle d_1, d_2, ..., d_6\rangle = \langle 4\rangle$ (so $|\langle d_1, d_2, ..., d_6\rangle| \neq 12$), we can divide

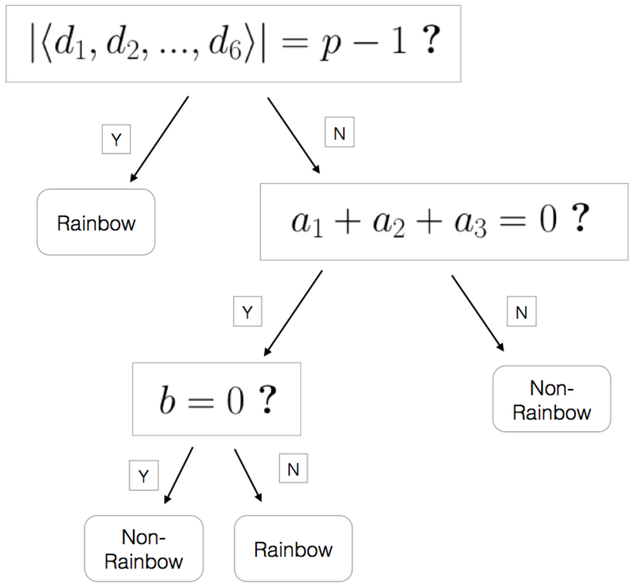

Figure 1: Test to determine if a given equation is rainbow, where $\langle d_1, d_2, ..., d_6\rangle$ is the subgroup of $\mathbb{Z}_p^*$ generated by $\{d_1, \ldots, d_6\}$.

$\mathbb{Z}_{13} \setminus \{0\} = B \cup C$ in such a way that both B and C are $\langle d_1, d_2, ..., d_6\rangle$–periodic sets. We let $B = \{1, 3, 4, 9, 10, 12\}$ and $C = \{2, 5, 6, 7, 8, 11\}$. In this case, also any translation of such coloring will be rainbow–free.

Example 10. Consider $\mathbb{Z}_{17}$ and the equation $x+8y-2z = 3$. Since $1+8-2 = 7 \neq 0$ then, in order to construct a rainbow–free coloring, we let $A = \{(3)(7^{-1})\} = \{15\}$. Since $\langle d_1, d_2, ..., d_6\rangle = \langle 2\rangle$ (so $|\langle d_1, d_2, ..., d_6\rangle| \neq 16$) we can divide $\mathbb{Z}_{17} \setminus \{15\} = B \cup C$ in such a way that both B and C are translations of $\langle d_1, d_2, ..., d_6\rangle$–periodic sets. We let $B = \{16, 0, 2, 6, 7, 11, 13, 14\}$ and $C = \{1, 3, 4, 5, 8, 9, 10, 12\}$.

4. The Case $|A| = 1$

In this section we describe all rainbow–free colorings of $\mathbb{Z}_p$ concerning Equation (2) with a color class of cardinality one.

Lemma 11. *Let $\mathbb{Z}_p = \{s\} \cup B \cup C$ be a rainbow–free coloring for Equation (2). Then $B \subseteq T_i(B) \cup \{d_i s + t_i\}$ and $C \subseteq T_i(C) \cup \{d_i s + t_i\}$ for every $i \in \{1, 2, \ldots, 6\}$.*

Proof. Consider the partition $\mathbb{Z}_p = T_i(s) \cup T_i(B) \cup T_i(C)$, and suppose $B \not\subseteq T_i(B) \cup$

$T_i(s)$ then $B \cap T_i(C) \neq \emptyset$. Thus, there exist $u \in B$ and $v \in C$ such that $u = d_i v + t_i$. It is not hard to see then, that $\{u, v, s\}$ will be a rainbow solution of Equation (2). □

The aim of this section is to prove that, in fact, a rainbow–free coloring with a color class of cardinality one is such that the other color classes are invariant under T_i for every $i \in \{1, \ldots, 6\}$. It is not difficult to see that $\mathbb{Z}_p = \{s\} \cup B \cup C$ with $T_i(B) = B$ and $T_i(C) = C$ is a rainbow–free coloring. We will prove this and the converse. The converse provides a restriction on s in terms of a_1, a_2, a_3 and b.

Lemma 12. *A 3–coloring $\mathbb{Z}_p = \{s\} \cup B \cup C$ is rainbow–free for Equation (2) if and only if s is such that*

$$s(a_1 + a_2 + a_3) = b \tag{3}$$

and

$$\begin{aligned} T_i(B) &= B \\ T_i(C) &= C \end{aligned} \tag{4}$$

for every $i \in \{1, 2, \ldots, 6\}$.

Proof. Assume without loss of generality that $a_2 \neq a_3$. The "if" part of the statement follows since, for both $X \in \{B, C\}$, every solution $\{s_1, s_2, s_3\}$ of Equation (2) that has one element in A and other element in X is such that $\{s_1, s_2, s_3\} \subseteq A \cup X$ so the 3–coloring is rainbow free.

Conversely, assume that the 3–coloring is rainbow–free. Suppose first that $d_i s + t_i = s$ for some $i \in \{1, 2, \ldots, 6\}$. Then, it is not hard to see that Equation (3) is satisfied. Since the coloring is rainbow–free, any solution $\{s, u, v\}$ with $u \in B$ is such that $v \in B \cup A$, but Lemma 11 and Equation (3) indicate that actually $v \in B$. Therefore $T_i(B) \subseteq B$ for every $i \in \{1, 2, \ldots, 6\}$. The same is true for C, thus we get (4) by using the cardinalities of the sets.

Assume now that $d_i s + t_i \neq s$ for every $i \in \{1, 2, \ldots, 6\}$. Without loss of generality let $d_1 s + t_1 \in C$. Since $B \subseteq T_1(B) \cup \{d_1 s + t_1\}$ by Lemma 11, we have $B \subseteq T_1(B)$. Thus $T_1(B) = B$, that is

$$B = d_1 B + t_1. \tag{5}$$

Note that $d_1 s + t_1 = -a_3 a_1^{-1} s + b a_1^{-1} - a_2 a_1^{-1} s = d_2 s + t_2$ then, by the same arguments, we get $T_2(B) = B$, that is

$$B = d_2 B + t_2. \tag{6}$$

By a dilation of Equation (6), we get

$$d_1 B = d_2 d_1 B + d_1 t_2. \tag{7}$$

By a translation of Equation (6), we get $B - t_1 = d_2B + t_2 - t_1$ which can be expressed as

$$B - t_1 = d_2(B - t_1) + (t_2 - t_1 + d_2t_1). \tag{8}$$

From Equation (5) we know that $B - t_1 = d_1B$. By substituting $B - t_1$ with d_1B in Equation (8) we get: $d_1B = d_2d_1B + (t_2 - t_1 + d_2t_1)$. Now we can use Equation (7) to conclude that:

$$d_1t_2 = t_2 - t_1 + d_2t_1. \tag{9}$$

By simple calculations it follows that Equation (9) is equivalent to Equation (3), a contradiction by the assumption that $d_is + t_i \neq s$ for every $i \in \{1, 2, \ldots, 6\}$. □

5. Additive Tools

Before treating the remaining cases $|A| = 2$ and $|A| = 3$, we give some results in additive number theory. These results have been used previously in solving arithmetic anti-Ramsey problems [3, 6]. As usual, for sets $X, Y \subseteq \mathbb{Z}_p$, let $X + Y = \{x + y : x \in X, y \in Y\}$. The well-known Cauchy–Davenport Theorem [8] states that for any $X, Y \subseteq \mathbb{Z}_p$ with $X + Y \neq \mathbb{Z}_p$, it happens that $|X + Y| \geq |X| + |Y| - 1$.

Lemma 13. *Let $\mathbb{Z}_p = A \cup B \cup C$ be a rainbow–free coloring for Equation (2). Then, for every $\{i, j, k\} = \{1, 2, 3\}$ and $\{X, Y, Z\} = \{A, B, C\}$ we have*

$$a_iX + a_jY \subseteq -a_k(X \cup Y) + b.$$

In particular

$$|X| + |Y| - 1 \leq |a_iX + a_jY| \leq |X| + |Y|.$$

Proof. Since $\mathbb{Z}_p = A \cup B \cup C$ is a rainbow–free coloring for Equation (2), we get that

$$a_iX + a_jY \cap -a_kZ + b = \emptyset$$

for every $\{i, j, k\} = \{1, 2, 3\}$ and $\{X, Y, Z\} = \{A, B, C\}$. Consequently:

$$a_iX + a_jY \subseteq \mathbb{Z}_p \setminus (-a_kZ + b). \tag{10}$$

Since $p = |X| + |Y| + |Z|$ and $|-a_kZ + b| = |Z|$, from Inclusion (10) on one side, and Cauchy–Davenport on the other we obtain:

$$|X| + |Y| - 1 \leq |a_iX + a_jY| \leq |X| + |Y|$$

and the proof is completed. □

The next two important results characterize the structure of subsets in $\mathbb{Z}_p$ with $|X+Y| = |X| + |Y| - 1$, and $|X+Y| = |X| + |Y|$, respectively.

Theorem 14 (Vosper [9])**.** *Let $X, Y \subseteq \mathbb{Z}_p$ with $|X|, |Y| \geq 2$, and*

$$|X+Y| = |X| + |Y| - 1 \leq p - 2.$$

Then both X and Y are arithmetic progressions with the same common difference.

An *almost arithmetic progression* with *difference* d in $\mathbb{Z}_p$ is an arithmetic progression with difference d and one term removed. Observe that an arithmetic progression is an almost arithmetic progression, if the term removed is the initial or the final term of the original progression.

Theorem 15 (Hamidoune–Rødseth [2])**.** *Let $X, Y \subseteq \mathbb{Z}_p$ with $|X|, |Y| \geq 3$ and*

$$7 \leq |X+Y| = |X| + |Y| \leq p - 4.$$

Then both X and Y are almost arithmetic progressions with the same common difference.

We will also need the following technical lemma.

Lemma 16. *Let $X \subseteq \mathbb{Z}_p$ with $5 \leq |X| \leq p-5$. If both X and tX are the union of at most two arithmetic progressions with the same common difference d, then $t \in \{0, \pm 1, \pm 2, \pm 2^{-1}\}$.*

Proof. We may assume without loss of generality that $5 \leq |X| \leq \frac{p-1}{2}$; otherwise we take $\mathbb{Z}_p \setminus X$. Also we suppose that $d = 1$, that is, X is the union of at most two intervals; otherwise we analyze $d^{-1}X$. By hypothesis $Y := tX$ is also the union of at most two intervals named $Y_1 = [y_1, y_2]$ and $Y_2 = [y_3, y_4]$. Note that $|(X+1) \setminus X| \leq 2$, and so

$$|(Y+t) \setminus Y| \leq 2, \tag{11}$$

thus

$$|(Y_1+t) \setminus (Y_1 \cup Y_2)| + |(Y_2+t) \setminus (Y_1 \cup Y_2)| \leq 2. \tag{12}$$

From here we consider two cases. First suppose that either $Y_1 \cap (Y_1 + t) \neq \emptyset$ or $Y_2 \cap (Y_2 + t) \neq \emptyset$. Assume without loss of generality that $Y_1 \cap (Y_1 + t) \neq \emptyset$. In this case we will prove that $|t| \leq 2$, and therefore $t \in \{0, \pm 1, \pm 2\}$. We proceed by contradiction. Suppose without loss of generality that $3 \leq t \leq \frac{p-1}{2}$. By Inequality (12) we know

$$\min\{t, |y_3 - y_2|\} \leq |(Y_1+t) \setminus (Y_1 \cup Y_2)| \leq 2,$$

and thus $|y_3 - y_2| \leq 2$. In other words, Y must be an arithmetic progression with 0, 1 or 2 consecutive terms removed. In all cases, since $5 \leq |Y| \leq \frac{p-1}{2}$, it follows that $|(Y+t) \setminus Y| \geq 3$, a contradiction to Inequality (11).

Suppose now that $Y_1 \cap (Y_1 + t) = \emptyset$ and $Y_2 \cap (Y_2 + t) = \emptyset$. Then, by Inequality (11) it follows that:

$$|(Y_1 + t) \setminus Y_2| + |(Y_2 + t) \setminus Y_1| \leq 2. \tag{13}$$

We shall note that Inequality (13) implies that the cardinalities of Y_1 and Y_2 differ at most by 2. Moreover, it must be that: $y_1 + t = y_3 + \epsilon_1$ and $y_3 + t = y_1 + \epsilon_2$, with $|\epsilon_1| \leq 2$ and $|\epsilon_2| \leq 2$ where $|\epsilon_1| + |\epsilon_2| \leq 2$. In all possible cases we get $t \in \{0, \pm 1, \pm 2, \pm 2^{-1}\}$. □

6. The Case $|A| = 2$

In this section we prove that there are no rainbow-free colorings of $\mathbb{Z}_p$ concerning Equation (2), such that the smallest color class has two elements. First let us note a useful fact.

Lemma 17. *Let $\mathbb{Z}_p = A \cup B \cup C$ be a rainbow–free coloring with $|A| = 2 \leq |B| \leq |C|$. For any choice of $\{i, j, k\} = \{1, 2, 3\}$ the sets a_iB, a_iC, a_jB and a_jC are unions of at most two arithmetic progressions with difference d, where d is the difference between the two elements in a_kA.*

Proof. It follows from Lemma 13 that for any choice of $\{i, j, k\} = \{1, 2, 3\}$ we have:

$$|B| + 1 \leq |a_iA + a_jB| \leq |B| + 2.$$

Suppose that the difference between the two elements in a_iA is d. Since $|a_jB| = |B|$ then necessarily a_jB is the union of at most two arithmetic progressions with difference d, and the same will be true for a_kB. By repeating this argument we conclude the claim. □

Next we consider the case where two of the coefficients in Equation (2) are equal. That is, without loss of generality, we handle equation: $x + y + cz = b$ where $c \notin \{0, 1\}$.

Proposition 18. *Every 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $|A| = 2$ contains a rainbow solution of $x + y + cz = b$ where $c \notin \{0, 1\}$.*

Proof. By Lemma 4 and Theorem 3 the statement is true for $c \neq -2$. If $c = -2$, then Lemma 17 states that B, C, $-2B$ and $-2C$ are sets which are union of at most two arithmetic progressions with difference d, where d is the difference between the two elements of A. For the sake of comprehension we will assume that $d = 1$; otherwise we can analyze the partition $\mathbb{Z}_p = d^{-1}A \cup d^{-1}B \cup d^{-1}C$. Recall that we called an arithmetic progression with difference one an *interval*. Let $A = \{t, t + 1\}$.

`Case 1.` Some $X \in \{B, C\}$, say B, is an interval. Since $2 \leq |B| \leq p-4$ and $-2B$ is the union of at most two intervals then necessarily $|B| = 2$. Moreover, since $-2C$ is the union of at most two intervals then, actually, $B = \{t+2^{-1}, t+2^{-1}+1\}$ or $B = \{t+2^{-1}, t+2^{-1}-1\}$. Note that in both cases $-2B$ is a two element set whose difference is 2. Recall now that a rainbow–free coloring for $x+y-2z = b$ satisfies $2B + b \subseteq \mathbb{Z}_p \setminus (A+C)$, which is a contradiction since $\mathbb{Z}_p \setminus (A+C)$ is a two element set whose difference is 2^{-1}, and $2B + b$ (as well as $-2B$) is a two element set whose difference is 2.

`Case 2.` Both B and C are not intervals, thus they are unions of exactly two intervals. Suppose without loss of generality that $t + 2^{-1} \in B$. Then, it is not hard to see that one of the following must hold: $B = \{t+2^{-1}+i\}_{i=0}^{i=k} \cup \{t+2+i\}_{i=0}^{i=k-1}$, $B = \{t+2^{-1}+i\}_{i=0}^{i=k+1} \cup \{t+2+i\}_{i=0}^{i=k-1}$, $B = \{t+2^{-1}-i\}_{i=0}^{i=k} \cup \{t-1-i\}_{i=0}^{i=k-1}$ or $B = \{t+2^{-1}-i\}_{i=0}^{i=k+1} \cup \{t-1-i\}_{i=0}^{i=k-1}$ where $1 \leq k \leq \frac{p+1}{2} - 3$. In any case $-2C$ is an interval. Recall now that a rainbow–free coloring for $x+y-2z = b$ satisfies $2C + b \subseteq \mathbb{Z}_p \setminus (A+B)$ which is impossible because of the shape of A, B and C. □

Next we consider two more specific equations that arise naturally from the proofs of Lemmas 20 and 21 below.

Proposition 19. *Every 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $|A| = 2$ contains a rainbow solution of $x - y + 2z = b$ (respectively, $x - y + 2^{-1}z = b$).*

Proof. The proof is analogous to the proof of the previous proposition. Concerning the equation $x - y + 2z = b$ (respectively, $x - y + 2^{-1}z = b$) by Lemma 17 we know B, C, $2B$ and $2C$ (respectively, B, C, $2^{-1}B$ and $2C^{-1}$) are union of at most two arithmetic progressions with difference d where d is the difference between the two elements of $-A$. □

Now we are ready to prove the lemmas which dismiss in general the existence of rainbow–free colorings with the smallest color class of size two.

Lemma 20. *Every 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $|A| = 2$ and $3 \leq |B| \leq |C|$ contains a rainbow solution of Equation (2).*

Proof. Suppose for a contradiction that $\mathbb{Z}_p = A \cup B \cup C$ is a rainbow–free coloring for Equation (2) with $|A| = 2$ and $3 \leq |B| \leq |C|$. Then $5 \leq |C| \leq p-5$ and, by Lemma 17, both a_1C and a_2C are union of at most two arithmetic progressions with the same common difference. From Lemma 16, since $a_1C = a_1a_2^{-1}(a_2C)$, we conclude that $a_1a_2^{-1} \in \{\pm 1, \pm 2, \pm 2^{-1}\}$. With similar arguments we obtain that:

$$\{a_1a_2^{-1}, a_2a_3^{-1}, a_3a_1^{-1}\} \subseteq \{\pm 1, \pm 2, \pm 2^{-1}\}. \tag{14}$$

If $a_i = a_j$ for some distinct $i, j \in \{1, 2, 3\}$, we get a contradiction by Proposition 18. Assume then without loss of generality that $a_1 = 1$ and all three coefficients are different from each other. Hence, by Inclusion (14) we have $a_2, a_3 \in \{-1, \pm 2, \pm 2^{-1}\}$. If $a_2 = -1$ then $a_3 \in \{\pm 2, \pm 2^{-1}\}$. Note that $a_3 = 2$ gives an equivalent equation to the one obtained by letting $a_3 = -2$, and the same is true for $a_3 = 2^{-1}$ or $a_3 = -2^{-1}$, thus we obtain either $x - y + 2z = b$ or $x - y + 2^{-1}z = b$. In both cases we obtain a contradiction by Proposition 19. The remaining cases where $a_1 = 1$ and $a_2, a_3 \in \{\pm 2, \pm 2^{-1}\}$ all give an equation equivalent to one of the considered equations in Propositions 18 and 19. □

Lemma 21. *Every 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $|A| = |B| = 2$ and $|B| \leq |C|$ contains a rainbow solution of Equation (2).*

Proof. Suppose for a contradiction that $\mathbb{Z}_p = A \cup B \cup C$ is a rainbow–free coloring for Equation (2) with $|A| = |B| = 2$. By Lemma 13 we get $3 \leq |a_iA + a_jB| \leq 4$.

Assume first that for some pair of coefficients, say a_1 and a_2, we have $|a_1A + a_2B| = 3$. Then Vosper's Theorem (Theorem 14) establishes that the sets a_1A and a_2B are arithmetic progressions with same common difference. Let $a_1A = \{t_1, t_1 + d\}$ and $a_2B = \{t_2, t_2 + d\}$. Consider now the set $a_2A + a_1B$, which can be written as:

$$\{a_2a_1^{-1}t_1, a_2a_1^{-1}(t_1 + d)\} + \{a_1a_2^{-1}t_2, a_1a_2^{-1}(t_2 + d)\} \tag{15}$$

because $a_2A = a_2a_1^{-1}(a_1A)$ and $a_1B = a_1a_2^{-1}(a_2B)$. By Lemma 13 both $a_1A + a_2B$ and $a_2A + a_1B$ are contained in $\mathbb{Z}_p \setminus (-a_3C + b)$ which is a four elements set. If $|a_2A + a_1B| = 3$ then Vosper Theorem establishes $a_2a_1^{-1} \in \{\pm a_1a_2^{-1}\}$ from which it follows that $a_2a_1^{-1} = 1$, providing a contradiction by Proposition 18. If $|a_2A + a_1B| = 4$ then $a_1A + a_2B$ is contained in $a_2A + a_1B$. Since $a_1A + a_2B$ is a three–term arithmetic progression with difference d, by analyzing the set of differences in (15) we get that either $a_2a_1^{-1}d \in \{\pm d\}$ or $a_1a_2^{-1}d \in \{\pm d\}$. In any case $a_2 \in \{\pm a_1\}$ and $|a_2A + a_1B| = 3 \neq 4$.

Assume now that $|a_iA + a_jB| = 4$ for all distinct $i, j \in \{1, 2, 3\}$. Let $a_1A = \{t_1, t_1 + d_1\}$ and $a_2B = \{t_2, t_2 + d_2\}$ with $d_1 \notin \{\pm d_2\}$. As in the previous paragraph we note that $a_2A + a_1B$ can be written as:

$$\{a_2a_1^{-1}t_1, a_2a_1^{-1}(t_1 + d_1)\} + \{a_1a_2^{-1}t_2, a_1a_2^{-1}(t_2 + d_2)\}. \tag{16}$$

Again, by comparing the set of differences in (16) with the set of differences in $a_1A + a_2B$ we deduce that either $a_2a_1^{-1}d_1 \in \{\pm d_1\}$ or $a_1a_2^{-1}d_2 \in \{\pm d_1\}$. In the first case we get a contradiction by Proposition 18. In the second case $(a_1A + a_2B \cup a_2A + a_1B) \subseteq \mathbb{Z}_p \setminus (-a_3C + b)$ implies $a_1A + a_2B = a_2A + a_1B$, hence $t_1 + t_2 = a_2a_1^{-1}t_1 + a_1a_2^{-1}t_2$, thus $A = B$ which is impossible. □

7. The Case $|A| = 3$

In this section we prove that there are no rainbow-free colorings of $\mathbb{Z}_p$ concerning Equation (2), such that the smallest color class has three elements. In the case $|A| = 3$ and $4 \le |B| \le |C|$ we will follow a similar line of argument than in the previous section. For the case $|A| = |B| = 3$ we use some other technical lemmas. First let us note a useful fact.

Observation 22. *If $p \ge 11$, $|X| = 3$ and X is an almost arithmetic progression with difference d, then X is an almost arithmetic progression with difference d' if and only if $d' \in \{\pm d\}$.*

Lemma 23. *Let $\mathbb{Z}_p = A \cup B \cup C$ be a rainbow–free coloring with $|A| = 3$ and $4 \le |B| \le |C|$. For any choice of $i, j \in \{1, 2, 3\}$ with $i \ne j$, the sets a_iB, a_iC, a_jB, a_jC are almost arithmetic progression with the same common difference.*

Proof. By Lemma 13 we know that for any choice of $\{i, j, k\} = \{1, 2, 3\}$ and $\{X, Y, Z\} = \{A, B, C\}$, either $|a_iX + a_jY| = |a_iX| + |a_jY| - 1$ or $|a_iX + a_jY| = |a_iX| + |a_jY|$. In the first case it follows from Vosper's Theorem that both a_iX and a_jY are arithmetic progressions with the same common difference. In the second case Theorem 15 implies that both a_iX and a_jY are almost arithmetic progressions with the same common difference. In both cases we obtain that the sets a_iA and a_jY, with $Y \in \{B, C\}$, are almost arithmetic progressions with the same common difference. By repeating this argument and the use of the previous observation we conclude the claim. □

As in the previous section we first handle specific cases that will arise from the lemmas.

Proposition 24. *Every 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $|A| = 3$ and $4 \le |B| \le |C|$ contains a rainbow solution of $x + y + cz = b$ where $c \ne 1$.*

Proof. By Lemma 4 and Theorem 3 the statement is true for $c \ne -2$. If $c = -2$ and we assume that there are not rainbow solutions then Lemma 23 states that B and $-2B$ are almost arithmetic progression with difference d. For the sake of simplicity assume $d = 1$. Hence $B = \{t + i\}_{i=1}^{i=j-1} \cup \{t + i\}_{i=j+1}^{i=k}$ for some $t \in \mathbb{Z}_p$, $4 \le k \le p - 6$ and $1 < j \le k$; thereby $-2B = \{-2t - 2i\}_{i=1}^{i=j-1} \cup \{-2t - 2i\}_{i=j+1}^{i=k}$ which clearly is not an almost arithmetic progression of difference 1. □

Proposition 25. *Every 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $|A| = 3$ and $4 \le |B| \le |C|$ contains a rainbow solution of $x - y + 2z = b$ (respectively, $x - y + 2^{-1}z = b$).*

Proof. The proof is analogous to the proof of the previous proposition. Concerning the equation $x - y + 2z = b$ (respectively, $x - y + 2^{-1}z = b$) by Lemma 23 we know

B and $2B$ (respectively, B and $2^{-1}B$) are almost arithmetic progressions with the same common difference. □

Now we are ready to prove the lemma who dismiss the existence of rainbow–free colorings in the case $|A| = 3 < |B|$.

Lemma 26. *Every 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $|A| = 3$ and $4 \leq |B| \leq |C|$ contains a rainbow solution of Equation (2).*

Proof. Suppose for a contradiction that $\mathbb{Z}_p = A \cup B \cup C$ is a rainbow–free coloring for Equation (2) with $|A| = 3$ and $4 \leq |B| \leq |C|$. By Lemma 23 we know that a_2B, a_2C, a_3B and a_3C are almost arithmetic progressions with the same common difference. Hence, from Lemma 16, we conclude that $a_2a_3^{-1} \in \{\pm1, \pm2, \pm2^{-1}\}$; in the same way $a_1a_2^{-1}, a_3a_1^{-1} \in \{\pm1, \pm2, \pm2^{-1}\}$. If $a_i = a_j$ for some distinct $i, j \in \{1, 2, 3\}$, we get a contradiction by Proposition 24. Assume then, without loss of generality, that $a_1 = 1$ and all three coefficients are different from each other, thus $a_2, a_3 \in \{-1, \pm2, \pm2^{-1}\}$. If $a_2 = -1$ then $a_3 \in \{\pm2, \pm2^{-1}\}$. Note that $a_3 = 2$ gives an equivalent equation to the case where $a_3 = -2$, and the same is true for $a_3 = 2^{-1}$ and $a_3 = -2^{-1}$, in both cases we obtain a contradiction by Proposition 24. The remaining cases where $a_1 = 1$ and $a_2, a_3 \in \{\pm2, \pm2^{-1}\}$ all give an equation equivalent to one of the equations considered in Propositions 24 and 25. □

Next we prove a technical lemmas to conclude the remaining case $|A| = |B| = 3$.

Lemma 27. *Suppose $p \geq 11$ and $X, Y \subseteq \mathbb{Z}_p$. If $|X| = |Y| = 3$ and $|X+Y| \in \{5, 6\}$, then one of the following holds true*

i) $X = Y + u$ *for some* $u \in \mathbb{Z}_p$.

ii) $\{X, Y\} = \{\{w, w + d, w + 2d\}, \{u, u + d, u + 3d\}\}$ *for some* $w, u, d \in \mathbb{Z}_p$.

Proof. If $|X + Y| = 5$ then Theorem 14 implies that both tX and Y are arithmetic progressions with the same common difference, and therefore $X = Y + u$ for some $u \in \mathbb{Z}_p$. The other case, $|X + Y| = 6$, is tedious but not difficult to prove (for more details see [6]). □

We will need the analogous of Proposition 24 in the more specific case $|A| = |B| = 3$.

Proposition 28. *Every 3–coloring $\mathbb{Z}_p = A \cup B \cup C$ with $|A| = |B| = 3$ and $|B| \leq |C|$ contains a rainbow solution of $x + y + cz = b$ where $c \neq 1$.*

Proof. By Lemma 4 and Theorem 3 the statement is true for $c \neq -2$. So we consider the equation $x + y - 2z = b$. Suppose, by contradiction, that $\mathbb{Z}_p = A \cup B \cup C$ with $|A| = |B| = 3$ and $|B| \leq |C|$ is a rainbow–free coloring for $x + y - 2z = b$. We handle two cases.

`Case 1.` There is no element $w \in \mathbb{Z}_p$ such that, either $A = B + w$ or $A = -2B + w$. Assume first that there is not $w \in \mathbb{Z}_p$ such that $A = B + w$. By Lemma 13 we get $|A + B| \le 6$. Then by Lemma 27 we know that $\{A, B\} = \{\{v, v + d, v + 2d\}, \{u, u + d, u + 3d\}\}$ for some $u, v, d \in \mathbb{Z}_p$. It is not difficult to see that $\max\{|A - 2B|, |B - 2A|\} > 6$ which is a contradiction by Lemma 13. The case where there is not $w \in \mathbb{Z}_p$ such that $A = -2B + w$ is done in a very similar way.

`Case 2.` There are u_1 and $u_2 \in \mathbb{Z}_p$ such that $A = B + u_1$ and $A = -2B + u_2$. Then $B = -2B + u_2 - u_1$. By the third outcome of Observation 1 we know that there exists a $w \in \mathbb{Z}_p$ such that $B + w$ is invariant under a $\langle -2 \rangle$–dilation. Hence, $B + w = \{x, -2x, 4x\}$ for some $x \in \mathbb{Z}_p$, and thereby $(-2)^3 x = x$ which is a contradiction. □

Lemma 29. *Let $\lambda \in \mathbb{Z}_p$ be such that $\lambda^4 + \lambda^2 + 1 = 0$ and*

$$X := \{0, 1, 2, \lambda^2 + 1, \lambda^2 + 2, 2\lambda^2 + 2\}.$$

If $Y := \{w, w + 1, w + 2\} \subseteq \lambda X$, then either $p \le 7$ or

$$w = \begin{cases} 2\lambda & \text{if } \lambda^3 = 1 \\ 2\lambda - 2 & \text{if } \lambda^3 = -1. \end{cases}$$

Proof. Since

$$(\lambda^2 - 1)(1 + \lambda^2 + \lambda^4) = \lambda^6 - 1 = (\lambda^3 - 1)(\lambda^3 + 1),$$

we have $\lambda^3 \in \{\pm 1\}$. By hypothesis the set $\lambda^{-1} Y$ is contained in X. Note that if $p > 7$ then $\lambda^{-1} Y$ cannot contain $\{u, u + 1\}$ for some $u \in \mathbb{Z}_p$; in the same way, $\{0, 2\} \subseteq \lambda^{-1} Y$ implies $p \le 7$. We have the remaining cases:

- $\lambda^{-1} Y = \{0, \lambda^2 + 1, 2\lambda^2 + 2\}$. Then $Y = \{0, \lambda^3 + \lambda, 2\lambda^3 + 2\lambda\}$. By Observation 22, $\lambda^3 + \lambda \in \{\pm 1\}$, thus $-1 = \lambda^4 + \lambda^2 \in \{\pm \lambda\}$. Hence $p = 3$.

- $\lambda^{-1} Y = \{0, \lambda^2 + 2, 2\lambda^2 + 2\}$. Then $Y = \{0, \lambda^3 + 2\lambda, 2\lambda^3 + 2\lambda\}$. Thus $\lambda^3 \in \{\pm 1\}$ implies $\lambda = 0$ which is impossible.

- $\lambda^{-1} Y = \{1, \lambda^2 + 1, 2\lambda^2 + 2\}$. Then $Y = \{\lambda, \lambda^3 + \lambda, 2\lambda^3 + 2\lambda\}$. If $\lambda^3 = 1$ then $\lambda = -3$ and $p \le 7$; in the same way, $\lambda^3 = -1$ implies $p \le 7$.

- $\lambda^{-1} Y = \{1, \lambda^2 + 2, 2\lambda^2 + 2\}$. Then $Y = \{\lambda, \lambda^3 + 2\lambda, 2\lambda^3 + 2\lambda\}$. If $\lambda^3 = 1$ then $\lambda = -3$ and $p \le 7$; in the same way, $\lambda^3 = -1$ implies $p \le 7$

- $\lambda^{-1} Y = \{2, \lambda^2 + 1, 2\lambda^2 + 2\}$. Then $Y = \{2\lambda, \lambda^3 + \lambda, 2\lambda^3 + 2\lambda\}$. Hence $\lambda^3 \in \{\pm 1\}$ implies $\lambda = 0$ which is impossible or $p \le 3$.

- $\lambda^{-1} Y = \{2, \lambda^2 + 1, 2\lambda^2\}$. Then $Y = \{2\lambda, \lambda^3 + 2\lambda, 2\lambda^3 + 2\lambda\}$; if $\lambda^3 = 1$ then $w = 2\lambda$, and if $\lambda^3 = -1$, then $w = 2\lambda - 2$.

□

Finally, we are ready to prove the lemma who dismiss the existence of rainbow–free colorings with $|A| = |B| = 3$.

Lemma 30. *Every* 3*–coloring* $\mathbb{Z}_p = A \cup B \cup C$ *with* $|A| = |B| = 3$ *and* $|B| \leq |C|$ *contains a rainbow solution of Equation (2).*

Proof. By Proposition 28 we may assume without loss of generality that $a_1 \notin \{\pm a_2\}$. We will show that:

$$|a_1A + a_2B \cup a_1B + a_2A| > 6 \tag{17}$$

which implies the lemma since $a_1A + a_2B \cup a_1B + a_2A$ would not be contained in $\mathbb{Z}_p \setminus -a_3C$.

If $|a_1A + a_2B| = 5$, then by Theorem 14

$$a_1A = \{u, u + d, u + 2d\} \qquad \text{and} \qquad a_2B = \{w, w + d, w + 2d\}$$

for some $u, w, d \in \mathbb{Z}_p$. Lemma 27 and Observation 22 imply $|a_1B + a_2A| > 6$ insomuch as $a_1 \notin \{\pm a_2\}$. In the same way, if $|a_1B + a_2A| = 5$ then Inequality (17) follows.

Suppose there are not rainbow solutions of Equation (2). By Lemma 13 and last paragraph

$$|a_1A + a_2B| = |a_1B + a_2A| = 6 \tag{18}$$

and Inequality (17) needs to be false.

First assume either there is not $r \in \mathbb{Z}_p$ such that $a_1A = a_2B + r$ or there is not $r \in \mathbb{Z}_p$ such that $a_2A = a_1B + r$. Without loss of generality suppose that there is not $r \in \mathbb{Z}_p$ such that $a_1A = a_2B + r$, thus by Lemma 27 there are $u, w, d \in \mathbb{Z}_p$ such that

$$\{a_1A, a_2B\} = \{\{u, u + d, u + 2d\}, \{w, w + d, w + 3d\}\}.$$

Lemma 27 and Observation 22 imply $|a_1B + a_2A| > 6$ which contradicts our assumption.

Now take $r_1, r_2 \in \mathbb{Z}_p$ such that $a_1A = a_2B + r_1$ and $a_2A = a_1B + r_2$. Write $\lambda := a_2a_1^{-1}$ and $\mu := a_2a_1^{-2}r_1 - a_1^{-1}r_2$ so

$$B = \lambda^2 B + \mu. \tag{19}$$

If $\lambda^2 = -1$, B is an arithmetic progression Consequently a_2B and a_1A are arithmetic progressions with the same common difference contradicting Equation (18). From now on we assume $\lambda^2 \neq -1$. Write $B = \{b_1, b_2, b_3\}$. We claim there is not $b' \in B$ such that $b' = \lambda^2 b' + \mu$; indeed if there is a b' like this, then $b'' \neq \lambda^2 b'' + \mu$ for all $b'' \in B \setminus \{b'\}$ but Equation (19) yields

$$b'' = \lambda^2(\lambda^2 b'' + \mu) + \mu = \lambda^4 b'' + (\lambda^2 + 1)\mu$$

so $b'' = \lambda^2 b'' + \mu$ since $\lambda^2 \neq -1$. Thus without loss of generality

$$b_2 = \lambda^2 b_1 + \mu, \quad b_3 = \lambda^2 b_2 + \mu, \quad b_1 = \lambda^2 b_3 + \mu,$$

and particularly $1+\lambda^2+\lambda^4 = 0$. By Equation (18) and since we assumed Inequality (17) is false, we get that

$$a_1 A + a_2 B = a_1 B + a_2 A$$

so

$$\lambda B + \lambda B = B + B - (r_1 - r_2)a_1^{-1}.$$

Adding $-2b_1$ and multiplying by $\theta := ((\lambda^2-1)b_1+\mu)^{-1}$ this last equation we obtain

$$\begin{aligned}\lambda\{0,1,2,\lambda^2+1,\lambda^2+2,2\lambda^2+2\} = \{0,1,2,\lambda^2+1,\lambda^2+2,2\lambda^2+2\} \\ + ((r_2-r_1)a_1^{-1} + 2b_1(1-\lambda))\theta. \qquad (20)\end{aligned}$$

By Lemma 29 we have either

$$\lambda^3 = 1 \quad \text{and} \quad 2\lambda = ((r_2-r_1)a_1^{-1} + 2b_1(1-\lambda))\theta \qquad (21)$$

or

$$\lambda^3 = -1 \quad \text{and} \quad 2\lambda - 2 = ((r_2-r_1)a_1^{-1} + 2b_1(1-\lambda))\theta. \qquad (22)$$

If Equation (21) holds true, then

$$2\lambda\mu = (r_2 - r_1)a_1^{-1}$$

and thereby

$$r_2(1+2\lambda) = r_1(1+2\lambda^2). \qquad (23)$$

On the other hand

$$A = \lambda B + a_1^{-1} r_1$$

and

$$B = \lambda B + (\lambda^2+1)\mu$$

by Equation (19), however Equation (23) implies $A = B$ which contradicts the assumption. If Equation (22) holds true, then by Equation (20)

$$\{0, \lambda-1, \lambda\} = \{3\lambda-2, 3\lambda-1, 4\lambda\}$$

which is impossible and thereby Inequality (17) holds true. □

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