ON MIKI'S IDENTITY FOR BERNOULLI NUMBERS

Takashi Agoh
Department of Mathematics, Tokyo University of Science, Noda, Chiba, Japan
agoh_takashi@ma.noda.tus.ac.jp

Received: 2/3/16, Accepted: 10/16/16, Published: 11/11/16

Abstract

It is the main purpose of this paper to give an elementary proof of Miki's identity for Bernoulli numbers by making use of a certain linear recurrence relation obtained from either Faulhaber's formula for the power sum of the first k positive integers or special expressions of the nth harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

1. Introduction

Let B_n be the nth Bernoulli number defined by the generating function

$$f(t) := \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \quad (|t| < 2\pi). \tag{1.1}$$

It is easy to find the values $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30$, and so on. Since $f(t) + t/2$ is an even function, $B_{2n+1} = 0$ for all $n \geq 1$. Furthermore, we see that $(-1)^{n-1} B_{2n} > 0$ for all $n \geq 1$ by observing Euler's formula related to the values of the Riemann zeta function at positive even integers.

Various types of linear and nonlinear recurrence relations for these numbers have been studied for a long time. Among them, the most basic linear one is

$$\sum_{i=0}^{n} \binom{n+1}{i} B_i = 0 \ \ (n \geq 1), \quad B_0 = 1. \tag{1.2}$$

We now define $S_0(0) := 1$ and $S_n(k) := 1^n + 2^n + \cdots + k^n$ for integers $n \geq 0$ and $k \geq 1$. Then, as is well-known, Faulhaber's formula states that

$$S_n(k-1) = \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} k^i B_{n+1-i} = \sum_{i=1}^{n+1} \frac{1}{i} \binom{n}{i-1} k^i B_{n+1-i}. \tag{1.3}$$

This formula itself can be proved without difficulty by considering the functional identity $f(t)(e^{kt} - 1) = t \sum_{j=0}^{k-1} e^{jt}$ and comparing the coefficients of t^{n+1} on both sides after expanding into the Maclaurin power series. However, we should note that (1.3) is also an important consequence of the Euler-Maclaurin summation formula. Obviously, (1.3) reduces to (1.2) when $k = 1$.

On the other hand, the most basic quadratic recurrence relation (i.e., convolution identity) is the following one usually attributed to Euler:

$$\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n-1)B_n \quad (n \geq 1). \tag{1.4}$$

This identity can be also derived in the same way as for (1.3) by using the functional identity $f^2(t) = (1-t)f(t) - t\,\frac{d}{dt}f(t)$.

There are various kinds of extensions and generalizations of (1.2) and (1.4). However, they are mostly constructed from one type of convolutions with binomial or multinomial coefficients. Many such examples can be found in the handbook [11] or the classic books [16, 14]. See also [1, 4, 5, 6] for lacunary recurrence relations, in which some of the preceding Bernoulli numbers are missing.

Surprisingly, Miki [13] proved in 1978 an unusual identity using p-adic methods, in which two different types of convolutions are combined. Namely,

$$\sum_{i=2}^{n-2} \frac{B_i B_{n-i}}{i(n-i)} - \sum_{i=2}^{n-2} \binom{n}{i} \frac{B_i B_{n-i}}{i(n-i)} = \frac{2H_n B_n}{n} \quad (n \geq 4), \tag{1.5}$$

where H_n is the nth harmonic number defined by

$$H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \tag{1.6}$$

Note that (1.5) is significant only for an even $n \geq 4$, because both sides vanish if $n \geq 5$ is odd. A simple and elementary proof of (1.5) based on Crabb's intelligible idea in [7] can be found in [2, 3].

Subsequently, inspired by Miki's identity, Matiyasevich discovered the following convolution identity with the aid of the computer software system *Mathematica* and announced it as "Identity #0120" on his website [12] (1997) without proof:

$$\sum_{i=2}^{n-2} \frac{B_i B_{n-i}}{i} - \sum_{i=2}^{n-2} \binom{n}{i} \frac{B_i B_{n-i}}{i} = H_n B_n \quad (n \geq 4). \tag{1.7}$$

As will be easily seen in Theorem 2.4 below, this identity is actually equivalent to (1.5) (see also Pan and Sun's proof in [15, p.158]).

It is the main purpose of this paper to prove (1.7), in Section 2, by developing and utilizing the following *linear* recurrence relation for Bernoulli numbers which is as a matter of fact equivalent to Faulhaber's formula (1.3):

$$\sum_{i=1}^{n-1} \frac{B_i}{ik^i} - \sum_{i=1}^{n-1} \binom{n}{i} \frac{B_i}{ik^i} = \frac{1}{k^n} \sum_{j=1}^{k} \frac{(k-j)^n}{j} - H_k + H_n \quad (n, k \geq 1). \tag{1.8}$$

In Section 3, we deal with (1.8) once more from the point of view of harmonic numbers independently of (1.3) and show that (1.8) can be rederived by making use of special expressions of H_n.

As a result, we are able to give, in this paper, two kinds of quite elementary proofs of (1.8) (and eventually of (1.7)) without any use of intricate tools.

In particular, taking $k = 1$ in (1.8), we have the simple recurrence relation

$$\sum_{i=1}^{n-1} \frac{B_i}{i} - \sum_{i=1}^{n-1} \binom{n}{i} \frac{B_i}{i} = -1 + H_n, \tag{1.9}$$

which is of course equivalent to (1.3) with $k = 1$ and hence to (1.2). Considering the form of this identity, we perceive that (1.9) is just a linear version of (1.7), or equivalently, of (1.5). By the way, a third-order analogue of (1.5) can be found in the recent papers [9, 8] as well as [10]. However, to the best of our knowledge, it seems that more higher-order analogues of (1.5) are not known as yet.

2. Discussion Based on Faulhaber's Formula

The von Staudt-Clausen theorem asserts that if $n \geq 2$ is even, then

$$B_n + \sum_{p-1 \mid n} \frac{1}{p} \in \mathbb{Z}, \tag{2.1}$$

where the sum is taken over all primes p satisfying $p - 1 \mid n$. Therefore, from (2.1) we obtain immediately

$$pB_n \equiv \begin{cases} -1 \pmod{p}, & \text{if } p - 1 \mid n; \\ 0 \pmod{p}, & \text{otherwise.} \end{cases} \tag{2.2}$$

Based on (2.2), we can evaluate $S_n(p-1)$ as follows:

Lemma 2.1. *Let p be an odd prime and $n \geq 0$ be an integer. Then we have*

$$S_n(p-1) \equiv \begin{cases} -1 \pmod{p}, & \textit{if } p - 1 \mid n; \\ 0 \pmod{p}, & \textit{otherwise} \end{cases} \tag{2.3}$$

and

$$S_n(p-1) \equiv \begin{cases} p - 1 \pmod{p^2}, & \textit{if } n = 0; \\ -\dfrac{np}{2} \pmod{p^2}, & \textit{if } n \geq 3 \textit{ and } p - 1 \mid n - 1; \\ pB_n \pmod{p^2}, & \textit{otherwise.} \end{cases} \tag{2.4}$$

Proof. It is confirmed by direct calculation that $S_0(p-1) = p-1$, $S_1(p-1) = (p-1)p/2 \equiv -p/2 \equiv pB_1 \pmod{p^2}$ and

$$S_2(p-1) = \frac{1}{6}(p-1)p(2p-1) \equiv \frac{p}{6} \equiv pB_2 \pmod{p^2};$$

so (2.4) is valid for $n = 0, 1$ and 2. Next, assuming $n \geq 3$, put $k = p$ in (1.3):

$$S_n(p-1) = \sum_{i=1}^{n+1} \frac{1}{i}\binom{n}{i-1} p^i B_{n+1-i}. \tag{2.5}$$

For $i \geq 3$, if $p \nmid i$, then $p^i B_{n+1-i}/i = p^{i-1}\left(pB_{n+1-i}/i\right) \equiv 0 \pmod{p^2}$ by (2.2). Otherwise, if $i = p^a r \geq 3$ with $a \geq 1$, $r \geq 1$ and $p \nmid r$, then

$$\frac{1}{i}p^i = \frac{1}{r}p^{p^a r - a} \geq \frac{1}{r}p^{3^a - a} \geq \frac{1}{r}p^2;$$

and thus from (2.5),

$$S_n(p-1) \equiv pB_n + \frac{np^2}{2}B_{n-1} \pmod{p^2}. \tag{2.6}$$

This congruence reduces to (2.3) modulo p by (2.2) including also the cases of $n = 0, 1$ and 2. If $n \geq 3$ is odd, then $B_n = 0$. Hence we get from (2.6) and (2.2),

$$S_n(p-1) \equiv \frac{np^2}{2}B_{n-1} \equiv \begin{cases} -\dfrac{np}{2} \pmod{p^2}, & \text{if } p-1 \mid n-1; \\ 0 \equiv pB_n \pmod{p^2}, & \text{otherwise.} \end{cases}$$

On the other hand, if $n \geq 4$ is even, then $B_{n-1} = 0$; so (2.6) gives $S_n(p-1) \equiv pB_n \pmod{p^2}$. Summarizing these results, we conclude that $S_n(p-1) \equiv pB_n \pmod{p^2}$ for all $n \geq 1$ unless $n = 0$ or $p-1 \mid n-1$ for $n \geq 3$. □

First of all, we argue a mutual relationship between (1.3) and (1.8), and prove that they are as a matter of fact equivalent.

Theorem 2.2. *For integers $n, k \geq 1$, (1.3) is equivalent to (1.8).*

Proof. Since $B_0 = 1$ and $\frac{1}{i}\binom{n}{i-1} = \frac{1}{n+1-i}\binom{n}{n-i}$ $(1 \leq i \leq n)$, (1.3) can be rewritten by exchanging i for $n+1-i$ as

$$S_n(k-1) = \sum_{i=1}^{n} \frac{1}{i}\binom{n}{i-1} k^{n+1-i} B_i + \frac{k^{n+1}}{n+1}.$$

So dividing by k^{n+1}, we get, since $S_n(k-1) = \sum_{j=1}^{k}(k-j)^n$,

$$\sum_{i=1}^{n} \binom{n}{i-1} \frac{B_i}{ik^i} = \frac{1}{k^{n+1}} \sum_{j=1}^{k} (k-j)^n - \frac{1}{n+1}. \tag{2.7}$$

Using (2.7) instead of (1.3), we will prove below that (2.7) is equivalent to (1.8).

(i) (2.7) $\Rightarrow$ (1.8): Changing the notation n to m in (2.7), take $m = 1, 2, ..., n-1$ and add up all of them. Then we obtain

$$\begin{aligned}\sum_{m=1}^{n-1}\sum_{i=1}^{m}\binom{m}{i-1}\frac{B_i}{ik^i} &= \sum_{m=1}^{n-1}\frac{1}{k^{m+1}}\sum_{j=1}^{k}(k-j)^m - \sum_{m=1}^{n-1}\frac{1}{m+1}\\ &= \frac{1}{k}\sum_{j=1}^{k}\left(\sum_{m=0}^{n-1}\left(\frac{k-j}{k}\right)^m - 1\right) - (H_n - 1)\\ &= \frac{1}{k}\sum_{j=1}^{k}\frac{((k-j)/k)^n - 1}{((k-j)/k) - 1} - \frac{1}{k}\sum_{j=1}^{k}1 - (H_n - 1)\\ &= -\frac{1}{k^n}\sum_{j=1}^{k}\frac{(k-j)^n - k^n}{j} - H_n.\end{aligned} \tag{2.8}$$

Since $\sum_{r=0}^{n-1}\binom{r}{i-1} = \binom{n}{i}$; and so $\sum_{r=i}^{n-1}\binom{r}{i-1} = \binom{n}{i} - 1$ $(0 \le i < n)$, the first part of (2.8) becomes

$$\sum_{m=1}^{n-1}\sum_{i=1}^{m}\binom{m}{i-1}\frac{B_i}{ik^i} = \sum_{i=1}^{n-1}\sum_{r=i}^{n-1}\binom{r}{i-1}\frac{B_i}{ik^i} = \sum_{i=1}^{n-1}\binom{n}{i}\frac{B_i}{ik^i} - \sum_{i=1}^{n-1}\frac{B_i}{ik^i}.$$

So that (2.8) leads to

$$\begin{aligned}\sum_{i=1}^{n-1}\binom{n}{i}\frac{B_i}{ik^i} - \sum_{i=1}^{n-1}\frac{B_i}{ik^i} &= -\frac{1}{k^n}\sum_{j=1}^{k}\frac{(k-j)^n - k^n}{j} - H_n\\ &= -\frac{1}{k^n}\sum_{j=1}^{k}\frac{(k-j)^n}{j} + H_k - H_n,\end{aligned}$$

which coincides with (1.8) by changing the signs of all terms.

(ii) (1.8) $\Rightarrow$ (2.7): Conversely, assuming that (1.8) holds for $n, k \ge 1$, we have

$$\begin{aligned}&\sum_{i=1}^{n+1}\binom{n+1}{i}\frac{B_i}{ik^i} - \sum_{i=1}^{n}\binom{n}{i}\frac{B_i}{ik^i}\\ &\quad= \left(\sum_{i=1}^{n+1}\frac{B_i}{ik^i} - \frac{1}{k^{n+1}}\sum_{j=1}^{k}\frac{(k-j)^{n+1} - k^{n+1}}{j} - H_{n+1}\right)\\ &\qquad - \left(\sum_{i=1}^{n}\frac{B_i}{ik^i} - \frac{1}{k^n}\sum_{j=1}^{k}\frac{(k-j)^n - k^n}{j} - H_n\right)\\ &\quad= \frac{B_{n+1}}{(n+1)k^{n+1}} + \frac{1}{k^{n+1}}\sum_{j=1}^{k}(k-j)^n - \frac{1}{n+1}.\end{aligned} \tag{2.9}$$

Since $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$ and $\frac{1}{n+1}\binom{n+1}{i} = \frac{1}{i}\binom{n}{i-1}$ $(1 \le i \le n)$, the first part of (2.9) can be written as

$$\begin{aligned}\sum_{i=1}^{n+1}\binom{n+1}{i}\frac{B_i}{ik^i} - \sum_{i=1}^{n}\binom{n}{i}\frac{B_i}{ik^i} &= \sum_{i=1}^{n+1}\left\{\binom{n}{i} + \binom{n}{i-1}\right\}\frac{B_i}{ik^i} - \sum_{i=1}^{n}\binom{n}{i}\frac{B_i}{ik^i}\\ &= \frac{B_{n+1}}{(n+1)k^{n+1}} + \sum_{i=1}^{n}\binom{n}{i-1}\frac{B_i}{ik^i}.\end{aligned}$$

Substituting this into (2.9), we finally get

$$\sum_{i=1}^{n}\binom{n+1}{i-1}\frac{B_i}{ik^i} = \frac{1}{k^{n+1}}\sum_{j=1}^{k}(k-j)^n - \frac{1}{n+1},$$

which is exactly the same as (2.7), as desired. □

Next, by making use of (1.8) we will prove Matiyasevich's identity.

Theorem 2.3. *For an integer $n \ge 4$, (1.7) holds.*

Proof. As mentioned in Section 1, (1.7) is trivial for an odd $n \ge 5$. So in what follows, assume that $n \ge 4$ is even. Multiplying (1.8) by k^n,

$$\sum_{i=1}^{n-1}\frac{B_i}{i}k^{n-i} - \sum_{i=1}^{n-1}\binom{n}{i}\frac{B_i}{i}k^{n-i} = \sum_{j=1}^{k}\frac{(k-j)^n}{j} - k^nH_k + k^nH_n. \tag{2.10}$$

Take here $k = 1, 2, ..., p$ for an odd prime p and add up all of them. Since $p^nH_p \equiv 0 \pmod{p^2}$ if $n \ge 4$, we have

$$\begin{aligned}&\sum_{i=1}^{n-1}\frac{B_i}{i}S_{n-i}(p) - \sum_{i=1}^{n-1}\binom{n}{i}\frac{B_i}{i}S_{n-i}(p)\\ &\quad= \sum_{k=1}^{p}\sum_{j=1}^{k}\frac{(k-j)^n}{j} - \sum_{k=1}^{p}k^nH_k + S_n(p)H_n\\ &\quad= \sum_{k=1}^{p-1}(p-k)^nH_k - \sum_{k=1}^{p-1}k^nH_k + S_n(p)H_n\\ &\quad= \sum_{k=1}^{p-1}((p-k)^n - k^n)H_k + S_n(p)H_n\\ &\quad\equiv -np\sum_{k=1}^{p-1}k^{n-1}H_k + S_n(p)H_n \pmod{p^2}.\end{aligned} \tag{2.11}$$

The sum $\sum_{k=1}^{p-1} k^{n-1}H_k$ appearing in the last line of (2.11) can be evaluated modulo p as follows: letting $\mu := (p-1)/2$ for short,

$$\begin{aligned}\sum_{k=1}^{p-1} k^{n-1}H_k &= \sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=k}^{p-1} j^{n-1} \\ &= \sum_{k=1}^{\mu} \frac{1}{k} \sum_{j=k}^{p-1} j^{n-1} + \sum_{k=\mu+1}^{p-1} \frac{1}{k} \sum_{j=k}^{p-1} j^{n-1} \\ &= \sum_{k=1}^{\mu} \frac{1}{k} \sum_{j=k}^{p-1} j^{n-1} + \sum_{k=1}^{\mu} \frac{1}{p-k} \sum_{j=1}^{k} (p-j)^{n-1} \\ &\equiv \sum_{k=1}^{\mu} \frac{1}{k} \left(\sum_{j=k}^{p-1} j^{n-1} - (-1)^{n-1} \sum_{j=1}^{k} j^{n-1} \right) \\ &\equiv \sum_{k=1}^{\mu} \frac{1}{k} \left(\sum_{j=1}^{p-1} j^{n-1} + k^{n-1} \right) \\ &\equiv H_\mu S_{n-1}(p-1) + S_{n-2}(\mu) \pmod{p}. \end{aligned} \tag{2.12}$$

Since $p-1 \nmid n-1$ for an even $n \geq 4$, we have $S_{n-1}(p-1) \equiv 0 \pmod{p}$ by (2.3). If $p-1 \nmid n-2$, then $S_{n-2}(\mu) \equiv S_{n-2}(p-1)/2 \equiv 0 \pmod{p}$ again by (2.3); so the last expression in (2.12) completely vanishes. Further, by (2.4) we have $B_{n-1} = 0$ and $S_{n-1}(p) \equiv S_{n-1}(p-1) \equiv pB_{n-1} \equiv 0 \pmod{p^2}$ for an even $n \geq 4$. Consequently, since $S_r(p) \equiv S_r(p-1) \pmod{p^2}$ for all $r \geq 2$, by using (2.4) we can deduce from (2.11) that

$$p \sum_{i=2}^{n-2} \frac{B_i B_{n-i}}{i} - p \sum_{i=2}^{n-2} \binom{n}{i} \frac{B_i B_{n-i}}{i} \equiv pH_n B_n \pmod{p^2},$$

and hence dividing by p,

$$\sum_{i=2}^{n-2} \frac{B_i B_{n-i}}{i} - \sum_{i=2}^{n-2} \binom{n}{i} \frac{B_i B_{n-i}}{i} \equiv H_n B_n \pmod{p}.$$

This congruence holds for infinitely many odd primes p, and moreover, both sides do not depend on p. These facts imply that (1.7) must unconditionally follow; and thus the proof of Theorem 2.3 is now complete. □

We note that the above proof of Theorem 2.3 shows us, by elementary methods, how to transform a linear recurrence relation into a convolution identity in the Bernoulli number case.

A mutual relationship between (1.5) and (1.7) can be stated by the following:

Theorem 2.4. *For an integer $n \geq 4$, (1.5) is equivalent to (1.7).*

Proof. By the symmetry of i and $n-i$, we can deduce from (1.7) that

$$\sum_{i=2}^{n-2} \frac{B_{n-i}B_i}{n-i} - \sum_{i=2}^{n-2} \binom{n}{n-i} \frac{B_{n-i}B_i}{n-i} = H_n B_n.$$

Adding this to (1.7), we have, since $\binom{n}{i} = \binom{n}{n-i}$ $(0 \leq i \leq n)$,

$$\begin{aligned}
&\sum_{i=2}^{n-2} \left(\frac{1}{i} + \frac{1}{n-i}\right) B_i B_{n-i} - \sum_{i=2}^{n-2} \binom{n}{i} \left(\frac{1}{i} + \frac{1}{n-i}\right) B_i B_{n-i} \\
&\quad = n \sum_{i=2}^{n-2} \frac{B_i B_{n-i}}{i(n-i)} - n \sum_{i=2}^{n-2} \binom{n}{i} \frac{B_i B_{n-i}}{i(n-i)} = 2H_n B_n,
\end{aligned}$$

which gives (1.5) dividing by n, and vice versa. □

3. Observation of (1.8) Based on Harmonic Numbers

In this section, we observe (1.8) once more, this time from the point of view of harmonic numbers. As a result, it is possible to give another proof of (1.8) by making use of special expressions of the nth harmonic number H_n.

At first, we present an unusual expression of H_n that is very important in our subsequent discussions.

Lemma 3.1. *For an integer $n \geq 1$ and a real or complex variable x, we have*

$$H_n = \sum_{i=1}^{n} \frac{x^i}{i} - \sum_{i=1}^{n} \binom{n}{i} \frac{(x-1)^i}{i}. \tag{3.1}$$

Proof. For brevity, let us denote by $g(x)$ the right-hand side of (3.1). Then, using the fact that $\binom{n}{i} = \sum_{m=0}^{n-1} \binom{m}{i-1}$ $(1 \leq i \leq n)$, we have

$$\begin{aligned}
\frac{d}{dx} g(x) &= \sum_{i=1}^{n} x^{i-1} - \sum_{i=1}^{n} \binom{n}{i} (x-1)^{i-1} \\
&= \sum_{i=1}^{n} x^{i-1} - \sum_{i=1}^{n} \sum_{m=0}^{n-1} \binom{m}{i-1} (x-1)^{i-1} \\
&= \sum_{i=1}^{n} x^{i-1} - \sum_{i=1}^{n} \sum_{j=0}^{i-1} \binom{i-1}{j} (x-1)^{j} \\
&= \sum_{i=1}^{n} x^{i-1} - \sum_{i=1}^{n} x^{i-1} = 0,
\end{aligned}$$

which implies that $g(x)$ is a constant function. Since $H_n = g(1)$ from the definition in (1.6), we get immediately (3.1). $\square$

Incidentally, it may be worth mentioning that by taking arbitrary values for x in (3.1), we can deduce various kinds of expressions of H_n. For instance,

$$\begin{aligned}
x = 0 \;&\Rightarrow\; H_n = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i}\binom{n}{i};\\
x = 1 \;&\Rightarrow\; H_n = \sum_{i=1}^{n} \frac{1}{i} \quad \text{(the definition in (1.6))};\\
x = 2 \;&\Rightarrow\; H_n = \sum_{i=1}^{n} \frac{1}{i}\left(2^i - \binom{n}{i}\right);\\
x = -1 \;&\Rightarrow\; H_n = \sum_{i=1}^{n} \frac{(-1)^i}{i}\left(1 - \binom{n}{i}2^i\right);\\
x = 1/2 \;&\Rightarrow\; H_n = \sum_{i=1}^{n} \frac{1}{2^i i}\left(1 + (-1)^{i-1}\binom{n}{i}\right).
\end{aligned}$$

Returning to the main subject, we will give below another proof of (1.8) by making use of special expressions of H_n obtained from (3.1).

Another proof of (1.8). Putting $x = m/k$ in (3.1) for integers $m, k \geq 1$, we have

$$H_n = \sum_{i=1}^{n} \frac{1}{i}\left(\frac{m}{k}\right)^i - \sum_{i=1}^{n}\binom{n}{i}\frac{1}{i}\left(\frac{m}{k} - 1\right)^i.$$

Summing up over $m = 1, 2, ..., p$ for an odd prime p,

$$pH_n = \sum_{i=1}^{n} \frac{1}{ik^i}\sum_{m=1}^{p} m^i - \sum_{i=1}^{n}\binom{n}{i}\frac{1}{ik^i}\sum_{m=1}^{p}(m-k)^i. \tag{3.2}$$

We now evaluate two double sums on the right-hand side individually modulo p^2. Since $S_1(p) \equiv pB_1 + p \pmod{p^2}$ and $S_i(p) \equiv S_i(p-1) \equiv pB_i \pmod{p^2}$ for $i \geq 2$ by (2.4), the first double sum can be calculated as follows:

$$\sum_{i=1}^{n} \frac{1}{ik^i}\sum_{m=1}^{p} m^i = \sum_{i=1}^{n} \frac{1}{ik^i}S_i(p) \equiv p\sum_{i=1}^{n}\frac{B_i}{ik^i} + \frac{p}{k} \pmod{p^2}. \tag{3.3}$$

Next, for evaluating the second double sum modulo p^2, we first calculate the inner sum $\sum_{m=1}^{p}(m-k)^i$ $(1 \leq k \leq p-1)$. Since $(-a)^i \equiv (p-a)^i - ip(-a)^{i-1} \pmod{p^2}$

for integers $a, i \geq 1$, by using (2.4) we obtain

$$\begin{aligned}
\sum_{m=1}^{p}(m-k)^i &= \sum_{m=1}^{k-1}(m-k)^i + (k-k)^i + \sum_{m=k+1}^{p}(m-k)^i \\
&\equiv \sum_{m=1}^{k-1}(p+m-k)^i - ip\sum_{m=1}^{k-1}(m-k)^{i-1} + \sum_{m=k+1}^{p}(m-k)^i \\
&\equiv \left(\sum_{m=1}^{k-1}(p+m-k)^i + \sum_{m=k+1}^{p}(m-k)^i\right) - ip\sum_{m=1}^{k-1}(m-k)^{i-1} \qquad (3.4) \\
&\equiv S_i(p-1) - ip\sum_{m=1}^{k-1}(m-k)^{i-1} \\
&\equiv pB_i - ip\sum_{j=1}^{k-1}(-j)^{i-1} \pmod{p^2}.
\end{aligned}$$

As the second step, put $x = -j/k$ $(j, k \geq 1)$ in the familiar identity

$$\sum_{i=1}^{n}\binom{n}{i}x^{i-1} = \frac{(x+1)^n - 1}{x} \quad (x \neq 0)$$

to deduce the identity

$$\sum_{i=1}^{n}\binom{n}{i}\left(-\frac{j}{k}\right)^{i-1} = \frac{(-j/k+1)^n - 1}{-j/k} = \frac{k^n - (k-j)^n}{jk^{n-1}}.$$

Take here $j = 1, 2, ..., k$ and add up all of them. Then we get, dividing by k,

$$\begin{aligned}
\sum_{i=1}^{n}\binom{n}{i}\frac{1}{k^i}\sum_{j=1}^{k}(-j)^{i-1} &= \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{n}\binom{n}{i}\left(-\frac{j}{k}\right)^{i-1} \\
&= \frac{1}{k}\sum_{j=1}^{k}\frac{k^n - (k-j)^n}{jk^{n-1}} \qquad (3.5) \\
&= H_k - \frac{1}{k^n}\sum_{j=1}^{k}\frac{(k-j)^n}{j}.
\end{aligned}$$

Making use of (3.4) in conjunction with (3.5), the second double sum is eventually evaluated modulo p^2 as follows:

$$\begin{aligned}\sum_{i=1}^{n}\binom{n}{i}\frac{1}{ik^i}\sum_{m=1}^{p}(m-k)^i &\equiv p\sum_{i=1}^{n}\binom{n}{i}\frac{B_i}{ik^i}-p\sum_{i=1}^{n}\binom{n}{i}\frac{1}{k^i}\sum_{j=1}^{k-1}(-j)^{i-1}\\ &\equiv p\sum_{i=1}^{n}\binom{n}{i}\frac{B_i}{ik^i}-p\sum_{i=1}^{n}\binom{n}{i}\frac{1}{k^i}\left(\sum_{j=1}^{k}(-j)^{i-1}-(-k)^{i-1}\right)\\ &\equiv p\sum_{i=1}^{n}\binom{n}{i}\frac{B_i}{ik^i}-p\sum_{i=1}^{n}\binom{n}{i}\frac{1}{k^i}\sum_{j=1}^{k}(-j)^{i-1}+\frac{p}{k}\sum_{i=1}^{n}\binom{n}{i}(-1)^{i-1}\\ &\equiv p\sum_{i=1}^{n}\binom{n}{i}\frac{B_i}{ik^i}+p\left(\frac{1}{k^n}\sum_{j=1}^{k}\frac{(k-j)^n}{j}-H_k\right)+\frac{p}{k} \pmod{p^2}.\end{aligned}\tag{3.6}$$

Substituting (3.3) and (3.6) simultaneously into (3.2) and dividing it by p, we get

$$H_n \equiv \sum_{i=1}^{n}\frac{B_i}{ik^i}-\sum_{i=1}^{n}\binom{n}{i}\frac{B_i}{ik^i}-\frac{1}{k^n}\sum_{j=1}^{k}\frac{(k-j)^n}{j}+H_k \pmod{p}.$$

As it is similar to the proof of (1.7) in Theorem 2.3, this congruence also holds for infinitely many odd primes p and both sides do not depend on p; so that (1.8) follows unconditionally, as desired. □

At the end of this paper, we wish to incidentally mention that the first sum on the left-hand side of (1.8) is just the partial sum of the infinite series appearing in the well-known asymptotic formula

$$\begin{aligned}H_k &\sim \log k+\gamma+\frac{1}{2k}-\sum_{i=1}^{\infty}\frac{B_{2i}}{2ik^{2i}}\\ &\sim \log k+\gamma-\sum_{i=1}^{\infty}\frac{B_i}{ik^i}\quad (k\to\infty),\end{aligned}$$

where $\gamma := \lim_{k\to\infty}(H_k-\log k)$ is the Euler-Mascheroni constant.

Acknowledgement. The author is grateful to Bruce Landman for his valuable suggestions and comments as to the improvement of the paper presentation.

References

[1] T. Agoh, Recurrences for Bernoulli and Euler polynomials and numbers, *Expo. Math.* **18** (2000), 197–214.

[2] T. Agoh, On the Miki and Matiyasevich identities for Bernoulli numbers, *Integers* **14** (2014), Article #A17.

[3] T. Agoh, Convolution identities for Bernoulli and Genocchi polynomials, *Electron. J. Combin.* **21** (2014), Paper P1.65, 14 pp.

[4] T. Agoh and K. Dilcher, Convolution identities and lacunary recurrences for Bernoulli numbers, *J. Number Theory* **124** (2007), 105–122.

[5] T. Agoh and K. Dilcher, Shortened recurrence relations for Bernoulli numbers, *Discrete Math.* **309** (2009), 887–898.

[6] T. Agoh and K. Dilcher, Higher-order recurrences for Bernoulli numbers, *J. Number Theory* **129** (2009), 1837–1847.

[7] M. C. Crabb, The Miki-Gessel Bernoulli number identity, *Glasg. Math. J.* **47** (2005), 327–328.

[8] K. Dilcher and C. Vignat, General convolution identities for Bernoulli and Euler polynomials, *J. Math. Anal. Appl.* **435** (2015), 1478–1498.

[9] G. V. Dunne and C. Schubert, Bernoulli number identities from quantum field theory and topological string theory, *Comm. Number Theory and Phys.* **7** (2013), 225–249.

[10] I. M. Gessel, On Miki's identity for Bernoulli numbers, *J. Number Theory* **110** (2005), 75–82.

[11] E. R. Hansen, *A table of series and products*, Prentice Hall, Englewood Cliffs, NJ, 1975.

[12] Y. Matiyasevich, Identity #0102 with Bernoulli numbers. Available at http://logic.pdmi.ras.ru/~yumat/personaljournal/identitybernoulli/iden0102.htm.

[13] H. Miki, A relation between Bernoulli numbers, *J. Number Theory* **10** (1978), 297–302.

[14] N. Nielsen, *Traité élémentaire des nombres de Bernoulli*, Gauthier-Villars, Paris, 1923.

[15] H. Pan and Z-W. Sun, New identities involving Bernoulli and Euler polynomials, *J. Comb. Theory,* Ser. A, **113** (2006), 156–175.

[16] L. Saalschütz, *Vorlesungen über die Bernoullischen Zahlen, ihren Zusammenhang mit den Secanten-Coefficienten und ihre wichtigeren Anwendungen*, Springer-Verlag, Berlin, 1893.