DISCRIMINATORS AND k-REGULAR SEQUENCES

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Abstract

The discriminator of an integer sequence $\mathbf{s} = (s(i))_{i \geq 0}$, introduced by Arnold, Benkoski, and McCabe in 1985, is the map $D_\mathbf{s}(n)$ that sends $n \geq 1$ to the least positive integer m such that the n numbers $s(0), s(1), \ldots, s(n-1)$ are pairwise incongruent modulo m. In this note we consider the discriminators of a certain class of sequences, the k-regular sequences. We compute the discriminators of two such sequences, the so-called "evil" and "odious" numbers, and show they are 2-regular. We give an example of a k-regular sequence whose discriminator is not k-regular. Finally, we examine sequences that are their own discriminators, and count the number of length-n finite sequences with this property.

1. Discriminators

Let $\mathbf{s} = (s(i))_{i \geq 0}$ be a sequence of distinct integers. For each $n \geq 1$, if the n numbers $s(0), s(1), \ldots, s(n-1)$ are pairwise incongruent modulo m, we say that m *discriminates* them. For $n \geq 1$ we define $D_\mathbf{s}(n)$ to be the least positive integer m that discriminates the numbers $s(0), s(1), \ldots, s(n-1)$; such an m always exists because of the distinctness requirement. Furthermore, we set $D_\mathbf{s}(0) = 0$, but usually this will be of no consequence. The function (or sequence) $D_\mathbf{s}(n)$ is called the *discriminator* of the sequence $\mathbf{s}$, and was introduced by Arnold, Benkoski, and McCabe [3]. They proved that the discriminator $D_{\mathrm{sq}}(n)$ of the sequence $(n+1)^2_{n \geq 0} =$

$1, 4, 9, 16, \ldots$ of positive integer squares is given by

$$
D_{\mathrm{sq}}(n) = \begin{cases}
1, & \text{if } n = 1; \\
2, & \text{if } n = 2; \\
6, & \text{if } n = 3; \\
9, & \text{if } n = 4; \\
\min\{k \ : \ k \geq 2n \text{ and } (k = p \text{ or } k = 2p \text{ for some prime } p) \}, & \text{if } n > 4.
\end{cases}
$$

More recently, discriminators of various sequences were studied by Schumer and Steinig [14], Barcau [4], Schumer [13], Bremser, Schumer, and Washington [7], Moree and Roskam [10], Moree [8], Moree and Mullen [9], Zieve [17], Sun [16], and Moree and Zumalacárrequi [11].

In this paper we recall the definition of k-regular sequences, an interesting class of sequences that has been widely studied. We introduce two well-known 2-regular sequences, the so-called "evil" and "odious" numbers. We prove that their discriminators are 2-regular. Finally, we give an example of a k-regular sequence whose discriminator sequence is not k-regular.

In what follows, we use the following notation. Let Σ_k denote the k-letter alphabet $\{0, 1, \ldots, k - 1\}$. If $x \in \Sigma_k^*$ is a string of digits, then $[x]_k$ denotes the value of x when considered as a base-k number. If n is an integer, then $(n)_k$ is the string giving the canonical base-k representation of n (with no leading zeroes). If x is a string of digits, then $|x|$ denotes the length of the string x, and $|x|_a$ denotes the number of occurrences of the letter a in x. Finally, $x^n = \overbrace{xx \cdots x}^{n}$ for $n \geq 0$.

By $S + i$, for S a set of integers and i an integer, we mean the set $\{x + i \ : \ x \in S\}$. For sets S and T, we write $S \sqcup T$ to denote the union of S and T, *and* the assertion that this union is actually disjoint.

2. k-regular Sequences

Let $k \geq 2$ be an integer. The k-regular sequences are an interesting class of sequences with pleasant closure properties [1, 2]. They can be defined in several equivalent ways, and here we give three:

– They are the class of sequences $(s(n))_{n \geq 0}$ such that the set of subsequences of the form

$$\{(s(k^e n + i))_{n \geq 0} \ : \ e \geq 0 \text{ and } 0 \leq i < k^e\}$$

is a subset of a finitely-generated $\mathbb{Z}$-module.

– They are the class of sequences $(s(n))_{n \geq 0}$ for which there exist an integer $r \geq 1$, a $1 \times r$ row vector u, an $r \times 1$ column vector w, and an $r \times r$ matrix-valued morphism μ with domain Σ_k^* such that $s(n) = u\mu(v)w$ for all strings v with $[v]_k = n$.

– They are the class of sequences such that there are a finite number of recurrence relations of the form

$$s(k^e n + i) = \sum_j a_j s(k^{e_j} n + i_j)$$

where $e \geq 0$, $e_j < e$, $0 \leq i < k^e$, and $0 \leq i_j < k^{e_j}$, that completely determine all but finitely many values of s.

We recall the following closure properties of k-regular sequences (see [1]):

Theorem 1. *Let* $\mathbf{r} = (r_i)_{i \geq 0}$ *and* $\mathbf{s} = (s_i)_{i \geq 0}$ *be two k-regular sequences of integers, and let* $m \geq 1$ *be an integer. Then so are*

(a) $\mathbf{r} + \mathbf{s} = (r_i + s_i)_{i \geq 0}$;

(b) $\mathbf{rs} = (r_i s_i)_{i \geq 0}$;

(c) $\mathbf{r} \bmod m = (r_i \bmod m)_{i \geq 0}$.

3. The Evil and Odious numbers

The so-called "evil" and "odious" numbers are two examples of 2-regular sequences; they are sequences A001969 and A000069 in Sloane's *On-Line Encyclopedia of Integer Sequences*, respectively. These numbers were named by Richard K. Guy c. 1976, and appear in the classic book *Winning Ways* [6, p. 431].

The evil numbers $(\mathrm{ev}(n))_{n \geq 0}$ are

$$0, 3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30, 33, 34, 36, 39, 40, 43, \dots$$

and are those non-negative numbers having an even number of 1's in their base-2 expansion.

The odious numbers $(\mathrm{od}(n))_{n \geq 0}$ are

$$1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31, 32, 35, 37, 38, 41, \dots$$

and are those non-negative numbers having an odd number of 1's in their base-2 expansion.

The names "evil" and "odious" are puns derived from "even" and "odd". Clearly the union of these two sequences is $\mathbb{N}$, the set of all non-negative integers.

To see that these two sequences are 2-regular, note that both sequences satisfy the recurrence relations

$$f(4n) = -2f(n) + 3f(2n)$$
$$f(4n + 1) = -2f(n) + 2f(2n) + f(2n + 1)$$
$$f(4n + 2) = \frac{2}{3}f(n) + \frac{5}{3}f(2n + 1)$$
$$f(4n + 3) = 6f(n) - 3f(2n) + 2f(2n + 1),$$

which can be proved by an induction using the characterization in [1, Example 12].

Let $\mathcal{O}_n = \{\mathrm{od}(i) : \mathrm{od}(i) < n\}$ (resp., $\mathcal{E}_n = \{\mathrm{ev}(i) : \mathrm{ev}(i) < n\}$) denote the set of all odious (resp., evil) numbers that are strictly less than n.

Lemma 1. *(a) For $i \geq 1$ we have $|\mathcal{O}_{2^i}| = |\mathcal{E}_{2^i}| = 2^{i-1}$.*

(b) For $i \geq 1$ we have $\mathcal{O}_{2^{i+1}} = \mathcal{O}_{2^i} \sqcup (\mathcal{E}_{2^i} + 2^i)$.

(c) For $i \geq 1$ we have $\mathcal{E}_{2^{i+1}} = \mathcal{E}_{2^i} \sqcup (\mathcal{O}_{2^i} + 2^i)$.

Proof. (a) Let $0 \leq n < 2^i$. These n can be placed in 1–1 correspondence with the binary strings w of length i, using the correspondence $[w]_2 = n$. For each binary string x of length $i - 1$, either $x0$ is odious and $x1$ is evil, or vice versa. Thus there are 2^{i-1} odious numbers less than 2^i, and 2^{i-1} evil numbers less than 2^i.

(b) Let $2^i \leq n < 2^{i+1}$. Consider $n - 2^i$. Since the base-2 expansion of $n - 2^i$ differs from that of n by omitting the first bit, clearly $n - 2^i$ is evil if and only if n is odious.

(c) Just like (b).

$\square$

This gives the following corollary:

Corollary 1. *For integers $n \geq 0$ and $i \geq 1$ we have $\mathrm{od}(n) \in \mathcal{O}_{2^i}$ and $\mathrm{ev}(n) \in \mathcal{E}_{2^i}$ if and only if $n < 2^{i-1}$. Furthermore*

$$\mathrm{od}(2^{i-1}) = 2^i; \tag{1}$$
$$\mathrm{ev}(2^{i-1}) = 2^i + 1. \tag{2}$$

3.1. Discriminator of the Odious Numbers

We now turn our attention to the discriminators for the evil and odious numbers, starting with the odious numbers. First, we need to prove the following useful lemma.

Lemma 2. *Let $i \geq 1$ and $1 \leq m < 2^i$. Then there exist two odious numbers j, ℓ with $1 \leq j < \ell \leq 2^i$ such that $m = \ell - j$.*

Proof. Let $w = (m)_2$. There are three cases according to the form of w.

1. No 1 follows a 0 in w. Then $w = 1^a 0^b$, where $a \geq 1$, $b \geq 0$, and $a + b \leq i$. So $m = 2^b(2^a - 1)$. Take $\ell = 2^{a+b}$ and $j = 2^b$.

2. $w = x01y$, where $|xy|_1$ is odd. Take $j = 2^{|y|+1}$ and $\ell = m + 2^{|y|+1}$. Now $(\ell)_2 = x11y$, and clearly $|x11y|_1$ is odd, so ℓ is odious.

3. $w = x01y$, where $|xy|_1$ is even. Take $j = 2^{|y|}$ and $\ell = m + 2^{|y|}$. Now $(\ell)_2 = x10y$, and clearly $|x10y|_1$ is odd, so ℓ is odious.

$\square$

With the help of this lemma, we can compute the discriminator for the sequence of odious numbers.

Theorem 2. *For the sequence of odious numbers, the discriminator $D_{\mathrm{od}}(n)$ satisfies the equation*

$$D_{\mathrm{od}}(n) = 2^{\lceil \log_2 n \rceil} \tag{3}$$

for $n \geq 1$.

Proof. The cases $n = 1, 2$ are left to the reader. Otherwise, let $i \geq 1$ be such that $2^i < n \leq 2^{i+1}$. We show $D_{\mathrm{od}}(n) = 2^{i+1}$. There are two cases:

Case 1: $n = 2^i + 1$. We must compute the discriminator of $\mathrm{od}(0), \mathrm{od}(1), \ldots, \mathrm{od}(2^i) = 2^{i+1}$. By Lemma 2, for each $m < 2^{i+1}$, there exist two odious numbers j, ℓ with $1 \leq j \leq \ell \leq 2^{i+1}$ with $\ell - j = m$. So the numbers $\mathrm{od}(0), \mathrm{od}(1), \ldots, \mathrm{od}(2^i)$ cannot be pairwise incongruent modulo m for $m < 2^{i+1}$. On the other hand, since 0 is not odious and each of the numbers $\mathrm{od}(0), \mathrm{od}(1), \ldots, \mathrm{od}(2^i)$ are less than 2^{i+1} except the very last (which is congruent to 0 modulo 2^{i+1}), clearly 2^{i+1} discriminates $\mathrm{od}(0), \mathrm{od}(1), \ldots, \mathrm{od}(2^i)$.

Case 2: $2^i + 1 < n \leq 2^{i+1}$. Since the discriminator is nondecreasing, we know $D_{\mathrm{od}}(n) \geq 2^{i+1}$. It suffices to show that 2^{i+1} discriminates

$$\mathcal{O}_{2^{i+2}} = \{\mathrm{od}(0), \mathrm{od}(1), \ldots, \mathrm{od}(2^{i+1} - 1)\}.$$

Now from Lemma 1(b), we have

$$\mathcal{O}_{2^{i+2}} = \mathcal{O}_{2^{i+1}} \sqcup (\mathcal{E}_{2^{i+1}} + 2^{i+1}).$$

If we now take both sides modulo 2^{i+1}, we see that the right-hand side is just $\mathcal{O}_{2^{i+1}} \sqcup \mathcal{E}_{2^{i+1}}$, which represents all integers in the range $[0, 2^{i+1})$.

$\square$

Empirically, many interesting sequences of positive integers seem to have discriminator $2^{\lceil \log_2 n \rceil}$. However, of all such sequences, the odious numbers play a special role: they are the lexicographically least.

Theorem 3. *The sequence of odious numbers is the lexicographically least increasing sequence of positive integers $\mathbf{s}$ such that $D_s(n) = 2^{\lceil \log_2 n \rceil}$.*

Proof. We prove this by contradiction. Suppose there exists a sequence of increasing positive integers, $s(0), s(1), \ldots$, that is lexicographically smaller than the sequence of odious numbers but shares the same discriminator, $D_s(n) = 2^{\lceil \log_2 n \rceil}$.

Let j denote the first index such that $s(j) \neq \text{od}(j)$, i.e., $s(j) < \text{od}(j)$, since s is a lexicographically smaller sequence than the odious numbers. We can see that $s(j)$ must be evil, because $\text{od}(j)$ is the next odious number after $\text{od}(j-1) = s(j-1)$. Note that since $\text{od}(0) = 1$ is the smallest positive integer, necessarily $j \geq 1$.

Now let $i \geq 0$ be such that $2^i \leq j < 2^{i+1}$. In that case, the discriminator of the sequence $s(0), s(1), \ldots, s(j)$ is $D_s(j+1) = 2^{\lceil \log_2(j+1) \rceil} = 2^{i+1}$. However, $s(j)$ also discriminates this sequence, which implies that $s(j) \geq D_s(j+1) = 2^{i+1}$. Note that by the definition of j, this means that all odious numbers less than 2^{i+1} are present in the sequence $s(0), s(1), \ldots, s(j)$.

Furthermore, we have $s(j) < \text{od}(j) < \text{od}(2^{i+1}) = 2^{i+2}$. So $2^{i+1} \leq s(j) < 2^{i+2}$, which means that the largest power of 2 appearing in the binary representation of $s(j)$ is 2^{i+1}. Therefore $s(j) \bmod 2^{i+1} = s(j) - 2^{i+1}$ is odious. However, $s(j) \bmod 2^{i+1} < 2^{i+1}$. But the sequence $s(0), s(1), \ldots, s(j)$ contains all odious numbers less than 2^{i+1}, which therefore includes the result of $s(j) \bmod 2^{i+1}$. In other words, $s(j)$ is congruent to another number in this sequence modulo 2^{i+1}, i.e., $D_s(j+1) \neq 2^{i+1}$, which is a contradiction. □

3.2. Discriminator of the Evil Numbers

We now focus on the discriminator for the sequence of evil numbers. Here, we need to utilize a similar lemma as before.

Lemma 3. *Let $i \geq 3$ and $1 \leq m < 2^i - 3$. Then there exist two evil numbers j, ℓ with $0 \leq j < \ell \leq 2^i + 1$ such that $m = \ell - j$.*

Proof. Let $w = (m)_2$. There are several cases according to the form of w.

1. The number m is evil. Take $\ell = m$ and $j = 0$.

2. There are no 0's in w. Then $m = 2^a - 1$ where $0 < a < i$. Note that $a \neq i$. If $m = 1$, then take $\ell = 6$ and $j = 5$. Otherwise, take $\ell = 2^a + 2$ and $j = 3$.

3. No 1 follows a 0 in w and $|w|_0 > 0$. Then $w = 1^a 0^b$, where $a \geq 1$, $b \geq 1$, and $a + b \leq i$. So $m = 2^b(2^a - 1)$. Take $\ell = 2^{a+b} + 1$ and $j = 2^b + 1$.

4. There is exactly one 0 in w and w ends with 01. Then $w = 1^a 01$, where $1 \leq a \leq i - 3$. So $m = 2^{a+2} - 3$. Take $\ell = 2^{a+2} + 2$ and $j = 5$.

5. There is exactly one 0 in w and w ends with 11. Then $w = 1^a 01^b$, where $a \geq 1$, $b \geq 2$, and $a + b \leq i - 1$. So $m = 2^{a+b+1} - 2^b - 1$. Take $\ell = 2^{a+b+1} + 1$ and $j = 2^b + 2$.

6. $w = x01y0z$, where $|xyz|_1$ is even. Take $j = 2^{|y|+|z|+1} + 2^{|z|}$ and $\ell = m + 2^{|y|+|z|+1} + 2^{|z|}$. So $(\ell)_2 = x10y1z$. We can see $|x10y1z|_1$ is even, so ℓ is evil.

7. $w = x0y01z$, where $|xyz|_1$ is even. Take $j = 2^{|y|+|z|+2} + 2^{|z|}$ and $\ell = m + 2^{|y|+|z|+2} + 2^{|z|}$. So $(\ell)_2 = x1y10z$. We can see $|x1y10z|_1$ is even, so ℓ is evil.

$\square$

With the help of this lemma, we can compute the discriminator for the sequence of evil numbers.

Theorem 4. *For the sequence of evil numbers, the discriminator $D_{\mathrm{ev}}(n)$ satisfies the equation*

$$D_{\mathrm{ev}}(n) = \begin{cases} 2^{i+1} - 3, & \text{if } n = 2^i + 1 \text{ for odd } i \geq 2; \\ 2^{i+1} - 1, & \text{if } n = 2^i + 1 \text{ for even } i \geq 2; \\ 2^{\lceil \log_2 n \rceil}, & \text{otherwise,} \end{cases} \tag{4}$$

for $n \geq 1$.

Proof. The cases $n = 1, 2, 3, 4$ are left to the reader. Otherwise, let $i \geq 2$ be such that $2^i < n \leq 2^{i+1}$. We show $D_{\mathrm{ev}}(n)$ satisfies the given equation. There are three cases presented in the equation:

`Case 1:` $n = 2^i + 1$ for odd $i \geq 2$. We must compute the discriminator of $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i) = 2^{i+1} + 1$. By Lemma 3, for each $m < 2^{i+1} - 3$, there exist two evil numbers j, ℓ with $1 \leq j \leq \ell \leq 2^{i+1} + 1$ with $\ell - j = m$. So the numbers $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)$ cannot be pairwise incongruent modulo m for $m < 2^{i+1} - 3$. Note that for odd $i \geq 2$, the only evil numbers in the range $[2^{i+1} - 3, 2^{i+1} + 1]$ are $2^{i+1} - 1$ and $2^{i+1} + 1$, easily observed from their binary representations. We can see that $2^{i+1} - 1 \equiv 2 \pmod{2^{i+1} - 3}$ and $2^{i+1} + 1 \equiv 4 \pmod{2^{i+1} - 3}$, where neither 2 nor 4 are evil. All the other numbers in the sequence $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)$ are less than $2^{i+1} - 3$, and thus it is clear that $2^{i+1} - 3$ discriminates $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)$.

`Case 2:` $n = 2^i + 1$ for even $i \geq 2$. We must compute the discriminator of $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i) = 2^{i+1} + 1$. Just as in the previous case, Lemma 3 ensures that the numbers $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)$ cannot be pairwise incongruent modulo m for $m < 2^{i+1} - 3$. For even $i \geq 2$, we can see that both $2^{i+1} - 3$ and $2^{i+1} - 2$ are evil from their binary representations. Neither of them can discriminate the sequence since $m \bmod m = 0$ for either $m = 2^{i+1} - 3$ or $m = 2^{i+1} - 2$, while 0 is evil. Thus the discriminator must be at least $2^{i+1} - 1$. Since neither $2^{i+1} - 1$ nor 2^{i+1} are evil, we can see that each of the numbers $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)$ are all less than $2^{i+1} - 1$ except the very last, which is $2^{i+1} + 1 = 2 \bmod (2^{i+1} - 1)$, where 2 is not evil. Therefore, it is clear that $2^{i+1} - 1$ discriminates $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^i)$.

Case 3: $2^i + 1 < n \le 2^{i+1}$. From the previous two cases, we know that $D_{\mathrm{ev}}(2^i + 1)$ is either $2^{i+1} - 3$ or $2^{i+1} - 1$. Since the discriminator is nondecreasing, we know $D_{\mathrm{ev}}(n) \ge 2^{i+1} - 3$. We can see that the sequence $\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(n-1)$ must include $\mathrm{ev}(2^i + 2) = 2^{i+1} + 2$, the next evil number after $2^{i+1} + 1$. We then observe that

$$2^{i+1} + 2 \equiv 5 \pmod{2^{i+1} - 3},$$
$$2^{i+1} + 1 \equiv 3 \pmod{2^{i+1} - 2},$$
$$2^{i+1} + 2 \equiv 3 \pmod{2^{i+1} - 1},$$

where the numbers 3 and 5 are evil. Therefore, the discriminator must be at least 2^{i+1}. It suffices to show that 2^{i+1} discriminates $\mathcal{E}_{2^{i+2}} = \{\mathrm{ev}(0), \mathrm{ev}(1), \ldots, \mathrm{ev}(2^{i+1} - 1)\}$. Now from Lemma 1(c), we have

$$\mathcal{E}_{2^{i+2}} = \mathcal{E}_{2^{i+1}} \sqcup (\mathcal{O}_{2^{i+1}} + 2^{i+1}).$$

If we now take both sides modulo 2^{i+1}, we see that the right-hand side is just $\mathcal{E}_{2^{i+1}} \sqcup \mathcal{O}_{2^{i+1}}$, which represents all integers in the range $[0, 2^{i+1})$. Thus we have $D_{\mathrm{ev}}(n) = 2^{i+1} = 2^{\lceil \log_2 n \rceil}$ for $2^i + 1 < n \le 2^{i+1}$.

$\square$

4. A k-regular Sequence Whose Discriminator is Not k-regular

Consider the sequence $1, 4, 9, 16, \ldots$ of perfect squares. From [1, Example 5], this sequence is k-regular for all integers $k \ge 2$. We show

Theorem 5. *The discriminator sequence of the perfect squares is not k-regular for any k.*

Proof. We use the characterization of the discriminator sequence $D_{\mathrm{sq}}(n)$ given above in Section 1. Suppose $D_{\mathrm{sq}}(n)$ is k-regular. Then from Theorem 1 (c) we know that the sequence A given by $A(n) = D_{\mathrm{sq}}(n) \bmod 2$ is k-regular. From Theorem 1 (b) we know that the sequence $F(n) = A(n)D_{\mathrm{sq}}(n)$ is k-regular. From Theorem 1 (a) we know that the sequence $B(n) = 2 - 2A(n)$ is k-regular. From Theorem 1 (a) we know that the sequence $E(n) = F(n) + B(n)$ is k-regular. It is now easy to see that for $n > 4$ we have $E(n) = 2$ if $B(n)$ is even, while $E(n) = D_{\mathrm{sq}}(n)$ if $D_{\mathrm{sq}}(n)$ is odd. Thus $E(n)$ takes only prime values for $n > 4$.

We now argue that $E(n)$ is unbounded. To see this, it suffices to show that there are infinitely many indices n such that $D_{\mathrm{sq}}(n)$ is prime. By Dirichlet's theorem on primes in arithmetic progressions there are infinitely many primes p for which $p \equiv 1 \pmod 4$. For these primes consider $n = (p-1)/2$. Then $2n = p - 1$ is divisible

by 4 and hence not twice a prime, but $2n + 1 = p$. Hence for these n we have $D_{\mathrm{sq}}(n) = p = 2n + 1$, and hence $E(n) = D_{\mathrm{sq}}(n)$. Thus $(E(n))$ is unbounded.

Finally, we apply a theorem of Bell [5] to the sequence E. Bell's theorem states that any unbounded k-regular sequence must take infinitely many composite values. However, the sequence $(E(n))$ is unbounded and takes only prime values for $n > 4$. This contradiction shows that $D_{\mathrm{sq}}(n)$ cannot be k-regular. $\qquad\square$

5. Discriminator of the Cantor Numbers

Consider the Cantor numbers $(C(n))_{n \geq 0}$

$$0, 2, 6, 8, 18, 20, 24, 26, 54, 56, 60, 62, 72, 74, 78, 80, 162, 164, 168, 170, 180, \ldots$$

which are the numbers having only 0's and 2's in their base-3 expansion. This is sequence A005823 in Sloane's *On-Line Encyclopedia of Integer Sequences*. It is 2-regular, as it satisfies the recurrence relations

$$C(2n) = 3C(n)$$
$$C(2n + 1) = 3C(n) + 2.$$

We have the following conjecture about the discriminator sequence $D_C(n)$ of the Cantor numbers:

$$D_C(8n) = \frac{13}{3}D_C(4n) - 2D_C(4n + 1) + \frac{2}{3}D_C(4n + 2)$$

$$D_C(8n + 1) = \frac{3}{2}D_C(2n) + \frac{7}{2}D_C(4n) - 2D_C(4n + 1) + D_C(4n + 2)$$

$$D_C(8n + 2) = \frac{10}{3}D_C(4n) - 2D_C(4n + 1) + \frac{5}{3}D_C(4n + 2)$$

$$D_C(8n + 3) = \frac{9}{2}D_C(2n) + \frac{11}{6}D_C(4n) - 3D_C(4n + 1) + \frac{8}{3}D_C(4n + 2)$$

$$D_C(8n + 4) = 6D_C(2n) - 2D_C(4n) + 2D_C(4n + 1) + D_C(4n + 2)$$

$$D_C(8n + 5) = 6D_C(2n) - 2D_C(4n) + D_C(4n + 1) + 2D_C(4n + 2)$$

$$D_C(8n + 6) = \frac{3}{2}D_C(2n) - \frac{1}{2}D_C(4n) - D_C(4n + 1) + 4D_C(4n + 2)$$

$$D_C(16n + 7) = -3D_C(2n) + D_C(4n) + 7D_C(4n + 1) + 2D_C(4n + 2)$$

$$D_C(16n + 15) = -9D_C(n) + \frac{27}{2}D_C(2n) - \frac{15}{2}D_C(4n) + 9D_C(4n + 1)$$
$$- 6D_C(4n + 2) + 10D_C(4n + 3).$$

If true, this would mean that $D_C(n)$ is also 2-regular.

6. Self-discriminators

In this section we change our indexing slightly. Let $\mathbf{s} = (s_1, s_2, s_3, \ldots)$ be an increasing sequence of positive integers and let $D_{\mathbf{s}} = (d_1, d_2, d_3, \ldots)$ be its associated discriminator sequence. When does $\mathbf{s} = D_{\mathbf{s}}$?

Theorem 6. *The sequence* $\mathbf{s}$ *is its own discriminator if and only if either* $s_i = i$ *for all* $i \geq 1$, *or the following three conditions hold:*

(a) *There exists an integer* $t \geq 1$ *such that* $\mathbf{s}$ *begins* $1, 2, 3, \ldots, t$ *but not* $1, 2, 3 \ldots, t + 1$; *and*

(b) $s_{t+1} \in \{t + 2, \ldots, 2t + 1\}$; *and*

(c) $s_{i+1} - s_i \in \{1, 2, \ldots, t\}$ *for* $i > t$.

Proof. The case where $s_i = i$ for all i is easy and is left to the reader. Similarly, if $s_1 > 1$ then $\mathbf{s}$ cannot be its own discriminator. Otherwise, assume condition (a) holds.

Let $(d_i)_{i \geq 1}$ be the discriminator of $\mathbf{s} = (s_i)_{i \geq 1}$. We show that conditions (b) and (c) hold if and only if $d_i = s_i$ for all i. If $1 \leq i \leq t$, this is clear. There are two cases to consider.

`Case 1:` $i = t + 1$: assume $d_{t+1} = s_{t+1}$. Since $(s_i)_{i \geq 1}$ is increasing, from (a) we have $s_{t+1} \geq t + 2$ and so $d_{t+1} \geq t + 2$. On the other hand, if $s_{t+1} \geq 2t + 2$, then the sequence $(1, 2, \ldots, t, s_{t+1})$ is discriminated by $s_{t+1} - (t + 1)$, a contradiction. So $s_{t+1} \leq 2t + 1$.

For the other direction, suppose $s_{t+1} \in \{t + 2, \ldots, 2t + 1\}$. Then contrary to what we want to prove, if $t \leq d_{t+1} < s_{t+1}$, then $s_{t+1} \equiv s_{t+1} - d_{t+1} \pmod{d_{t+1}}$. Unless $s_{t+1} = 2t + 1$, $d_{t+1} = t$, we have $1 \leq s_{t+1} - d_{t+1} \leq t$, a contradiction, since then $s_{t+1} \bmod d_{t+1}$ already occurred in $s_1 \bmod d_{t+1}, s_2 \bmod d_{t+1}, \ldots, s_t \bmod d_{t+1}$. In the exceptional case $s_{t+1} = 2t + 1$, $d_{t+1} = t$, which gives $s_{t+1} \bmod d_{t+1} = 1 = s_1$, a contradiction. So $d_{t+1} \geq s_{t+1}$. On the other hand, it is easy to see that s_{t+1} discriminates $s_1, s_2, \ldots, s_{t+1}$.

`Case 2:` $i > t + 1$: Suppose $d_i = s_i$ for all $i > t + 1$, and, to get a contradiction, let $i > t + 1$ be the smallest index for which $s_i - s_{i-1} \notin \{1, 2, \ldots, t\}$. We cannot have $s_i = s_{i-1}$ because the sequence $(s_i)_{i \geq 1}$ is strictly increasing. So $d_i = s_i \geq s_{i-1} + t + 1$. But then the sequence $(s_1, s_2, \ldots, s_i)$ is also discriminated by $s_i - (t + 1)$, a contradiction. So $s_i - s_{i-1} \in \{1, 2, \ldots, t\}$, as claimed.

For the other direction, assume $s_i - s_{i-1} \in \{1, 2, \ldots, t\}$ and $d_i < s_i$. Then $s_i \bmod d_i = s_i - d_i < t$, a contradiction. On the other hand, $s_i \bmod d_i = 0$, which is not an element of $\mathbf{s}$, so d_i discriminates $s_1, \ldots, s_i$, as desired. $\qquad \square$

Corollary 2. *There are uncountably many increasing sequences of positive integers that are their own discriminators.*

Corollary 3. *For $1 \le t \le n$, the number of length-n finite sequences, beginning with $1, 2, \ldots, t$ but not $1, 2, \ldots, t, t+1$, that are self-discriminators is t^{n-t}. Hence the total number of finite sequences of length n that are self-discriminators is $\sum_{1 \le t \le n} t^{n-t}$.*

Proof. Suppose $s_1, s_2, \ldots, s_t$ are fixed. If $t = n$, there is exactly one such sequence. Otherwise s_{t+1} is constrained to lie in $\{t+2, \ldots, 2t+1\}$, which is of cardinality t, and subsequent terms s_i (if there are any) are constrained to lie in $\{s_{i-1}+1, \ldots, s_{i-1}+t\}$, which is also of cardinality t. There are $n - t$ remaining terms, which gives t^{n-t} possible extensions of length n. ☐

Remark 1. The number of finite sequences of length n that are self-discriminators is given by sequence A026898 in Sloane's *On-Line Encyclopedia of Integer Sequences* [15].

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