ASYMPTOTICS FOR THE PARTIAL FRACTIONS OF THE RESTRICTED PARTITION GENERATING FUNCTION II

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Abstract

The generating function for $p_N(n)$, the number of partitions of n into at most N parts, may be written as a product of N factors. In an earlier paper, we studied the behavior of coefficients in the partial fraction decomposition of this product as $N \to \infty$ by applying the saddle-point method to get the asymptotics of the main terms. In this paper, we bound the error terms. This involves estimating products of sines and further saddle-point arguments. The saddle-points needed are associated with zeros of the analytically continued dilogarithm.

1. Introduction

1.1. Background

The generating function for $p_N(n)$, the number of partitions of n into at most N parts, and its partial fraction decomposition may be written as

$$\sum_{n=0}^{\infty} p_N(n) q^n = \prod_{j=1}^{N} \frac{1}{1-q^j} = \sum_{\substack{0 \leqslant h < k \leqslant N \\ (h,k)=1}} \sum_{\ell=1}^{\lfloor N/k \rfloor} \frac{C_{hk\ell}(N)}{(q - e^{2\pi i h/k})^{\ell}} \tag{1.1}$$

for coefficients $C_{hk\ell}(N)$ studied by Rademacher in [12]. Each $C_{hk\ell}(N)$ is in the field $\mathbb{Q}(e^{2\pi i h/k})$ by [8, Prop. 3.3]. Let Li_2 denote the dilogarithm. It is shown in [10, Sect. 1] that

$$\mathrm{Li}_2(w) - 2\pi i \log(w) = 0 \tag{1.2}$$

has a unique solution, namely $w_0 \approx 0.916198 - 0.182459i$. From this, define $z_0 := 1 + \log(1 - w_0)/(2\pi i) \approx 1.18147 + 0.255528i$. With

$$_N := \Big\{ h/k \ : \ 1 \leqslant k \leqslant N, \ 0 \leqslant h < k, \ (h,k) = 1 \Big\}$$

denoting the Farey fractions of order N in $[0,1)$, the asymptotic result

$$\sum_{h/k\in{}_{100}} C_{hk1}(N) = \mathrm{Re}\left[(-2z_0 e^{-\pi i z_0})\frac{w_0^{-N}}{N^2}\right] + O\left(\frac{|w_0|^{-N}}{N^3}\right) \tag{1.3}$$

is given in [9, Thm. 1.2]. This resolves an old conjecture of Rademacher in [12, p. 302] by showing that the limit of $C_{hk\ell}(N)$ as $N\to\infty$ does not exist in general since $|1/w_0| > 1$; see [9, Cor. 1.3].

Equation (1.3) is a special case of the more general theorem, [9, Thm. 1.4], which we state next. Note that $C_{01\ell}(N)$ is the coefficient of $1/(q-1)^\ell$ in (1.1).

Theorem 1.1. *There are explicit coefficients* $c_{\ell,0}, c_{\ell,1},\dots$ *so that*

$$C_{01\ell}(N) + \sum_{0<h/k\in{}_{100}} \sum_{j=1}^{\ell} (e^{2\pi i h/k}-1)^{\ell-j} C_{hkj}(N)$$
$$= \mathrm{Re}\left[\frac{w_0^{-N}}{N^{\ell+1}}\left(c_{\ell,0} + \frac{c_{\ell,1}}{N} + \cdots + \frac{c_{\ell,m-1}}{N^{m-1}}\right)\right] + O\left(\frac{|w_0|^{-N}}{N^{\ell+m+1}}\right) \tag{1.4}$$

where $c_{\ell,0} = -2z_0 e^{-\pi i z_0}(2\pi i z_0)^{\ell-1}$ *and the implied constant depends only on* ℓ *and* m.

The main term of Theorem 1.1 is shown in [9]. The proof that the size of the error term above is $O\left(|w_0|^{-N}/N^{\ell+m+1}\right)$ is sketched in [9], due to its length, and the detailed proof of this error bound is the main result of this paper.

Rademacher's coefficients $C_{hk\ell}(N)$ are fascinating numbers and their properties have been coming into focus with the recent papers [2, 1, 6, 14, 3, 8]. Andrews gave the first formulas for them in [1, Thm. 1]. Further expressions were given in [8] with, for example, the relatively simple

$$C_{01\ell}(N) = \frac{(-1)^N(\ell-1)!}{N!} \sum_{j_0+j_1+j_2+\cdots+j_N=N-\ell} \left\{ {\ell+j_0 \atop \ell} \right\} \frac{B_{j_1}B_{j_2}\cdots B_{j_N}}{(\ell-1+j_0)!} \frac{1^{j_1}2^{j_2}\cdots N^{j_N}}{j_1!j_2!\cdots j_N!}$$

where B_n is the nth Bernoulli number and $\left\{ {n \atop m} \right\}$ is the Stirling number, denoting the number of ways to partition a set of size n into m non-empty subsets. Also, with $s_m(N) := 1^m + 2^m + \cdots + N^m$,

$$C_{01\ell}(N) = \frac{(-1)^N}{N!} \sum_{j_0+1j_1+2j_2+\cdots+Nj_N=N-\ell} \frac{1}{j_0!j_1!j_2!\cdots j_N!}$$
$$\times \left(\frac{B_1}{1\cdot 1!}\big(s_1(N)+1-\ell\big)\right)^{j_1} \cdots \left(\frac{(-1)^{N-1}B_N}{N\cdot N!}\big(s_N(N)+1-\ell\big)\right)^{j_N}.$$

These results are [8, Eq. (2.12), Prop 2.4] and in that paper the close connection is described between Rademacher's coefficients $C_{hk\ell}(N)$ and Sylvester's waves. In forthcoming work we develop this link and obtain the asymptotics of the individual waves in Sylvester's decomposition of the (unrestricted) partition function $p(n)$.

It is also shown in [8, Thm. 7.3] that, for $r \geqslant 1$,

$$P_{01r}(N) := (-1)^N N! \cdot (-4)^r r! \cdot C_{01(N-r)}(N)$$

is a monic polynomial in N of degree $2r$ with 0 and 1 as roots. This proved part of Conjecture 7.1 in [14]. In the remaining part, Sills and Zeilberger conjecture that $P_{01r}(N)$ is convex and has coefficients that alternate in sign.

Rademacher realized, already in the 1937 paper [11], that his celebrated formula for $p(n)$ leads to a decomposition similar to (1.1):

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j} = \sum_{\substack{0 \leqslant h < k \\ (h,k)=1}} \sum_{\ell=1}^{\infty} \frac{C_{hk\ell}(\infty)}{(q-e^{2\pi i h/k})^\ell} \qquad (|q|<1), \tag{1.5}$$

with numbers $C_{hk\ell}(\infty)$ computed explicitly in [12, Eq. (130.6)]. Using limited numerical evidence he conjectured that $\lim_{N\to\infty} C_{hk\ell}(N) = C_{hk\ell}(\infty)$. Numerical computations were extended in [1, 2, 14] with the results in [14] indicating clearly that Rademacher's conjecture was almost certainly false. Confirmation of this was given independently in [3] and [9]. The work of Drmota and Gerhold in [3] gives the main term in the asymptotics of $C_{01\ell}(N)$ as $N \to \infty$ using techniques involving the Mellin transform. The proof of our Theorem 1.1, in [9] and this paper, is based on a different, conceptually simple idea that is described in the next subsection. Though certainly very long when all details are included, our proof results in the complete asymptotic expansion of a finite average containing $C_{01\ell}(N)$. With further improvements it should be possible to replace the average on the left side of (1.4) with just $C_{01\ell}(N)$; see [9, Conj. 1.5].

We highlight two further interesting directions for investigation leading from this paper.

(i) It should be possible to obtain the asymptotics for all coefficients $C_{hk\ell}(N)$ with k small. Based on Theorem 1.6 below, the asymptotic expansion of $C_{121}(N)$ was conjectured in [8, Conj. 6.3] and [9, Conj. 6.4]. Elements possibly leading to the asymptotic expansion of $C_{131}(N)+C_{231}(N)$ are given in [9, Eq. (6.12)].

(ii) Rademacher's original conjecture on the relationship between the sequence $C_{hk\ell}(1), C_{hk\ell}(2), \dots$ and $C_{hk\ell}(\infty)$ was too simplistic. However, it seems clear that there is indeed a close relationship between them, as shown in [14, Sect. 4] and [8, Table 2]. The precise nature of this link remains to be found.

1.2. Proof of Theorem 1.1

We introduce some notation and results from [9, Sect. 1.3] to describe the proof of Theorem 1.1. Define the numbers

$$Q_{hk\sigma}(N) := 2\pi i \operatorname*{Res}_{z=h/k} \frac{e^{2\pi i\sigma z}}{(1-e^{2\pi iz})(1-e^{2\pi i2z})\cdots(1-e^{2\pi iNz})}. \tag{1.6}$$

The Rademacher coefficients $C_{hk\ell}(N)$ are related to them by

$$C_{hk\ell}(N) = \sum_{\sigma=1}^{\ell} \binom{\ell-1}{\sigma-1}(-e^{2\pi ih/k})^{\ell-\sigma} Q_{hk\sigma}(N) \tag{1.7}$$

and for σ a positive integer they satisfy

$$\sum_{h/k\in {}_N} Q_{hk\sigma}(N) = 0 \tag{1.8}$$

for $N(N+1)/2 > \sigma$. Put

$$\mathcal{A}(N) := \Big\{h/k \ : \ N/2 < k \leqslant N, \ h=1 \ \text{ or } \ h=k-1\Big\} \subseteq {}_N \tag{1.9}$$

and decompose (1.8) into

$$\sum_{h/k\in {}_{100}} Q_{hk\sigma}(N) + \sum_{h/k\in {}_N-({}_{100}\cup\mathcal{A}(N))} Q_{hk\sigma}(N) + \sum_{h/k\in\mathcal{A}(N)} Q_{hk\sigma}(N) = 0. \tag{1.10}$$

Theorem 1.1 breaks into two natural parts. The first is proved in [9]:

Theorem 1.2. *With $b_0 = 2z_0e^{-\pi iz_0}$ and explicit $b_1(\sigma)$, $b_2(\sigma), \dots$ depending on $\sigma \in \mathbb{Z}$ we have*

$$\sum_{h/k\in\mathcal{A}(N)} Q_{hk\sigma}(N) = \operatorname{Re}\left[\frac{w_0^{-N}}{N^2}\left(b_0 + \frac{b_1(\sigma)}{N} + \cdots + \frac{b_{m-1}(\sigma)}{N^{m-1}}\right)\right] + O\left(\frac{|w_0|^{-N}}{N^{m+2}}\right)$$

for an implied constant depending only on σ and m.

The proof of the second part is sketched in [9]:

Theorem 1.3. *There exists $W < U := -\log|w_0| \approx 0.068076$ so that*

$$\sum_{h/k\in {}_N-({}_{100}\cup\mathcal{A}(N))} Q_{hk\sigma}(N) = O\left(e^{WN}\right)$$

for an implied constant depending only on $\sigma \in \mathbb{Z}$. We may take $W = 0.055$.

Theorem 1.1 follows from combining Theorems 1.2 and 1.3 with (1.10) and (1.7). This is done in [9, Sect. 5.4].

1.3. Main Results

In this paper we give the details of the proof of Theorem 1.3. This therefore completes the proof of Theorem 1.1 and (1.3). The work in this paper and [9] will also be useful in describing the asymptotics of Sylvester waves and restricted partitions; this corresponds to estimating $Q_{hk\sigma}(N)$ for $\sigma < 0$ as discussed in [9, Sect. 6.2]. Further natural extensions and possible generalizations of our results are given there as well.

Define the sine product

$$\textstyle\prod_m(\theta) := \displaystyle\prod_{j=1}^{m} 2\sin(\pi j \theta) \tag{1.11}$$

with $\prod_0(\theta) := 1$. In Section 3 we show

Proposition 1.4. *For $2 \leqslant k \leqslant N$, $\sigma \in \mathbb{R}$ and $s := \lfloor N/k \rfloor$*

$$|Q_{hk\sigma}(N)| \leqslant \frac{3}{k^3} \exp\left(N\frac{2+\log(1+3k/4)}{k} + \frac{|\sigma|}{N}\right) \left| \textstyle\prod_{N-sk}^{-1}(h/k) \right|.$$

In Section 2 we find sharp general bounds for $\prod_m^{-1}(h/k)$. This requires the interesting sum

$$S(m;h,k) := \sum_{(\beta,\gamma)\in Z(h,k)} \frac{\sin(2\pi m\gamma/k)}{|\beta\gamma|} \tag{1.12}$$

for

$$Z(h,k) := \Big\{(\beta,\gamma) \in \mathbb{Z}\times\mathbb{Z} \ : \ 1 \leqslant |\beta| < k, \ 1 \leqslant \gamma < k, \ \beta h \equiv \gamma \bmod k\Big\}. \tag{1.13}$$

We will see that $\prod_m^{-1}(h/k)$ and $S(m;h,k)$ may be bounded in terms of $1/|\beta_0\gamma_0|$ where (β_0,γ_0) is a pair in $Z(h,k)$ with $|\beta_0\gamma_0|$ minimal.

Combining a refinement of Proposition 1.4 with our bound for $\prod_m^{-1}(h/k)$ allows us to prove Theorem 1.3 except for h/k in the following sets

$$\mathcal{C}(N) := \Big\{h/k \ : \ \frac{N}{2} < k \leqslant N, \ k \text{ odd}, \ h = 2 \ \text{ or } \ h = k-2\Big\}, \tag{1.14}$$

$$\mathcal{D}(N) := \Big\{h/k \ : \ \frac{N}{2} < k \leqslant N, \ k \text{ odd}, \ h = \frac{k-1}{2} \ \text{ or } \ h = \frac{k+1}{2}\Big\}, \tag{1.15}$$

$$\mathcal{E}(N) := \Big\{h/k \ : \ \frac{N}{3} < k \leqslant \frac{N}{2}, \ h = 1 \ \text{ or } \ h = k-1\Big\}. \tag{1.16}$$

For the next results we need a brief description of the zeros of the dilogarithm; see [9, Sect. 2.3] and [10] for a fuller discussion. Initially defined as

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad \text{ for } |z| \leqslant 1, \tag{1.17}$$

the dilogarithm has an analytic continuation given by $-\int_{C(z)} \log(1-u)\frac{du}{u}$ where the contour of integration $C(z)$ is a path from 0 to $z \in \mathbb{C}$. This makes the dilogarithm a multi-valued holomorphic function with branch points at 0, 1 and ∞. See for example [5], [16]. We let $\mathrm{Li}_2(z)$ denote the dilogarithm on its principal branch so that $\mathrm{Li}_2(z)$ is a single-valued holomorphic function on $\mathbb{C} - [1, \infty)$. It can be shown that the value of the analytically continued dilogarithm is always given by

$$\mathrm{Li}_2(z) + 4\pi^2 A + 2\pi i B \log(z) \tag{1.18}$$

for some A, $B \in \mathbb{Z}$.

Let $w(A, B)$ be a zero of (1.18). It is shown in [10, Thm. 1.1] that for $B \neq 0$, a zero $w(A, B)$ exists if and only if $-|B|/2 < A \leqslant |B|/2$ and is unique in this case. Each zero may be found to arbitrary precision using Newton's method according to [10, Thm. 1.3]. We already met $w_0 = w(0, -1)$ and we also need the two further zeros $w(1, -3) \approx -0.459473 - 0.848535i$, $w(0, -2) \approx 0.968482 - 0.109531i$ and the associated saddle-points

$$z_3 := 3 + \log\big(1 - w(1, -3)\big)/(2\pi i), \qquad z_1 := 2 + \log\big(1 - w(0, -2)\big)/(2\pi i).$$

Theorem 1.5. *With $c_0^* = -z_3 e^{-\pi i z_3}/4$ and explicit $c_1^*(\sigma)$, $c_2^*(\sigma), \dots$ depending on $\sigma \in \mathbb{Z}$ we have*

$$\sum_{h/k \in \mathcal{C}(N)} Q_{hk\sigma}(N) = \mathrm{Re}\left[\frac{w(1,-3)^{-N}}{N^2}\left(c_0^* + \frac{c_1^*(\sigma)}{N} + \cdots + \frac{c_{m-1}^*(\sigma)}{N^{m-1}}\right)\right] + O\left(\frac{|w(1,-3)|^{-N}}{N^{m+2}}\right) \tag{1.19}$$

for an implied constant depending only on σ and m.

Theorem 1.6. *Let $\overline{N}$ denote N modulo 2. With*

$$d_0\big(\overline{N}\big) = z_0 \sqrt{2e^{-\pi i z_0}\big(e^{-\pi i z_0} + (-1)^N\big)} \tag{1.20}$$

and explicit $d_1\big(\sigma, \overline{N}\big)$, $d_2\big(\sigma, \overline{N}\big), \dots$ depending on $\sigma \in \mathbb{Z}$ and $\overline{N}$, we have

$$\sum_{h/k \in \mathcal{D}(N)} Q_{hk\sigma}(N) = \mathrm{Re}\left[\frac{w_0^{-N/2}}{N^2}\left(d_0\big(\overline{N}\big) + \frac{d_1\big(\sigma, \overline{N}\big)}{N} + \cdots + \frac{d_{m-1}\big(\sigma, \overline{N}\big)}{N^{m-1}}\right)\right] + O\left(\frac{|w_0|^{-N/2}}{N^{m+2}}\right) \tag{1.21}$$

for an implied constant depending only on σ and m.

(By $w_0^{-N/2}$ we mean $\big(\sqrt{w_0}\big)^{-N}$ where $\sqrt{w_0}$ is chosen as usual with $\mathrm{Re}(\sqrt{w_0}) > 0$.)

Theorem 1.7. *With* $e_0 = -3z_1 e^{-\pi i z_1}/2$ *and explicit* $e_1(\sigma)$, $e_2(\sigma), \dots$ *depending on* $\sigma \in \mathbb{Z}$ *we have*

$$\sum_{h/k \in \mathcal{E}(N)} Q_{hk\sigma}(N) = \operatorname{Re}\left[\frac{w(0,-2)^{-N}}{N^2}\left(e_0 + \frac{e_1(\sigma)}{N} + \cdots + \frac{e_{m-1}(\sigma)}{N^{m-1}}\right)\right] + O\left(\frac{|w(0,-2)|^{-N}}{N^{m+2}}\right) \quad (1.22)$$

for an implied constant depending only on σ *and* m*.*

The above three estimates are the final elements required for Theorem 1.3, and its proof is given near the end of Section 8. Theorems 1.5, 1.6 and 1.7 above are proved using the techniques developed in [9] for Theorem 1.2, though they each present new challenges. These techniques use the saddle-point method described in the next subsection.

In fact, Theorems 1.5, 1.6 and 1.7 are more than is needed for Theorem 1.3, but we included them for two reasons. First, they allow us to check our work numerically; see Tables 1 – 4. Secondly, their asymptotic expansions point the way to further results and a better understanding of relations in the left side of the identity (1.8). Examples of these relations, from [9, Sect. 6.2], are

$$Q_{011}(N) \sim -\sum_{h/k \in \mathcal{A}(N)} Q_{hk1}(N), \quad (1.23)$$

$$Q_{121}(N) \sim -\sum_{h/k \in \mathcal{D}(N)} Q_{hk1}(N) \quad (1.24)$$

where by (1.23) and (1.24) (and (1.25)) we mean that, at least numerically, the asymptotic expansions of both sides seem to be identical. With Theorems 1.5 and 1.7 we discover another asymptotic relation. To describe it, let $\mathcal{C}'(N)$ be all $h/k \in \mathcal{C}(N)$ with $2N/3 < k \leqslant N$, so that $\mathcal{C}'(N)$ is about two thirds of $\mathcal{C}(N)$. Then

$$3\sum_{h/k \in \mathcal{C}'(N)} Q_{hk\sigma}(N) \sim \sum_{h/k \in \mathcal{E}(N)} Q_{hk\sigma}(N). \quad (1.25)$$

See the end of Section 8 for more about (1.25).

1.4. The Saddle-point Method

The next result is a simpler version of [7, Theorem 7.1, p. 127] that was used in [9, Sect. 5.1].

Theorem 1.8 (Saddle-point method)**.** *Let* $\mathcal{P}$ *be a finite length path, made of closed line segments in* $\mathbb{C}$*, with* $p(z)$*,* $q(z)$ *holomorphic functions in a neighborhood of* $\mathcal{P}$*. Assume* p*,* q *and* $\mathcal{P}$ *are independent of a parameter* $N > 0$*. Suppose* $p'(z)$ *has a* *simple zero at $z_0 \in \mathcal{P}$ with $\mathrm{Re}(p(z) - p(z_0)) > 0$ for $z \in \mathcal{P}$ except at $z = z_0$. We also require z_0 to not be an endpoint of any line segment. Then there exist explicit numbers a_{2s} depending on p, q, z_0 and $\mathcal{P}$ so that we have*

$$\int_{\mathcal{P}} e^{-N\cdot p(z)} q(z)\, dz = 2e^{-N\cdot p(z_0)} \left(\sum_{s=0}^{S-1} \Gamma(s+1/2) \frac{a_{2s}}{N^{s+1/2}} + O\left(\frac{1}{N^{S+1/2}}\right)\right) \tag{1.26}$$

as $N \to \infty$ where S is an arbitrary positive integer.

Write the power series for p and q near z_0 as

$$p(z) = p(z_0) + p_0(z - z_0)^2 + p_1(z - z_0)^3 + \cdots , \tag{1.27}$$

$$q(z) = q_0 + q_1(z - z_0) + q_2(z - z_0)^2 + \cdots . \tag{1.28}$$

Choose $\omega \in \mathbb{C}$ giving the direction of the path $\mathcal{P}$ through z_0: near z_0, $\mathcal{P}$ looks like $z = z_0 + \omega t$ for small $t \in \mathbb{R}$ increasing. Wojdylo in [15, Theorem 1.1] found an explicit formula for the numbers a_{2s}:

$$a_{2s} = \frac{\omega}{2(\omega^2 p_0)^{1/2}} \sum_{i=0}^{2s} q_{2s-i} \sum_{j=0}^{i} p_0^{-s-j} \binom{-s-1/2}{j} \hat{B}_{i,j}(p_1, p_2, \dots) \tag{1.29}$$

where we must choose the square root $(\omega^2 p_0)^{1/2}$ in (1.29) so that $\mathrm{Re}\big((\omega^2 p_0)^{1/2}\big) > 0$ and $\hat{B}_{i,j}$ is the *partial ordinary Bell polynomial*. The first cases are

$$a_0 = \frac{\omega}{2(\omega^2 p_0)^{1/2}} q_0, \qquad a_2 = \frac{\omega}{2(\omega^2 p_0)^{1/2}} \left(\frac{q_2}{p_0} - \frac{3}{2} \frac{p_1 q_1 + p_2 q_0}{p_0^2} + \frac{15}{8} \frac{p_1^2 q_0}{p_0^3} \right), \tag{1.30}$$

agreeing with [7, p. 127].

We will be applying Theorem 1.8 to functions p of the form

$$p_d(z) := \frac{-\operatorname{Li}_2\left(e^{2\pi i z}\right) + \operatorname{Li}_2(1) + 4\pi^2 d}{2\pi i z} \tag{1.31}$$

with $p(z) := p_0(z)$ the most important. Recall that $\operatorname{Li}_2(z)$ is holomorphic on $\mathbb{C} - [1, \infty)$. Hence $p_d(z)$ is a single-valued holomorphic function away from the vertical branch cuts $(-i\infty, n]$ for $n \in \mathbb{Z}$. (We use $(-i\infty, n]$ to indicate all points in $\mathbb{C}$ with real part n and imaginary part at most 0.) The next result is shown in [9, Sect. 2.3]. The notation $w(A, B)$ for the dilogarithm zeros is defined after (1.18).

Theorem 1.9. *Fix integers m and d with $-|m|/2 < d \leqslant |m|/2$. Then there is a unique solution to $p_d'(z) = 0$ for $z \in \mathbb{C}$ with $m - 1/2 < \mathrm{Re}(z) < m + 1/2$ and $z \notin (-i\infty, m]$. Denoting this saddle-point by z^*, it is given by*

$$z^* = m + \frac{\log\big(1 - w(d, -m)\big)}{2\pi i} \tag{1.32}$$

and satisfies

$$p_d(z^*) = \log\big(w(d, -m)\big). \tag{1.33}$$

2. The Maxima and Minima of $\prod_m(h/k)$

Recall the set $Z(h,k)$ from (1.13). We will also need Clausen's integral,

$$\mathrm{Cl}_2(\theta) := -\int_0^\theta \log|2\sin(x/2)|\,dx \qquad (\theta \in \mathbb{R}) \tag{2.1}$$

$$= \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}. \tag{2.2}$$

The maximum value of $\mathrm{Cl}_2(\theta)$ is $\mathrm{Cl}_2(\pi/3) \approx 1.0149416$.

Theorem 2.1. *For all m, h, $k \in \mathbb{Z}$ with $1 \leqslant h < k$, $(h,k) = 1$ and $0 \leqslant m < k$ we have*

$$\frac{1}{k}\log\Big|\prod\nolimits_m^{-1}(h/k)\Big| = \frac{\mathrm{Cl}_2(2\pi m\gamma_0 h/k)}{2\pi|\beta_0\gamma_0|} + O\left(\frac{\log k}{\sqrt{k}}\right) \tag{2.3}$$

where (β_0,γ_0) is a pair in $Z(h,k)$ with $|\beta_0\gamma_0|$ minimal. The implied constant in (2.3) *is absolute and in fact this error is bounded by $(16.05 + \sqrt{2}/\pi\log k)/\sqrt{k}$.*

We prove Theorem 2.1 in the following subsections, assuming throughout that m, h, k satisfy its conditions. Define $D(h,k)$ to be the above minimal value $|\beta_0\gamma_0|$. For example, it is easy to see that

$$D(h,k) = 1 \quad \text{if and only if} \quad h \equiv \pm 1 \bmod k \tag{2.4}$$

and if $D(h,k) \neq 1$ then

$$D(h,k) = 2 \quad \text{if and only if} \quad h \text{ or } h^{-1} \equiv \pm 2 \bmod k \tag{2.5}$$

with k necessarily odd. Since $(1,h) \in Z(h,k)$ we have $D(h,k) \leqslant h < k$. We will see later in Lemma 2.9 that there is a unique $(\beta_0,\gamma_0) \in Z(h,k)$ with $|\beta_0\gamma_0|$ minimal if $|\beta_0\gamma_0| < \sqrt{k/2}$.

The corollary we will need, Corollary 2.11, says there exists an absolute constant τ such that

$$\frac{1}{k}\Big|\log\big|\prod\nolimits_m(h/k)\big|\Big| \leqslant \frac{\mathrm{Cl}_2(\pi/3)}{2\pi D(h,k)} + \tau\frac{\log k}{\sqrt{k}}. \tag{2.6}$$

For example, Figure 1 compares both sides of (2.6) with $k = 101$, $\tau = 0$ and

$$\Psi(h,k) := \max_{0 \leqslant m < k}\left\{\frac{1}{k}\Big|\log\big|\prod\nolimits_m(h/k)\big|\Big|\right\}. \tag{2.7}$$

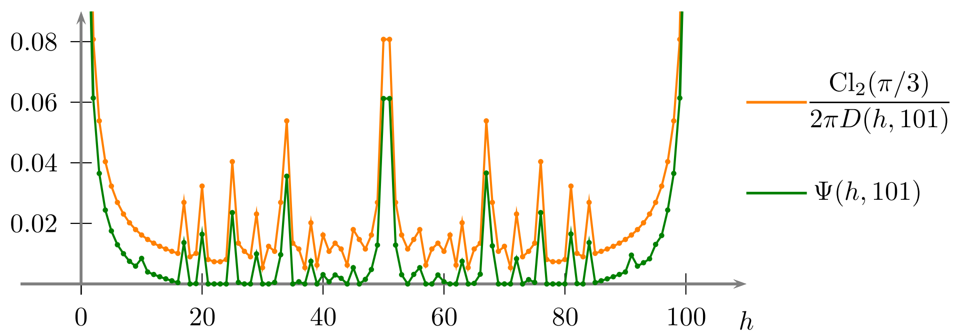

Figure 1: Bounding $\Psi(h,k)$ for $1 \leqslant h \leqslant k-1$ and $k = 101$

2.1. Relating $\prod_m^{-1}(h/k)$ to $S(m;h,k)$

By (2.1) we have $\mathrm{Cl}_2'(\theta) = -\log|2\sin(\theta/2)|$ and

$$\log\left|\prod\nolimits_m^{-1}(h/k)\right| = \sum_{j=1}^{m} \mathrm{Cl}_2'(2\pi jh/k). \tag{2.8}$$

With the sum $S(m;h,k)$ defined in (1.12), our first goal is to prove:

Proposition 2.2. *For $0 \leqslant m < k$ and an absolute implied constant*

$$\sum_{j=1}^{m} \mathrm{Cl}_2'(2\pi jh/k) = \frac{k}{2\pi} S(m;h,k) + O\left(\log^2 k\right).$$

With $x \in \mathbb{R}$, let

$$f_L(x) := \sum_{n=1}^{L} \frac{\cos(nx)}{n} \tag{2.9}$$

and define $\|x\|$ as the distance from x to the nearest integer, so that $0 \leqslant \|x\| \leqslant 1/2$.

Lemma 2.3. *For $L \geqslant 1$ and $x \in \mathbb{R}$, $x \notin \mathbb{Z}$ we have*

$$\mathrm{Cl}_2'(2\pi x) = f_L(2\pi x) + O\left(\frac{1}{L\,\|x\|}\right).$$

Proof. We first claim that

$$\left|\sum_{r=L}^{M} \frac{\cos(2\pi rx)}{r}\right| \leqslant \frac{1}{L\,\|x\|} \tag{2.10}$$

for $x \notin \mathbb{Z}$. Let $A_m(2\pi x) := \sum_{r=1}^{m} e^{2\pi i r x}$. Then this geometric series evaluates to

$$A_m(2\pi x) = -\frac{i}{2}\frac{e^{2\pi i(m+1/2)x} - e^{\pi i x}}{\sin \pi x}$$

and the inequality $|\sin \pi x| \geqslant 2\|x\|$ implies $|A_m(2\pi x)| \leqslant 1/(2\|x\|)$. By partial summation

$$\sum_{r=L}^{M} \frac{e^{2\pi i r x}}{r} = \frac{A_M}{M} - \frac{A_{L-1}}{L} + \sum_{d=L}^{M-1} \frac{A_d}{d(d+1)}.$$

Taking real parts, using the bound for A_m and evaluating the telescoping sum shows (2.10).

Now $\sum_{n=1}^{L} \sin(nx)n^{-2}$ as $L \to \infty$ converges uniformly to $\mathrm{Cl}_2(x)$. The derivative of the above partial sum is $f_L(x)$. As $L \to \infty$, (2.10) implies that $f_L(2\pi x)$ converges uniformly for x in any closed interval not containing an integer. Hence, with [13, Thm. 7.17], $\lim_{L\to\infty} f_L(2\pi x) = \mathrm{Cl}_2'(2\pi x)$ for $x \notin \mathbb{Z}$ and the lemma follows. □

Corollary 2.4. *We have*

$$\sum_{j=1}^{m} \mathrm{Cl}_2'(2\pi jh/k) = \sum_{j=1}^{m} f_k(2\pi jh/k) + O(\log k).$$

Proof. Use

$$\sum_{j=1}^{m} \frac{1}{\|jh/k\|} \leqslant \sum_{j=1}^{k-1} \frac{1}{\|jh/k\|} \leqslant 2\sum_{j=1}^{k/2} \frac{1}{\|j/k\|} = 2\sum_{j=1}^{k/2} \frac{k}{j}.$$

With $\sum_{j=1}^{k} 1/j \leqslant 1 + \log k$ we get

$$\sum_{j=1}^{m} \frac{1}{k\,\|jh/k\|} \ll \log k$$

and the corollary now follows from Lemma 2.3. (We use $\ll$ as an equivalent form of the big-O notation.) □

Lemma 2.5. *For $0 \leqslant m < k$ and $L = k^2$,*

$$\sum_{j=1}^{m} \mathrm{Cl}_2'(2\pi jh/k) = \frac{k}{2\pi} \sum_{l=-L}^{L} \sum_{n=1}^{k-1} \frac{\sin(2\pi m(nh+lk)/k)}{n(nh+lk)} + O(\log k). \tag{2.11}$$

Proof. Apply Euler-Maclaurin summation, in the form of [4, Corollary 4.3], to find

$$\sum_{j=1}^{m} f_k(2\pi jh/k) = \sum_{l=-L}^{L} \int_0^m f_k(2\pi xh/k)e^{2\pi i l x}\,dx + \frac{1}{2}f_k(2\pi mh/k) - \frac{1}{2}f_k(0) + O\left(\int_0^m \frac{|f_k'(2\pi xh/k)2\pi h/k|}{1+L\|x\|}\,dx\right) \tag{2.12}$$

where the implied constant is absolute. Clearly we see $|f_k(x)| \leqslant 1 + \log k$ and $|f_k'(x)| \leqslant k$. To bound the error term in (2.12) note that

$$\int_0^m \frac{dx}{1+L\,\|x\|} \leqslant \int_0^k \frac{dx}{1+L\,\|x\|} = 2k\int_0^{1/2} \frac{dx}{1+Lx} = \frac{2k\log(1+L/2)}{L}.$$

Hence, on choosing $L = k^2$, (2.12) implies

$$\sum_{j=1}^{m} f_k(2\pi jh/k) = \sum_{l=-L}^{L} \int_0^m f_k(2\pi xh/k) e^{2\pi ilx}\, dx + O(\log k). \tag{2.13}$$

Use $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ to evaluate the right side of (2.13) as follows.

$$\begin{aligned}\sum_{l=-L}^{L} \int_0^m f_k(2\pi xh/k) e^{2\pi ilx}\, dx &= \sum_{l=-L}^{L}\sum_{n=1}^{k} \int_0^m \frac{\cos(2\pi nxh/k)}{n} e^{2\pi ilx}\, dx \\ &= \frac{m}{k} + \frac{1}{4\pi i}\sum_{l=-L}^{L}\sum_{n=1}^{k-1}\left(\frac{e^{2\pi im(nh/k+l)}-1}{n(nh/k+l)} + \frac{e^{2\pi im(-nh/k+l)}-1}{n(-nh/k+l)}\right) \\ &= \frac{m}{k} + \frac{1}{4\pi i}\sum_{l=-L}^{L}\sum_{n=1}^{k-1}\frac{e^{2\pi im(nh/k+l)} - e^{-2\pi im(nh/k+l)}}{n(nh/k+l)}.\end{aligned}$$

Combining this with Corollary 2.4 completes the proof. □

To simplify the right of (2.11) set

$$H(d) = H(d, L; h, k) := \#\Big\{(l, n) \;:\; nh + lk = d, 1 \leqslant n \leqslant k-1, -L \leqslant l \leqslant L\Big\}.$$

Then the double sum equals

$$\sum_{d\in\mathbb{Z}} H(d)\frac{\sin(2\pi md/k)}{(dh^{-1} \bmod k)d} \tag{2.14}$$

where we exclude ds that are multiples of k, since $H(d)$ is necessarily 0 if $k|d$, and we understand here and throughout that $0 \leqslant (* \bmod k) \leqslant k-1$.

Lemma 2.6. *Recall that $L = k^2$. For all $d \in \mathbb{Z}$ we have $H(d) = H(d, L; h, k)$ equalling 0 or 1. Also*

$$H(d) = 1 \quad \textit{for} \quad 1 \leqslant |d| < k, \tag{2.15}$$

$$H(d) = 0 \quad \textit{for} \quad |d| > 2k^3. \tag{2.16}$$

Proof. Since $(h, k) = 1$ there exist n_0, l_0 such that $n_0h + l_0k = 1$. Then for all $t \in \mathbb{Z}$

$$(n_0 + tk)h + (l_0 - th)k = 1$$

and we may choose n_0, l_0 satisfying $1 \leqslant n_0 < k$ and $-h < l_0 \leqslant -1$. Similarly, for fixed h, k, d, all solutions (n, l) of $nh + lk = d$ are given by

$$n = dn_0 + tk, \quad l = dl_0 - th \quad (t \in \mathbb{Z}). \tag{2.17}$$

Hence, for $k \nmid d$, there is exactly one solution (n, l) with $1 \leqslant n \leqslant k - 1$. Then $H(d) = 1$ if the corresponding l satisfies $-L \leqslant l \leqslant L$ and $H(d) = 0$ otherwise.

In (2.17), if $1 \leqslant n \leqslant k - 1$ then $t = -\lfloor dn_0/k \rfloor$. Therefore

$$l = dl_0 - th = dl_0 + h\lfloor dn_0/k \rfloor$$

and l satisfies $-k^2 < l < k^2$ for $|d| < k$. This proves (2.15). Finally, to show (2.16), note that $|n| < k$, $|l| \leqslant L$ implies $|nh + lk| < k(h + L) < 2k^3$. □

The sum (2.14) with indices d restricted to $|d| < k$ is

$$\sum_{-k<d<k,\ d\neq 0} \frac{\sin(2\pi md/k)}{(dh^{-1} \bmod k)d}. \tag{2.18}$$

Replacing d by $dh \bmod k$ if $d > 0$, and d by $-(dh \bmod k) \equiv (-dh) \bmod k$ if $d < 0$, allows us to write (2.18) as

$$\sum_{-k<d<k,\ d\neq 0} \frac{\sin(2\pi mdh/k)}{(dh \bmod k)|d|} = S(m; h, k).$$

Proof of Proposition 2.2. With Lemmas 2.5 and 2.6 we have demonstrated that

$$\sum_{j=1}^{m} \mathrm{Cl}_2'(2\pi jh/k) = \frac{k}{2\pi} S(m; h, k) + \frac{k}{2\pi} \sum_{d\in\mathbb{Z}\ :\ k<|d|<2k^3} H(d) \frac{\sin(2\pi md/k)}{(dh^{-1} \bmod k)d} + O(\log k). \tag{2.19}$$

To estimate the sum on the right of (2.19), write $d = uk + r$ and use Lemma 2.6 to see that it is bounded by

$$\sum_{\substack{-2k^2\leqslant u\leqslant 2k^2 \\ u\neq 0,-1}} \sum_{r=1}^{k-1} \frac{1}{|uk + r|(rh^{-1} \bmod k)}. \tag{2.20}$$

For $u \geqslant 1$ the inner sum is less than

$$\sum_{r=1}^{k-1} \frac{1}{uk(rh^{-1} \bmod k)} = \frac{1}{uk} \sum_{r=1}^{k-1} \frac{1}{r} < \frac{1 + \log k}{uk}.$$

Similarly for $u \leqslant -2$ and therefore (2.20) is bounded by

$$2\frac{1 + \log k}{k} \sum_{u=1}^{2k^2} \frac{1}{u} \ll \frac{\log^2 k}{k}. \quad \square$$

2.2. Relating $S(m;h,k)$ to Clausen's Integral

With (2.8) and Proposition 2.2 we have proved that

$$\frac{1}{k}\log\Big|\prod_m^{-1}(h/k)\Big| = \frac{S(m;h,k)}{2\pi} + O\left(\frac{\log^2 k}{k}\right). \tag{2.21}$$

Remark 2.7. The implied constant in (2.21) is absolute and we may find it explicitly. In Corollary 2.4 the error is bounded by $2(1+\log(k/2))$. In (2.12) the implied constant can be $1/2+1/\pi$ which follows (see [4, Eq. (4.18)]) from

$$\left|x-\lfloor x\rfloor - 1/2 + \sum_{j=1}^{L}\frac{\sin(2\pi j x)}{\pi j}\right| \leqslant \frac{T}{1+L\,\|x\|} \qquad (T=1/2+1/\pi). \tag{2.22}$$

To prove (2.22), show that the left is bounded by $1/2$ and, with a similar proof to Lemma 2.3, also bounded by $1/(\pi L\,\|x\|)$. This yields (2.22). (It seems that $T=1/2$ should be possible.) Hence the error in Lemma 2.5 is bounded by

$$2(1+\log(k/2)) + 1 + \log k + 4\pi T(1+\log(k^2/2)).$$

For Proposition 2.2 we add $(1+\log k)(1+\log(2k^2))/\pi$. Altogether this shows the error in (2.21) is bounded by

$$(5.31 + 24.75\log k + 2/\pi \log^2 k)/k < 40.18(\log^2 k)/k \qquad (k\geqslant 2). \tag{2.23}$$

For the proof of Theorem 2.1 we therefore need to estimate $S(m;h,k)$ in (2.21). To do this, note that the largest terms in the sum (1.12) should occur when $|\beta|$ and γ are both small. We introduce a parameter R to the set $Z(h,k)$ to control the size of the elements:

$$Z_R(h,k) := \Big\{(\beta,\gamma)\in\mathbb{Z}\times\mathbb{Z}\ :\ 1\leqslant|\beta|<R,\ 1\leqslant\gamma<R,\ \beta h\equiv\gamma \bmod k\Big\}. \tag{2.24}$$

Then $Z(h,k)$ is $Z_k(h,k)$ in this notation.

Lemma 2.8. *For an absolute implied constant*

$$\sum_{(\beta,\gamma)\in Z_k(h,k)-Z_R(h,k)} \frac{\sin(2\pi m\gamma/k)}{|\beta\gamma|} = O\left(\frac{\log R}{R}\right). \tag{2.25}$$

Proof. We may partition the terms of the sum on the left of (2.25) into the three cases where $|\beta|\geqslant R$ or $\gamma\geqslant R$ or both. The first two corresponding sums are each bounded by $2(1+\log R)/R$. With the Cauchy-Schwarz inequality, the third is bounded by

$$2\left(\sum_{\beta=R}^{k-1}\frac{1}{\beta^2}\right)^{1/2}\left(\sum_{\gamma=R}^{k-1}\frac{1}{\gamma^2}\right)^{1/2} < 2\left(\sum_{d=R}^{\infty}\frac{1}{d^2}\right) < \frac{2}{R}\left(1+\frac{1}{R}\right). \qquad \square$$

Lemma 2.9. *Suppose $Z_R(h,k)$ is non-empty and $k \geqslant 2R^2$. Let (β_1, γ_1) be a pair in $Z_R(h,k)$ with $|\beta_1\gamma_1|$ minimal. Then for each $(\beta,\gamma) \in Z_R(h,k)$ there exists a positive integer λ such that $(\beta,\gamma) = (\lambda\beta_1, \lambda\gamma_1)$.*

Proof. The number β may not have an inverse modulo k so write $\beta = \beta' k'$ with $k'|k$ and $\gcd(\beta', k) = 1$. Necessarily we also have $\gamma = \gamma' k'$ with $\gcd(\gamma', k) = 1$. Similarly, there exists $k_1|k$ so that

$$\beta_1 = \beta_1' k_1, \quad \gamma_1 = \gamma_1' k_1, \quad \gcd(\beta_1', k) = \gcd(\gamma_1', k) = 1.$$

Then

$$h \equiv (\beta')^{-1}\gamma' \bmod k/k', \quad h \equiv (\beta_1')^{-1}\gamma_1' \bmod k/k_0$$

and letting $k^* = \gcd(k/k', k/k_1)$ we obtain

$$(\beta')^{-1}\gamma' \equiv (\beta_1')^{-1}\gamma_1' \bmod k^*$$

so that

$$\beta_1'\gamma' - \beta'\gamma_1' \equiv 0 \bmod k^*. \tag{2.26}$$

Now

$$|\beta_1'\gamma' - \beta'\gamma_1'| < \frac{2R^2}{k_1 k'} \leqslant \frac{k}{k_1 k'} \leqslant k^* \tag{2.27}$$

so that (2.26) and (2.27) imply

$$\beta_1'\gamma' - \beta'\gamma_1' = 0$$

which, in turn, shows that $\beta/\beta_1 = \gamma/\gamma_1$. Hence $(\beta,\gamma) = (\mu\beta_1, \mu\gamma_1)$ for $\mu := \gamma/\gamma_1 \in \mathbb{Q}_{>0}$. Write $\mu = \lambda + \delta$ with $\lambda \in \mathbb{Z}$ and $0 \leqslant \delta < 1$. If $0 < \delta < 1$ then

$$(\beta,\gamma) - \lambda(\beta_1,\gamma_1) = (\beta - \lambda\beta_1, \gamma - \lambda\gamma_1) = (\delta\beta_1, \delta\gamma_1) \in Z_k(h,k),$$

but $|\delta^2\beta_1\gamma_1| < |\beta_1\gamma_1|$ and $|\beta_1\gamma_1|$ was supposed to be minimal. We must have $\delta = 0$, as required. □

Proposition 2.10. *Let (β_0, γ_0) be a pair in $Z_k(h,k)$ with $|\beta_0\gamma_0|$ minimal, and so equalling $D(h,k)$ as defined after Theorem 2.1. Then for an absolute implied constant*

$$S(m;h,k) = \frac{\mathrm{Cl}_2(2\pi m\gamma_0/k)}{|\beta_0\gamma_0|} + O\left(\frac{\log k}{\sqrt{k}}\right). \tag{2.28}$$

Proof. By Lemma 2.8 with $R = \sqrt{k/2}$

$$S(m;h,k) = \sum_{(\beta,\gamma)\in Z_{\sqrt{k/2}}(h,k)} \frac{\sin(2\pi m\gamma/k)}{|\beta\gamma|} + O\left(\frac{\log k}{\sqrt{k}}\right). \tag{2.29}$$

Case (i) Assume first that $Z_{\sqrt{k/2}}(h,k)$ is empty. If $(\beta_0,\gamma_0)\notin Z_{\sqrt{k/2}}(h,k)$ it follows that $|\beta_0\gamma_0|\geqslant\sqrt{k/2}$ and so

$$\frac{\mathrm{Cl}_2(2\pi m\gamma_0/k)}{|\beta_0\gamma_0|}=O\left(\frac{1}{\sqrt{k}}\right). \tag{2.30}$$

Then (2.28) follows from (2.29) and (2.30).

Case (ii) Assume now that $Z_{\sqrt{k/2}}(h,k)$ is not empty. Apply Lemma 2.9 with the same $R=\sqrt{k/2}$, and $(\beta_1,\gamma_1)\in Z_{\sqrt{k/2}}(h,k)$ with $|\beta_1\gamma_1|$ minimal, to get

$$\begin{aligned}\sum_{(\beta,\gamma)\in Z_{\sqrt{k/2}}(h,k)}\frac{\sin(2\pi m\gamma/k)}{|\beta\gamma|}&=\frac{1}{|\beta_1\gamma_1|}\sum_{1\leqslant\lambda<\sqrt{k/2}/\max\{|\beta_1|,\gamma_1\}}\frac{\sin(2\pi m\lambda\gamma_1/k)}{\lambda^2}\\&=\frac{\mathrm{Cl}_2(2\pi m\gamma_1/k)}{|\beta_1\gamma_1|}+O\left(\frac{1}{|\beta_1\gamma_1|}\sum_{\lambda\geqslant\sqrt{k/2}/\max\{|\beta_1|,\gamma_1\}}\frac{1}{\lambda^2}\right)\\&=\frac{\mathrm{Cl}_2(2\pi m\gamma_1/k)}{|\beta_1\gamma_1|}+O\left(\frac{1}{\sqrt{k}}\right).\end{aligned} \tag{2.31}$$

Case (iia) If $(\beta_0,\gamma_0)\in Z_{\sqrt{k/2}}(h,k)$ then necessarily $(\beta_0,\gamma_0)=(\beta_1,\gamma_1)$ and so (2.29) and (2.31) prove the proposition in this case.

Case (iib) In the final case, $Z_{\sqrt{k/2}}(h,k)$ is not empty and doesn't contain (β_0,γ_0). Since $|\beta_1\gamma_1|\geqslant|\beta_0\gamma_0|\geqslant\sqrt{k/2}$ we find

$$\frac{\mathrm{Cl}_2(2\pi m\gamma_1/k)}{|\beta_1\gamma_1|}=O\left(\frac{1}{\sqrt{k}}\right) \tag{2.32}$$

so that (2.28) follows from (2.29), (2.30), (2.31) and (2.32).

We see that both sides of (2.28) are $O((\log k)/\sqrt{k})$ except in Case (iia), and in this case the pair $(\beta_0,\gamma_0)\in Z_{\sqrt{k/2}}(h,k)$ is unique. □

Proof of Theorem 2.1. The proof now follows directly from combining (2.21) and Proposition 2.10. Treating the error in (2.28) of Proposition 2.10 more carefully, we find it is bounded by

$$(2\sqrt{2}(5-\log 2+\mathrm{Cl}_2(\pi/3))+2\sqrt{2}\log k)/\sqrt{k}<(15.06+2\sqrt{2}\log k)/\sqrt{k}.$$

Combining this with the estimate (2.23) for the error in (2.21) shows that the error term in (2.3) of Theorem 2.1 is bounded by $(16.05+\sqrt{2}/\pi\log k)/\sqrt{k}$. □

Corollary 2.11. *There exists an absolute constant* τ *such that for all integers* m *with* $0 \leqslant m \leqslant k-1$

$$\frac{1}{k}\Big|\log\big|\prod\nolimits_m(h/k)\big|\Big| \leqslant \frac{\mathrm{Cl}_2(\pi/3)}{2\pi D(h,k)} + \tau\frac{\log k}{\sqrt{k}}.$$

Proof. We may take τ to be the absolute implied constant of Theorem 2.1 and note that $|\,\mathrm{Cl}_2(\theta)| \leqslant \mathrm{Cl}_2(\pi/3)$ for all $\theta \in \mathbb{R}$. Hence we may take any $\tau > \sqrt{2}/\pi$ for k large enough. □

3. Bounds for Most $Q_{hk\sigma}(N)$

In this section we continue to assume that h and k are integers with $1 \leqslant h < k$ and $(h,k) = 1$.

3.1. Initial Estimates

The next result, mentioned in the introduction, is proved in this subsection.

Proposition 1.4. For $2 \leqslant k \leqslant N$, $\sigma \in \mathbb{R}$ and $s := \lfloor N/k \rfloor$

$$|Q_{hk\sigma}(N)| \leqslant \frac{3}{k^3}\exp\left(N\frac{2+\log(1+3k/4)}{k} + \frac{|\sigma|}{N}\right)\Big|\prod\nolimits_{N-sk}^{-1}(h/k)\Big|. \tag{3.1}$$

Proof. From definition (1.6),

$$Q_{hk\sigma}(N) = \int_{\mathcal{L}} \frac{e^{2\pi i\sigma z}}{(1-e^{2\pi iz})(1-e^{2\pi i2z})\cdots(1-e^{2\pi iNz})}\,dz \tag{3.2}$$

where z traces a loop $\mathcal{L}$ of radius $1/(2\pi Nk\lambda)$ around h/k, i.e.,

$$z = h/k + w, \qquad |w| = \frac{1}{2\pi Nk\lambda}$$

and λ is large enough that only the pole of the integrand at h/k is inside $\mathcal{L}$. This is ensured when $\lambda > 1/2\pi$, since if a/b is any other pole $(1 \leqslant b \leqslant N)$ we have

$$\left|\frac{a}{b} - \frac{h}{k}\right| = \left|\frac{ak-bh}{bk}\right| \geqslant \frac{1}{bk} \geqslant \frac{1}{Nk} > |w|.$$

Therefore, letting $e^{2\pi i\sigma z}I_N(z)$ denote the integrand in (3.2),

$$|Q_{hk\sigma}(N)| \leqslant \int_{\mathcal{L}} \left|e^{2\pi i\sigma z}I_N(z)\right|\,dz \leqslant 2\pi\left(\frac{1}{2\pi Nk\lambda}\right)\sup\big\{|e^{2\pi i\sigma z}I_N(z)| : z \in \mathcal{L}\big\}. \tag{3.3}$$

It is easy to see that if $\lambda \geqslant 1/k$ then

$$|e^{2\pi i\sigma z}| \leqslant e^{|\sigma|/N} \qquad (z \in \mathcal{L},\ \sigma \in \mathbb{R}). \tag{3.4}$$

Now write $I_N(z) = I_N^*(z) \cdot I_N^{**}(z)$ for

$$I_N^*(z) := \prod_{\substack{1 \leqslant j \leqslant N \\ k|j}} \frac{1}{(1 - e^{2\pi ijz})}, \qquad I_N^{**}(z) := \prod_{\substack{1 \leqslant j \leqslant N \\ k\nmid j}} \frac{1}{(1 - e^{2\pi ijz})}.$$

We use the following simple bounds (better ones are proved in Lemma 3.3). For all $z \in \mathbb{C}$ with $|z| \leqslant 1$

$$|1 - e^z| \leqslant 2|z|, \tag{3.5}$$

$$|1 - e^z|^{-1} \leqslant 2/|z|, \tag{3.6}$$

$$|\log(1 - z/2)| \leqslant 3|z|/4. \tag{3.7}$$

Lemma 3.1. *For $z \in \mathcal{L}$ and $\lambda \geqslant 1/k$ we have*

$$|I_N^*(z)| \leqslant \frac{e}{\sqrt{2\pi}} \left(\frac{k}{N}\right)^{1/2} (2ek\lambda)^s. \tag{3.8}$$

Proof. Clearly

$$I_N^*(z) = \prod_{\substack{1 \leqslant j \leqslant N \\ k|j}} \frac{1}{(1 - e^{2\pi ij(h/k+w)})} = \prod_{1 \leqslant m \leqslant s} \frac{1}{(1 - e^{2\pi ikmw})}.$$

Also

$$|2\pi ikmw| = \frac{2\pi km}{2\pi Nk\lambda} \leqslant \frac{s}{N\lambda} \leqslant \frac{1}{k\lambda}, \tag{3.9}$$

so assuming $\lambda \geqslant 1/k$, we can apply (3.6) to get

$$|I_N^*(z)| \leqslant \prod_{1 \leqslant m \leqslant s} \frac{2}{2\pi km|w|} = \prod_{1 \leqslant m \leqslant s} \frac{2N\lambda}{m} = \frac{(2N\lambda)^s}{s!}.$$

It follows from Stirling's formula that $1/a! < \frac{1}{\sqrt{2\pi a}} \left(\frac{e}{a}\right)^a$ for $a \in \mathbb{Z}_{\geqslant 1}$. Hence the lemma is obtained with

$$\frac{1}{s!} = \frac{s+1}{(s+1)!} < \frac{s+1}{\sqrt{2\pi(s+1)}} \left(\frac{e}{s+1}\right)^{s+1}$$
$$= \frac{e}{\sqrt{2\pi(s+1)}} \left(\frac{e}{s+1}\right)^s < \frac{e}{\sqrt{2\pi N/k}} \left(\frac{ek}{N}\right)^s.$$

□

Lemma 3.2. *For $z \in \mathcal{L}$ and $\lambda \geqslant 1$ we have*

$$|I_N^{**}(z)| \leqslant \exp\left(\frac{N}{2k\lambda} + \frac{3N}{8\lambda}\right) \frac{1}{k^s} \left| \prod_{N-sk}^{-1}(h/k) \right|.$$

Proof. Write

$$\begin{aligned} I_N^{**}(z) &= \prod_{\substack{1 \leqslant j \leqslant N \\ k \nmid j}} \frac{1}{(1 - e^{2\pi i j(h/k + w)})} \\ &= \prod_{\substack{1 \leqslant j \leqslant N \\ k \nmid j}} e^{-2\pi i j w} \prod_{\substack{1 \leqslant j \leqslant N \\ k \nmid j}} \frac{1}{(1 - e^{2\pi i j h/k} - 1 + e^{-2\pi i j w})} \\ &= e^{-\pi i w(N(N+1) - ks(s+1))} \prod_{\substack{1 \leqslant j \leqslant N \\ k \nmid j}} \frac{1}{(1 - e^{2\pi i j h/k})} \prod_{\substack{1 \leqslant j \leqslant N \\ k \nmid j}} \frac{1}{(1 - \eta_{h/k}(j, w))} \end{aligned} \tag{3.10}$$

for

$$\eta_{h/k}(j, w) := \frac{1 - e^{-2\pi i j w}}{1 - e^{2\pi i j h/k}}.$$

To estimate the parts of (3.10), we start with

$$N(N+1) - ks(s+1) \leqslant N^2, \qquad (k \leqslant N) \tag{3.11}$$

to see that

$$\left| e^{-\pi i w(N(N+1) - ks(s+1))} \right| \leqslant \exp\left(\frac{N}{2k\lambda}\right). \tag{3.12}$$

With $(1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{k-1}) = k$ for ζ a primitive kth root of unity, (by [8, Lemma 4.4] for example), the middle product satisfies

$$\prod_{\substack{1 \leqslant j \leqslant N \\ k \nmid j}} \frac{1}{|(1 - e^{2\pi i j h/k})|} = \frac{1}{k^s} \left| \prod_{N-sk}^{-1}(h/k) \right|. \tag{3.13}$$

Next we estimate the right-hand product of (3.10). By (3.5)

$$\left|1 - e^{-2\pi i j w}\right| \leqslant 2 \cdot 2\pi j |w| = \frac{2j}{Nk\lambda} \tag{3.14}$$

provided $\lambda \geqslant 1/k$. We have

$$\frac{1}{|1 - e^{2\pi i \theta}|} = \frac{1}{2|\sin(\pi\theta)|} \leqslant \frac{1}{4|\theta|} \qquad (-1/2 \leqslant \theta \leqslant 1/2)$$

and it follows that

$$\frac{1}{\left|1 - e^{2\pi i j h/k}\right|} \leqslant \frac{1}{\left|1 - e^{2\pi i/k}\right|} \leqslant \frac{k}{4} \qquad (k \geqslant 2). \tag{3.15}$$

Consequently, (3.14), (3.15) show

$$|\eta_{h/k}(j,w)| \leqslant \frac{j}{2N\lambda}. \tag{3.16}$$

If $\lambda \geqslant 1$ then $|\eta_{h/k}(j,w)| \leqslant 1/2$ for all $j \leqslant N$ and we may apply (3.7):

$$\begin{aligned}
\prod_{\substack{1\leqslant j\leqslant N\\ k\nmid j}} \frac{1}{\left|1-\eta_{h/k}(j,w)\right|} &= \exp\left(-\sum_{1\leqslant j\leqslant N,\ k\nmid j} \log\left|1-\eta_{h/k}(j,w)\right|\right)\\
&\leqslant \exp\left(\sum_{1\leqslant j\leqslant N,\ k\nmid j} \left|\log(1-\eta_{h/k}(j,w))\right|\right)\\
&\leqslant \exp\left(\frac{3}{2}\sum_{1\leqslant j\leqslant N,\ k\nmid j} \left|\eta_{h/k}(j,w)\right|\right) \leqslant \exp\left(\frac{3N}{8\lambda}\right) \qquad (3.17)
\end{aligned}$$

where we used (3.11) in the last inequality. Combining the estimates (3.12), (3.13) and (3.17) for (3.10) finishes the proof. □

Inserting the bounds from (3.4) and Lemmas 3.1, 3.2 into (3.3), we obtain

$$\begin{aligned}
|Q_{hk\sigma}(N)| \leqslant{}& \frac{e}{\sqrt{2\pi}N^{3/2}k^{1/2}\lambda}\\
&\times \exp\left(N\left[\frac{1}{2k\lambda}+\frac{3}{8\lambda}+\frac{1+\log 2\lambda}{k}\right]+\frac{|\sigma|}{N}\right)\left|\textstyle\prod_{N-sk}^{-1}(h/k)\right|. \qquad (3.18)
\end{aligned}$$

For fixed k, the expression

$$\frac{1}{2k\lambda}+\frac{3}{8\lambda}+\frac{1+\log 2\lambda}{k}$$

has its minimum at $\lambda = 1/2 + 3k/8$. We may set λ to this value in (3.18) since all the conditions $\lambda \geqslant 1/(2\pi)$, $1/k$, 1 are satisfied when $k \geqslant 2$. This completes the proof of Proposition 1.4. □

An example of Proposition 1.4 is given in Figure 2 for $h = \sigma = 1$ and $N = 50$ where we denote the right side of (3.1) as $Q^*_{hk\sigma}(N)$. The numbers $Q_{hk\sigma}(N)$ are calculated using the methods of [8, Sect. 5] as follows. For N, $k \geqslant 1$, $m \geqslant 0$ and $0 \leqslant r \leqslant k-1$ define the rational numbers $E_k(N,m;r)$ recursively with $E_k(0,m;r)$ set as 1 if $m = r = 0$ and 0 otherwise. Also, with $N \geqslant 1$

$$E_k(N,m;r) := \sum_{a=0}^{m} \frac{N^a k^{a-1}}{a!} \sum_{j=0}^{k-1} E_k\big(N-1, m-a; (r-Nj) \bmod k\big) \cdot B_a(j/k)$$

for $B_a(x)$ the Bernoulli polynomial. Then

$$Q_{hk\sigma}(N) = \frac{(-1)^N}{N!} \sum_{r=0}^{k-1} e^{2\pi i(r+\sigma)h/k} \sum_{j=0}^{N-1} \frac{\sigma^j}{j!} E_k(N, N-1-j; r). \tag{3.19}$$

In particular, we see from (3.19) that $e^{-2\pi i\sigma h/k} Q_{hk\sigma}(N)$ is a polynomial in σ of degree $N-1$.

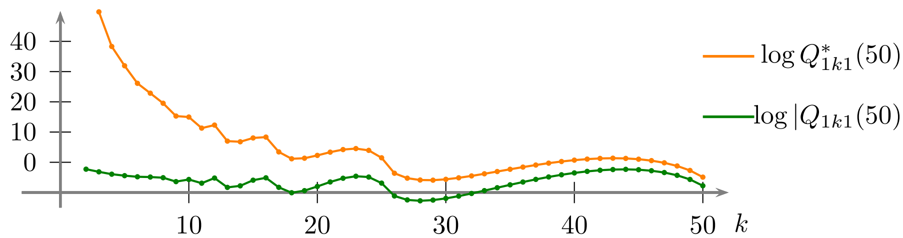

Figure 2: Bounding $Q_{hk\sigma}(50)$ for $h = \sigma = 1$ and $2 \leqslant k \leqslant 50$

3.2. Improved Estimates

By tightening up the bounds (3.5), (3.6), (3.7) and restricting the range of k we can improve Proposition 1.4 a little as follows.

Lemma 3.3. *For $z \in \mathbb{C}$ and $|z| \leqslant Y$ we have*

$$\left|\frac{1-e^z}{z}\right| \leqslant \alpha(Y) := \frac{e^Y - 1}{Y} \tag{3.20}$$

$$\left|\frac{z}{1-e^z}\right| \leqslant \beta(Y) := 2 + \frac{Y}{2}\left(1 - \cot\left(\frac{Y}{2}\right)\right) \qquad (Y < 2\pi) \tag{3.21}$$

$$\left|\frac{\log(1-z)}{z}\right| \leqslant \gamma(Y) := \frac{1}{Y}\log\left(\frac{1}{1-Y}\right) \qquad (Y < 1). \tag{3.22}$$

Proof. For $|z| \leqslant Y < 2\pi$ we have

$$\left|\frac{z}{1-e^z}\right| = \left|\sum_{n=0}^{\infty} B_n \frac{z^n}{n!}\right| \leqslant \sum_{n=0}^{\infty} |B_n| \frac{Y^n}{n!} = 1 + \frac{Y}{2} + \left(1 - \frac{Y}{2}\cot\left(\frac{Y}{2}\right)\right),$$

using [12, Eq. (11.1)]. The other two inequalities have similar proofs. Note that for $Y = 0$ we have $\alpha(0) = \beta(0) = \gamma(0) = 1$ in the limit, with $\alpha(Y)$, $\beta(Y)$ and $\gamma(Y)$ increasing for $Y \geqslant 0$. □

Start with a parameter $K \geqslant 2$. We assume

$$k \geqslant K, \qquad \lambda \geqslant 1/2 + K/8. \tag{3.23}$$

The quantity $1/(k\lambda)$ in (3.9) then satisfies

$$\frac{1}{k\lambda} \leqslant \frac{1}{K(1/2+K/8)} < 2\pi.$$

With (3.21) we may therefore replace the factor 2 in (3.8) by

$$\xi_1 = \xi_1(K) := \beta\left(\frac{1}{K(1/2+K/8)}\right). \tag{3.24}$$

Similarly, the factor 2 in (3.14) may be replaced by

$$\xi_2 = \xi_2(K) := \alpha\left(\frac{1}{K(1/2+K/8)}\right). \tag{3.25}$$

This improves the bound (3.16) to

$$|\eta_{h/k}(j,w)| \leqslant \frac{\xi_2 j}{4N\lambda}$$

so that for all $j \leqslant N$ we have $|\eta_{h/k}(j,w)| \leqslant \xi_2/(4\lambda) < 1$. The factor 3/2 in (3.17) can now be replaced by

$$\xi_3 = \xi_3(K) := \gamma\left(\frac{\xi_2}{4(1/2+K/8)}\right) \tag{3.26}$$

and we obtain

$$\prod_{\substack{1\leqslant j\leqslant N\\ k\nmid j}} \frac{1}{\left|1-\eta_{h/k}(j,w)\right|} \leqslant \exp\left(\frac{\xi_2\xi_3 N}{8\lambda}\right).$$

Hence

$$|Q_{hk\sigma}(N)| \leqslant \frac{e}{\sqrt{2\pi}N^{3/2}k^{1/2}\lambda} \times \exp\left(N\left[\frac{1}{2k\lambda}+\frac{\xi_2\xi_3}{8\lambda}+\frac{1+\log\xi_1\lambda}{k}\right]+\frac{|\sigma|}{N}\right)\left|\prod\nolimits_{N-sk}^{-1}(h/k)\right| \tag{3.27}$$

and setting $\lambda = 1/2 + \xi_2\xi_3 k/8$ minimizes (3.27). Note that $\xi_2\xi_3 \geqslant 1$ so that our initial inequality (3.23) for λ is true. We have proved

Proposition 3.4. *For $2 \leqslant K \leqslant k \leqslant N$ and $s := \lfloor N/k \rfloor$ we have*

$$|Q_{hk\sigma}(N)| \leqslant \frac{9}{k^3}\exp\left(N\frac{2+\log(\xi_1/2+\xi_1\xi_2\xi_3 k/8)}{k}+\frac{|\sigma|}{N}\right)\left|\prod\nolimits_{N-sk}^{-1}(h/k)\right| \tag{3.28}$$

for ξ_1, ξ_2, ξ_3 defined in (3.24), (3.25), (3.26) *and depending on K.*

Some examples of triples $(K,\ \xi_1,\ \xi_1\xi_2\xi_3)$ are

$$K = 2: \quad \xi_1 \approx 1.37065, \quad \xi_1\xi_2\xi_3 \approx 2.64070 \tag{3.29}$$

$$K = 61: \quad \xi_1 \approx 1.00101, \quad \xi_1\xi_2\xi_3 \approx 1.01778 \tag{3.30}$$

$$K = 82: \quad \xi_1 \approx 1.00057, \quad \xi_1\xi_2\xi_3 \approx 1.01297 \tag{3.31}$$

$$K = 101: \quad \xi_1 \approx 1.00038, \quad \xi_1\xi_2\xi_3 \approx 1.01041. \tag{3.32}$$

3.3. Final Bounds

Define $\mathcal{B}(K,N)$ to be the set

$$\Big\{h/k \ : \ K \leqslant k \leqslant N,\, 0 \leqslant h < k,\ (h,k)=1\Big\} \tag{3.33a}$$

but with the restrictions

$$h \not\equiv \pm 1 \bmod k \qquad \text{if} \qquad N/3 < k \leqslant N/2, \tag{3.33b}$$

$$h \not\equiv \pm 1, \pm 2, (k\pm 1)/2 \bmod k \qquad \text{if} \qquad N/2 < k \leqslant N. \tag{3.33c}$$

Theorem 3.5. *There exists* $W < U := -\log|w_0| \approx 0.068076$ *so that*

$$\sum_{h/k \in \mathcal{B}(101,N)} Q_{hk\sigma}(N) = O(e^{WN}).$$

We may take any $W > \mathrm{Cl}_2(\pi/3)/(6\pi) \approx 0.0538$ *and the implied constant depends only on* σ *and* W.

Proof. Recall from Corollary 2.11 that there exists an absolute constant τ such that for all $m,\, h,\, k \in \mathbb{Z}$ with $1 \leqslant h < k$, $(h,k)=1$ and $0 \leqslant m < k$ we have

$$\log \Big| \textstyle\prod_m^{-1}(h/k)\Big| \quad \leqslant \quad \frac{\mathrm{Cl}_2(\pi/3)}{2\pi D(h,k)} \cdot k + \tau\sqrt{k}\log k. \tag{3.34}$$

It follows from Proposition 3.4 and (3.34) that

$$Q_{hk\sigma}(N) \ll \frac{1}{k^3}\exp\left(N\frac{2+\log\left(\xi_1/2+\xi_1\xi_2\xi_3 k/8\right)}{k} + \frac{\mathrm{Cl}_2(\pi/3)}{2\pi D(h,k)}\cdot k + \tau\sqrt{N}\log N\right)$$

where $k \geqslant K = 101$ and $\xi_1,\ \xi_1\xi_2\xi_3$ are as in (3.32). Given any $\epsilon > 0$ we have $\tau\sqrt{N}\log N \leqslant \epsilon N$ for N large enough. For k in a range $0 < a \leqslant k \leqslant b$ where we know $D(h,k) \geqslant D^*$ for some D^*, the expression

$$N\frac{2+\log\left(\xi_1/2+\xi_1\xi_2\xi_3 k/8\right)}{k} + \frac{\mathrm{Cl}_2(\pi/3)}{2\pi D^*}\cdot k \tag{3.35}$$

has possible maxima only at the end points $k=a$ or $k=b$. For $h/k \in \mathcal{B}(101,N)$ with $101 \leqslant k \leqslant N/3$ we know $D(h,k) \geqslant 1$ and see the end points are bounded by

$$N\frac{2+\log\left(\xi_1/2+\xi_1\xi_2\xi_3 101/8\right)}{101} + \frac{\mathrm{Cl}_2(\pi/3)}{2\pi\cdot 1}\cdot 101 < 0.0454N + 16.315, \tag{3.36}$$

$$N\frac{2+\log\left(\xi_1/2+\xi_1\xi_2\xi_3 (N/3)/8\right)}{N/3} + \frac{\mathrm{Cl}_2(\pi/3)}{2\pi\cdot 1}\cdot\frac{N}{3} < 6 + \epsilon N + \frac{\mathrm{Cl}_2(\pi/3)}{6\pi}N.$$

Therefore

$$Q_{hk\sigma}(N) \ll \frac{1}{k^3}\exp\left(N\left[\frac{\mathrm{Cl}_2(\pi/3)}{6\pi}+2\epsilon\right]\right) \qquad (h/k \in \mathcal{B}(101,N),\ k \leqslant N/3).$$

Similarly, for $h/k \in \mathcal{B}(101, N)$ with $N/3 < k \leqslant N/2$ we have $D(h,k) \geqslant 2$ by (2.4). Hence (3.35) is bounded by the maximum of

$$6 + \epsilon N + \frac{\mathrm{Cl}_2(\pi/3)}{2\pi \cdot 2} \cdot \frac{N}{3}, \qquad 4 + \epsilon N + \frac{\mathrm{Cl}_2(\pi/3)}{2\pi \cdot 2} \cdot \frac{N}{2}.$$

For $h/k \in \mathcal{B}(101, N)$ with $N/2 < k \leqslant N$ we have $D(h,k) \geqslant 3$ by (2.5). Hence (3.35) is bounded by the maximum of

$$4 + \epsilon N + \frac{\mathrm{Cl}_2(\pi/3)}{2\pi \cdot 3} \cdot \frac{N}{2}, \qquad 2 + \epsilon N + \frac{\mathrm{Cl}_2(\pi/3)}{2\pi \cdot 3} \cdot N.$$

It follows that for any $W > \mathrm{Cl}_2(\pi/3)/(6\pi)$, we have

$$Q_{hk\sigma}(N) \ll e^{WN}/k^3 \qquad (h/k \in \mathcal{B}(101, N)).$$

Finally,

$$\begin{aligned} \sum_{h/k \in \mathcal{B}(101,N)} Q_{hk\sigma}(N) &\ll \sum_{h/k \in \mathcal{B}(101,N)} e^{WN}/k^3 \\ &\ll e^{WN} \sum_{k=1}^{N} \sum_{h=1}^{k} 1/k^3 = e^{WN} \sum_{k=1}^{N} 1/k^2 \ll e^{WN}. \end{aligned}$$ □

Remark 3.6. Theorem 3.5 is still true if we enlarge $\mathcal{B}(101, N)$ to $\mathcal{B}(82, N)$, i.e., allowing all $k \geqslant 82$. This is because we obtain $0.0535N + \dots$ on the right side of (3.36) when we replace 101 by $K = 82$ on the left (and use the corresponding ξ_is as in (3.31)). Furthermore, with $K = 61$ we find

$$\sum_{h/k \in \mathcal{B}(61,N)} Q_{hk\sigma}(N) = O(e^{WN}),$$

needing $W \approx 0.067403$, very close to U (see (3.30)). We expect that K can be pushed all the way back to 2 and that with improved techniques it should be possible to prove that for some $W < U$

$$\sum_{h/k \in \mathcal{B}(2,N)} Q_{hk\sigma}(N) = O(e^{WN}).$$

This would eliminate the $\sum_{0<h/k \in_{100}}$ term in (1.4) of Theorem 1.1.

What remains from ${}_N - \big({}_{100} \cup \mathcal{A}(N) \cup \mathcal{B}(101, N)\big)$ are the subsets $\mathcal{C}(N)$, $\mathcal{D}(N)$ and $\mathcal{E}(N)$ as defined in (1.14), (1.15) and (1.16). In the following sections we find the asymptotics for each of the corresponding $Q_{hk\sigma}(N)$ sums.

4. Further Required Results

We gather here some more results from [9] we will require for developing the asymptotic expansions in the next sections. Throughout we write $z = x + iy \in \mathbb{C}$.

4.1. Some Dilogarithm Results

In [9, Sect. 2.3] we saw the identity

$$\operatorname{Li}_2\left(e^{-2\pi i z}\right) = -\operatorname{Li}_2\left(e^{2\pi i z}\right) + 2\pi^2\left(z^2 - (2m+1)z + m^2 + m + 1/6\right) \tag{4.1}$$

for $m < \operatorname{Re}(z) < m+1$ where $m \in \mathbb{Z}$. Also

$$\operatorname{Cl}_2(2\pi z) = -i\operatorname{Li}_2\left(e^{2\pi i z}\right) + i\pi^2\left(z^2 - (2m+1)z + m^2 + m + 1/6\right) \tag{4.2}$$

for $m \leqslant z \leqslant m+1$.

Lemma 4.1. *Consider* $\operatorname{Im}(\operatorname{Li}_2(e^{2\pi i z}))$ *as a function of* $y \in \mathbb{R}$. *It is positive and decreasing for fixed* $x \in (0, 1/2)$ *and negative and increasing for fixed* $x \in (1/2, 1)$.

Lemma 4.2. *Consider* $\operatorname{Re}(\operatorname{Li}_2(e^{2\pi i z}))$ *as a function of* $y \geqslant 0$. *It is positive and decreasing for fixed* x *with* $|x| \leqslant 1/6$. *It is negative and increasing for fixed* x *with* $1/4 \leqslant |x| \leqslant 3/4$.

Lemma 4.3. *For* $y \geqslant 0$ *we have* $|\operatorname{Li}_2(e^{2\pi i z})| \leqslant \operatorname{Li}_2(1)$.

4.2. Approximating Products of Sines

In the following, let h and k be relatively prime integers with $1 \leqslant h < k$. From [9, Sect. 2.1] we have

Proposition 4.4. *For* $N/2 < k \leqslant N$

$$Q_{hk\sigma}(N) = \frac{(-1)^{k+1}}{k^2} \exp\left(\frac{-\pi i h(N^2 + N - 4\sigma)}{2k}\right) \times \exp\left(\frac{\pi i}{2}(2Nh + N + h + k - hk)\right) \prod_{N-k}^{-1}(h/k).$$

So estimating $Q_{hk\sigma}(N)$ requires these further results on sine products from [9, Sect. 3]:

Proposition 4.5. *For* $m,\ L \in \mathbb{Z}_{\geqslant 1}$ *and* $-1/m < \theta < 1/m$ *with* $\theta \neq 0$ *we have*

$$\prod_m(\theta) = \left(\frac{\theta}{|\theta|}\right)^m \left(\frac{2\sin(\pi m\theta)}{\theta}\right)^{1/2} \exp\left(-\frac{\operatorname{Cl}_2(2\pi m\theta)}{2\pi\theta}\right) \times \exp\left(\sum_{\ell=1}^{L-1} \frac{B_{2\ell}}{(2\ell)!}(\pi\theta)^{2\ell-1}\cot^{(2\ell-2)}(\pi m\theta)\right) \exp\bigl(T_L(m,\theta)\bigr) \tag{4.3}$$

with $\rho(z) := \log\bigl((\sin z)/z\bigr)$ *and*

$$T_L(m,\theta) := (\pi\theta)^{2L} \int_0^m \frac{B_{2L} - B_{2L}(x - \lfloor x \rfloor)}{(2L)!}\rho^{(2L)}(\pi x\theta)\,dx + \int_0^\infty \frac{B_{2L} - B_{2L}(x - \lfloor x \rfloor)}{2L(x+m)^{2L}}\,dx.$$

Lemma 4.6. *For* $1 \leqslant m < k/h$ *we have*

$$|T_1(m, h/k)| \leqslant \pi^2 h/18 + 1/12. \tag{4.4}$$

Proposition 4.7. *Let* $W > 0$. *For* δ *satisfying* $0 < \delta \leqslant 1/e$ *and* $\delta \log(1/\delta) \leqslant W$ *we have*

$$\prod_m^{-1}(h/k) \leqslant c(h) \exp\left(\frac{kW}{h}\right) \quad \text{for} \quad 0 \leqslant \frac{mh}{k} \leqslant \delta, \quad \frac{1}{2} - \delta \leqslant \frac{mh}{k} < 1$$

and

$$c(h) := h^{1/2} \exp(\pi^2 h/18 + 1/6)/2.$$

Proposition 4.8. *Suppose* Δ *and* W *satisfy* $0.0048 \leqslant \Delta \leqslant 0.0079$ *and* $\Delta \log 1/\Delta \leqslant W$. *For the integers* h, k, s *and* m *we require*

$$0 < h < k \leqslant s, \quad R_\Delta \leqslant s/h, \quad \Delta s/h \leqslant m \leqslant k/(2h).$$

Then for $L := \lfloor \pi e \Delta \cdot s/h \rfloor$ *we have*

$$\left| \prod_m^{-1}(h/k) T_L(m, h/k) \right| \leqslant (\pi^3/2) c(h) \cdot e^{sW/h}, \tag{4.5}$$

$$|T_L(m, h/k)| \leqslant \pi^3/2. \tag{4.6}$$

See [9, Sect. 3.4] for the definition of R_Δ. We will only use it in the case when $\Delta = 0.006$ and then $R_\Delta \approx 130.7$.

Corollary 4.9. *Let* W, Δ, s, h, k, m *and* L *be as in Proposition 4.8. Suppose also that* $0 < u/v \leqslant h/k$. *Then*

$$\left| \prod_m^{-1}(h/k) T_L(m, u/v) \right| \leqslant (\pi^3/2) c(h) \cdot e^{sW/h}, \tag{4.7}$$

$$|T_L(m, u/v)| \leqslant \pi^3/2. \tag{4.8}$$

The main consequence of Propositions 4.5 and 4.8 is:

Proposition 4.10. *For* W, Δ, s, h, k, m *and* L *as in Proposition 4.8 we have*

$$\begin{aligned} \prod_m^{-1}(h/k) = \left(\frac{h}{2k \sin(\pi m h/k)}\right)^{1/2} \exp\left(\frac{k}{2\pi h} \operatorname{Cl}_2\big(2\pi m h/k\big)\right) \\ \times \exp\left(-\sum_{\ell=1}^{L-1} \frac{B_{2\ell}}{(2\ell)!} \left(\frac{\pi h}{k}\right)^{2\ell-1} \cot^{(2\ell-2)}\left(\frac{\pi m h}{k}\right)\right) + O\left(e^{sW/h}\right) \end{aligned} \tag{4.9}$$

for an implied constant depending only on h.

5. The Sum $\mathcal{C}_1(N,\sigma)$

Let $\sigma \in \mathbb{Z}$. In this section and the next we prove Theorem 1.5, giving the asymptotic expansion as $N \to \infty$ of

$$\mathcal{C}_1(N,\sigma) := \sum_{h/k \in \mathcal{C}(N)} Q_{hk\sigma}(N) = 2\mathrm{Re} \sum_{\frac{N}{2} < k \leqslant N, \ k \text{ odd}} Q_{2k\sigma}(N). \tag{5.1}$$

The equality in (5.1) is straightforward to justify; see [9, Sect. 2.2]. For k odd, setting $h = 2$ in Proposition 4.4 yields

$$Q_{2k\sigma}(N) = \frac{1}{k^2} \exp\left(-\pi i \frac{N^2+N-4\sigma}{k}\right) \exp\left(\frac{\pi i}{2}(5N+2-k)\right) \textstyle\prod_{N-k}^{-1}(2/k). \tag{5.2}$$

The sum (5.1) corresponds to $2N/k \in [2,4)$ and we break it into two parts: $\mathcal{C}_2(N,\sigma)$ for $2N/k \in [2,3)$ and $\mathcal{C}_2^*(N,\sigma)$ with $2N/k \in [3,4)$.

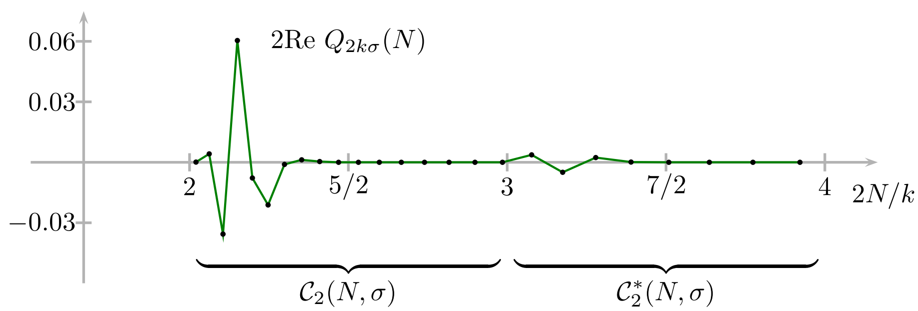

Figure 3: 2Re $Q_{2k\sigma}(N)$ for $\sigma = 1$ and $N = 100$

5.1. First Results for $\mathcal{C}_2(N,\sigma)$

With (5.2) we have

$$\mathcal{C}_2(N,\sigma) = \mathrm{Re} \sum_{k \text{ odd}, \ 2N/k \in [2,3)} \frac{-2}{k^2} \exp\left(N\left[\frac{\pi i}{2}\left(-\frac{2N}{k} + 5 - 2\frac{k}{2N}\right)\right]\right)$$
$$\times \exp\left(\frac{-\pi i}{2}\frac{2N}{k}\right) \exp\left(\frac{1}{N}\left[2\pi i \sigma \frac{2N}{k}\right]\right) \textstyle\prod_{N-k}^{-1}(2/k). \tag{5.3}$$

Define

$$g_\ell(z) := -\frac{B_{2\ell}}{(2\ell)!}(\pi z)^{2\ell-1} \cot^{(2\ell-2)}(\pi z) \tag{5.4}$$

and set $\hat{z} = \hat{z}(N,k) := 2N/k$. The analog of the sine product approximation, [9, Thm. 4.1], we need here is:

Theorem 5.1. *Fix $W > 0$. Let Δ be in the range $0.0048 \leqslant \Delta \leqslant 0.0079$ and set $\alpha = \Delta\pi e$. Suppose δ and δ' satisfy*

$$\frac{\Delta}{1-\Delta} < \delta \leqslant \frac{1}{e},\ 0 < \delta' \leqslant \frac{1}{e} \qquad \text{and} \qquad \delta \log 1/\delta,\quad \delta' \log 1/\delta' \leqslant W.$$

Then for all $N \geqslant 2 \cdot R_\Delta$ we have

$$\textstyle\prod_{N-k}^{-1}(2/k) = O\left(e^{WN/2}\right) \qquad \text{for} \qquad \hat{z} \in [2,\ 2+\delta] \cup [5/2-\delta',\ 3) \tag{5.5}$$

and

$$\textstyle\prod_{N-k}^{-1}(2/k) = \frac{1}{N^{1/2}} \exp\left(N \frac{\mathrm{Cl}_2(2\pi\hat{z})}{2\pi\hat{z}}\right) \left(\frac{\hat{z}}{2\sin(\pi\hat{z})}\right)^{1/2} \times \exp\left(\sum_{\ell=1}^{L-1} \frac{g_\ell(\hat{z})}{N^{2\ell-1}}\right) + O\left(e^{WN/2}\right) \qquad \text{for} \qquad \hat{z} \in (2+\delta,\ 5/2-\delta') \tag{5.6}$$

with $L = \lfloor \alpha \cdot N/2 \rfloor$. The implied constants in (5.5), (5.6) are absolute.

Proof. The bound (5.5) follows directly from Proposition 4.7 with $m = N - k$ and $h = 2$. Next, in Proposition 4.10, we set $s = N$ and again $m = N - k$ and $h = 2$. The condition on m in Proposition 4.10 is equivalent to

$$2 + \frac{\Delta}{1-\Delta/2} \leqslant \frac{2N}{k} \leqslant \frac{5}{2}.$$

So (5.6) follows from Proposition 4.10 if

$$\frac{\Delta}{1-\Delta/2} \leqslant \delta. \tag{5.7}$$

The inequality (5.7) is equivalent to $1/\Delta - 1/\delta \geqslant 1/2$. Since our assumption $\Delta/(1-\Delta) < \delta$ is equivalent to $1/\Delta - 1/\delta > 1$, we have that (5.7) is true. □

With (4.2) for $m = 2$ we obtain

$$\mathrm{Cl}_2(2\pi z) = -i\,\mathrm{Li}_2(e^{2\pi i z}) + i\pi^2(z^2 - 5z + 37/6) \qquad (2 < z < 3).$$

Therefore

$$\frac{\mathrm{Cl}_2(2\pi z)}{2\pi z} + \frac{\pi i}{2}\left(-z + 5 - \frac{2}{z}\right) = \frac{1}{2\pi i z}\Big[\mathrm{Li}_2(e^{2\pi i z}) - \mathrm{Li}_2(1) - 4\pi^2\Big], \tag{5.8}$$

with the right side of (5.8) now holomorphic in the strip $2 < \mathrm{Re}(z) < 3$.

To combine (5.3) and (5.6) we set, initially with $z \in (2,3)$,

$$r_{\mathcal{C}}(z) := \frac{1}{2\pi i z}\Big[\mathrm{Li}_2(e^{2\pi i z}) - \mathrm{Li}_2(1) - 4\pi^2\Big], \tag{5.9}$$

$$q_{\mathcal{C}}(z) := \left(\frac{z}{2\sin(\pi z)}\right)^{1/2} \exp(-\pi i z/2), \tag{5.10}$$

$$v_{\mathcal{C}}(z; N, \sigma) := \frac{2\pi i \sigma z}{N} + \sum_{\ell=1}^{L-1} \frac{g_\ell(z)}{N^{2\ell-1}} \qquad (L = \lfloor \alpha \cdot N/2 \rfloor). \tag{5.11}$$

Then define

$$\mathcal{C}_3(N,\sigma) := \frac{-2}{N^{1/2}}\mathrm{Re} \sum_{k \text{ odd: } \hat{z} \in (2+\delta, 5/2-\delta')} \frac{1}{k^2} \exp\big(N \cdot r_{\mathcal{C}}\left(\hat{z}\right)\big) q_{\mathcal{C}}\left(\hat{z}\right) \exp\big(v_{\mathcal{C}}(\hat{z}; N, \sigma)\big), \tag{5.12}$$

and it follows from (5.3) and Theorem 5.1 that for an absolute implied constant

$$\mathcal{C}_2(N,\sigma) = \mathcal{C}_3(N,\sigma) + O(e^{WN/2}). \tag{5.13}$$

5.2. Expressing $\mathcal{C}_3(N,\sigma)$ as an Integral

Proposition 5.2. *Suppose* $3/2 \leqslant \mathrm{Re}(z) \leqslant 5/2$ *and* $|z-2| \geqslant \varepsilon > 0$. *Also assume that*

$$\max\left\{1 + \frac{1}{\varepsilon},\ 16\right\} < \frac{\pi e}{\alpha}. \tag{5.14}$$

Then, for an implied constant depending only on ε, α *and* d,

$$\sum_{\ell=d}^{L-1} \frac{g_\ell(z)}{N^{2\ell-1}} \ll \frac{1}{N^{2d-1}} e^{-\pi|y|} \qquad (d \geqslant 2,\ L = \lfloor \alpha \cdot N/2 \rfloor). \tag{5.15}$$

Proof. For z in this range, Theorem 3.3 of [9] bounding derivatives of the cotangent allows us to show

$$\frac{g_\ell(z)}{N^{2\ell-1}} \ll F_{N,\varepsilon}(2\ell-1) \cdot e^{-\pi|y|}, \quad \text{for} \quad F_{N,\varepsilon}(\ell) := \left(\frac{\ell}{2\pi e N}\right)^{\ell} \left(\left(1 + \frac{2}{\varepsilon}\right)^{\ell} + 32^{\ell}\right). \tag{5.16}$$

This bound gets very large for ℓ large. The condition (5.14) ensures L is small enough that $g_\ell(z)/N^{2\ell-1}$ remains small. See [9, Sect. 4.2] for the details. □

We now fix some of the parameters in Theorem 5.1 and take

$$W = 0.05, \quad \alpha = 0.006\pi e \approx 0.0512, \quad 0.0061 \leqslant \delta,\ \delta' \leqslant 0.01, \quad N \geqslant 400. \tag{5.17}$$

Also, with $\varepsilon = 0.0061$, condition (5.14) is satisfied and Proposition 5.2 implies:

Corollary 5.3. *With $\delta, \delta' \in [0.0061, 0.01]$ and $z \in \mathbb{C}$ such that $2+\delta \leqslant \operatorname{Re}(z) \leqslant 5/2-\delta'$ we have*

$$v_{\mathcal{C}}(z;N,\sigma) = \frac{2\pi i \sigma z}{N} + \sum_{\ell=1}^{d-1} \frac{g_\ell(z)}{N^{2\ell-1}} + O\left(\frac{1}{N^{2d-1}}\right)$$

for $2 \leqslant d \leqslant L = \lfloor 0.006\pi e \cdot N/2 \rfloor$ and an implied constant depending only on d.

In the next theorem we assemble the results we need to convert the sum $\mathcal{C}_3(N,\sigma)$ in (5.12) into an integral.

Theorem 5.4. *The functions $r_{\mathcal{C}}(z)$, $q_{\mathcal{C}}(z)$ and $v_{\mathcal{C}}(z;N,\sigma)$ are holomorphic for $2 < \operatorname{Re}(z) < 5/2$. In this strip,*

$$\operatorname{Re}\left(r_{\mathcal{C}}\,(z) + \frac{2\pi i j}{z}\right) \leqslant \frac{1}{2\pi|z|^2}\left(x \operatorname{Cl}_2(2\pi x) + \pi^2|y|\left[\frac{1}{3}+4(j+1)\right]\right) \qquad (y \geqslant 0) \tag{5.18}$$

$$\operatorname{Re}\left(r_{\mathcal{C}}\,(z) + \frac{2\pi i j}{z}\right) \leqslant \frac{1}{2\pi|z|^2}\left(x \operatorname{Cl}_2(2\pi x) + \pi^2|y|\left[\frac{1}{3}-4j\right]\right) \qquad (y \leqslant 0) \tag{5.19}$$

for $j \in \mathbb{R}$. Also, in the box with $2+\delta \leqslant \operatorname{Re}(z) \leqslant 5/2-\delta'$ and $-1 \leqslant \operatorname{Im}(z) \leqslant 1$,

$$q_{\mathcal{C}}(z), \quad \exp\bigl(v_{\mathcal{C}}(z;N,\sigma)\bigr) \ll 1 \tag{5.20}$$

for an implied constant depending only on $\sigma \in \mathbb{R}$.

Proof. Since $\operatorname{Li}_2(e^{2\pi i z})$ is holomorphic away from the vertical branch cuts $(-i\infty, n]$ for $n \in \mathbb{Z}$, we see that $r_{\mathcal{C}}(z)$ is holomorphic for $2 < \operatorname{Re}(z) < 5/2$. Then in this strip, using (4.1),

$$\begin{aligned} r_{\mathcal{C}}\,(z) + \frac{2\pi i j}{z} &= \frac{1}{2\pi i z}\Bigl[\operatorname{Li}_2(e^{2\pi i z}) - \operatorname{Li}_2(1) - 4\pi^2(j+1)\Bigr] \\ &= \frac{1}{2\pi i z}\Bigl[-\operatorname{Li}_2(e^{-2\pi i z}) + \operatorname{Li}_2(1) - 4\pi^2(j-2)\Bigr] - \pi i(z-5). \end{aligned} \tag{5.21}$$

The inequalities (5.18) and (5.19) follow, as in [9, Sect. 4.3].

Check that for $w \in \mathbb{C}$,

$$-\pi/2 < \arg\bigl(\sin(\pi w)\bigr) < \pi/2 \quad \text{for} \quad 0 < \operatorname{Re}(w) < 1.$$

Consequently, $-\pi < \arg\bigl(z/\sin(\pi z)\bigr) < \pi$ for $2 < \operatorname{Re}(z) < 5/2$ and so $q_{\mathcal{C}}(z)$ is holomorphic in this strip. Also $v_{\mathcal{C}}(z;N,\sigma)$ is holomorphic here since the only poles of $g_\ell(z)$ are at $z \in \mathbb{Z}$.

Finally, $q_{\mathcal{C}}(z)$ is bounded on the compact box, as is $\exp\bigl(v_{\mathcal{C}}(z;N,\sigma)\bigr)$ by Corollary 5.3. ∎

By the calculus of residues, see for example [7, p. 300],

$$\sum_{a\leqslant k\leqslant b,\ k \text{ odd}} \varphi(k) = \frac{1}{2}\int_C \frac{\varphi(z)}{2i\tan(\pi(z-1)/2)}\,dz \tag{5.22}$$

for $\varphi(z)$ a holomorphic function and C a positively oriented closed contour surrounding the interval $[a,b]$ and not surrounding any integers outside this interval. Hence

$$\sum_{a\leqslant k\leqslant b,\ k \text{ odd}} \frac{1}{k^2}\varphi(2N/k) = \frac{-1}{4N}\int_C \frac{\varphi(z)}{2i\tan(\pi(2N/z-1)/2)}\,dz$$

for C now surrounding $\{2N/k \mid a \leqslant k \leqslant b\}$ with $a>0$. Therefore

$$\mathcal{C}_3(N,\sigma) = \frac{1}{2N^{3/2}}\mathrm{Re}\int_{C_1} \exp\bigl(N\cdot r_{\mathcal{C}}(z)\bigr)\frac{q_{\mathcal{C}}(z)}{2i\tan\bigl(\pi(2N/z-1)/2\bigr)}\exp\bigl(v_{\mathcal{C}}(z;N,\sigma)\bigr)\,dz \tag{5.23}$$

where C_1 is the positively oriented rectangle with horizontal sides C_1^+, C_1^- having imaginary parts $1/N^2$, $-1/N^2$ and vertical sides $C_{1,L}$, $C_{1,R}$ having real parts $2+\delta$ and $5/2-\delta'$ respectively, as shown in Figure 4. The next result shows that the

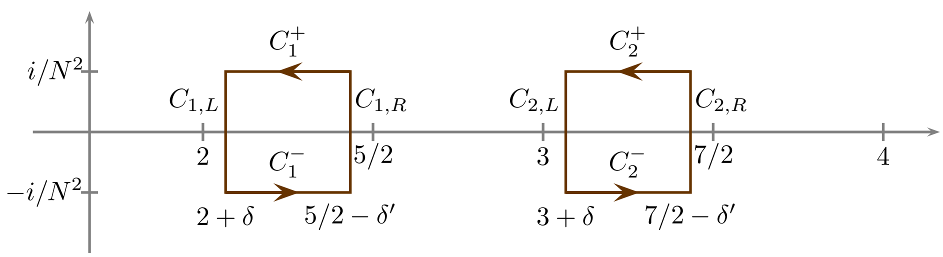

Figure 4: The rectangles C_1 and C_2

integrals over $C_{1,L}$, $C_{1,R}$ are small.

Proposition 5.5. *For N greater than an absolute constant, we may choose δ, $\delta' \in [0.0061, 0.01]$ so that*

$$\mathcal{C}_3(N,\sigma) = \frac{1}{2N^{3/2}}\mathrm{Re}\int_{C_1^+\cup C_1^-} \frac{\exp\bigl(N\cdot r_{\mathcal{C}}(z)\bigr)q_{\mathcal{C}}(z)\exp\bigl(v_{\mathcal{C}}(z;N,\sigma)\bigr)}{2i\tan\bigl(\pi(2N/z-1)/2\bigr)}\,dz + O(e^{WN/2})$$

for $W=0.05$ and an implied constant depending only on σ.

Proof. The proposition follows from (5.23) if we can show $\int_{C_{1,L}\cup C_{1,R}} = O(e^{WN/2})$. For N large enough, we may choose δ and δ' so that $C_{1,L}$ and $C_{1,R}$ pass midway between the poles of $1/\tan\bigl(\pi(2N/z-1)/2\bigr)$. Hence

$$\frac{1}{\tan\bigl(\pi(2N/z-1)/2\bigr)} \ll 1 \qquad (z\in C_{1,L}\cup C_{1,R}). \tag{5.24}$$

The bound (5.20) from Theorem 5.4 implies

$$q_{\mathcal{C}}(z)\exp\big(v_{\mathcal{C}}(z;N,\sigma)\big) \ll 1 \qquad (z \in C_{1,L} \cup C_{1,R}). \tag{5.25}$$

Theorem 5.4 with $j = 0$ also implies

$$\mathrm{Re}\big(r_{\mathcal{C}}(z)\big) < \frac{1}{8\pi}\left(x\,\mathrm{Cl}_2(2\pi x) + \frac{5\pi^2}{N^2}\right) \qquad (z \in C_{1,L} \cup C_{1,R}).$$

Note that

$$\mathrm{Cl}_2(2\pi x) < 0.24 \quad \text{if} \quad 2 \leqslant x \leqslant 2.01, \qquad \mathrm{Cl}_2(2\pi x) < 0.05 \quad \text{if} \quad 2.49 \leqslant x \leqslant 2.5. \tag{5.26}$$

Therefore

$$\mathrm{Re}\big(r_{\mathcal{C}}(z)\big) < \frac{1}{8\pi}\left(2.01 \times 0.24 + \frac{5\pi^2}{N^2}\right) < 0.025 \qquad (z \in C_{1,L},\ N \geqslant 25) \tag{5.27}$$

and we obtain (5.27) for $z \in C_{1,R}$ in the same way. Consequently

$$\exp\big(N \cdot r_{\mathcal{C}}(z)\big) \ll \exp(0.025N) \qquad (z \in C_{1,L} \cup C_{1,R}). \tag{5.28}$$

The proposition now follows from the bounds (5.24), (5.25) and (5.28). □

We have

$$\frac{1}{2i\tan(\pi(2N/z-1)/2)} = \begin{cases} 1/2 + \sum_{j \leqslant -1} (-1)^j e^{2\pi i j N/z} & \text{if } \ \mathrm{Im} z > 0 \\ -1/2 - \sum_{j \geqslant 1} (-1)^j e^{2\pi i j N/z} & \text{if } \ \mathrm{Im} z < 0 \end{cases} \tag{5.29}$$

and therefore

$$\int_{C_1^+} = \sum_{j \leqslant 0}{}' (-1)^j \int_{C_1^+} \exp\big(N[r_{\mathcal{C}}(z) + 2\pi i j/z]\big) q_{\mathcal{C}}(z)\exp\big(v_{\mathcal{C}}(z;N,\sigma)\big)\,dz, \tag{5.30}$$

$$\int_{C_1^-} = -\sum_{j \geqslant 0}{}' (-1)^j \int_{C_1^-} \exp\big(N[r_{\mathcal{C}}(z) + 2\pi i j/z]\big) q_{\mathcal{C}}(z)\exp\big(v_{\mathcal{C}}(z;N,\sigma)\big)\,dz \tag{5.31}$$

where $\sum'$ indicates the $j = 0$ term is taken with a $1/2$ factor. The terms with $j = 0, -1$ are the largest:

Proposition 5.6. *For $W = 0.05$ and an implied constant depending only on σ*

$$\mathcal{C}_3(N,\sigma) = \frac{-1}{2N^{3/2}} \sum_{j=0,-1} (-1)^j$$
$$\times\, \mathrm{Re} \int_{2.01}^{2.49} \exp\big(N[r_{\mathcal{C}}(z) + 2\pi i j/z]\big) q_{\mathcal{C}}(z)\exp\big(v_{\mathcal{C}}(z;N,\sigma)\big)\,dz + O(e^{WN/2}). \tag{5.32}$$

Proof. As in [9, Sect. 4.5], the total contribution to (5.30), (5.31) for all j with $|j| > N^2$ can be shown to be $O(N)$. Let D_1^+ be the three lines which, when added to C_1^+, make a rectangle with top side having imaginary part 1. Orient the path D_1^+ so that it has the same starting and ending points as C_1^+. Since the integrand is holomorphic we see that $\int_{C_1^+} = \int_{D_1^+}$. For integers j with $-N^2 \leqslant j < 0$ we consider

$$\int_{D_1^+} \exp\bigl(N[r_{\mathcal{C}}(z) + 2\pi i j/z]\bigr) q_{\mathcal{C}}(z) \exp\bigl(v_{\mathcal{C}}(z; N, \sigma)\bigr)\, dz. \tag{5.33}$$

We have $q_{\mathcal{C}}(z) \exp\left(v_{\mathcal{C}}(N, z)\right) \ll 1$ for $z \in D_1^+$ by Theorem 5.4. On the vertical sides of D_1^+ we have

$$\operatorname{Re}\left(r_{\mathcal{C}}(z) + \frac{2\pi i j}{z}\right) < \frac{x\,\mathrm{Cl}_2(2\pi x)}{8\pi} < 0.02$$

by Theorem 5.4 and (5.26) if $j \leqslant -2$. On the horizontal side of D_1^+, with $y = 1$, Theorem 5.4 implies

$$\operatorname{Re}\left(r_{\mathcal{C}}\,(z) + \frac{2\pi i j}{z}\right) \leqslant \frac{1}{2\pi|z|^2}\left(2.5\,\mathrm{Cl}_2(\pi/3) + \pi^2\left[\frac{1}{3} + 4(j+1)\right]\right) < 0$$

if $j \leqslant -2$. Hence, for each integer j with $-N^2 \leqslant j \leqslant -2$, (5.33) is $O\bigl(\exp(0.02N)\bigr)$. In a similar way, the terms in (5.31) for $1 \leqslant j \leqslant N^2$ are $O\bigl(\exp(0.02N)\bigr)$. Moving the lines of integration from C_1^- and C_1^+ to $[2.01, 2.49]$ is valid with (5.25), (5.28) and this completes the proof. □

A slightly more detailed argument shows that the $j = 0$ term in (5.32) is also $O(e^{WN/2})$:

Proposition 5.7. *For $W = 0.05$ and an implied constant depending only on σ*

$$\int_{2.01}^{2.49} \exp\bigl(N \cdot r_{\mathcal{C}}(z)\bigr) q_{\mathcal{C}}(z) \exp\bigl(v_{\mathcal{C}}(z; N, \sigma)\bigr)\, dz = O(e^{WN/2}). \tag{5.34}$$

Proof. Change the path of integration to the lines joining 2.01, $2.01 - i$, $2.49 - i$ and 2.49. The result follows if we can show $\operatorname{Re}(r_{\mathcal{C}}(z)) \leqslant W/2$ on these lines. For $y \leqslant 0$, by (5.21),

$$\begin{aligned}\operatorname{Re}\bigl(r_{\mathcal{C}}(z)\bigr) &= \pi y - \frac{y}{2\pi|z|^2}\left(\mathrm{Li}_2(1) - \operatorname{Re}(\mathrm{Li}_2(e^{-2\pi i z})) + 8\pi^2\right) - \frac{x\operatorname{Im}(\mathrm{Li}_2(e^{-2\pi i z}))}{2\pi|z|^2}\\ &\leqslant \pi y - \frac{y}{2\pi|z|^2}\left(\mathrm{Li}_2(1) - \operatorname{Re}(\mathrm{Li}_2(e^{-2\pi i z})) + 8\pi^2\right) + \frac{x\,\mathrm{Cl}_2(2\pi x)}{2\pi|z|^2}\end{aligned}$$

using Lemma 4.1. Recalling (5.26) we obtain the following bounds on each segment:

- $x = 2.01$, $-1 \leqslant y \leqslant 0$. By Lemma 4.2 we have $-\operatorname{Re}(\mathrm{Li}_2(e^{-2\pi i z})) \leqslant 0$ so that

$$\operatorname{Re}\bigl(r_{\mathcal{C}}(z)\bigr) \leqslant \pi y + \frac{1}{2\pi(x^2+y^2)}\left(-y(\mathrm{Li}_2(1) + 8\pi^2) + 0.24x\right) < 0.025.$$

- $x = 2.49$, $-1 \leqslant y \leqslant 0$. By Lemma 4.3 we have $-\mathrm{Re}(\mathrm{Li}_2(e^{-2\pi i z})) \leqslant \mathrm{Li}_2(1)$ so that

$$\mathrm{Re}\big(r_{\mathcal{C}}(z)\big) \leqslant \pi y + \frac{1}{2\pi(x^2+y^2)}\big(-y(2\,\mathrm{Li}_2(1)+8\pi^2)+0.05x\big) < 0.01.$$

- $2 \leqslant x \leqslant 2.5$, $y = -1$. With Lemma 4.3 again

$$\mathrm{Re}\big(r_{\mathcal{C}}(z)\big) \leqslant \pi y + \frac{1}{2\pi(2^2+y^2)}\big(-y(2\,\mathrm{Li}_2(1)+8\pi^2)+2.5\,\mathrm{Cl}_2(\pi/3)\big) < 0. \quad \square$$

Define $p(z)$ as $p_0(z)$ with (1.31). Since $p(z) = -(r_{\mathcal{C}}(z) - 2\pi i/z)$, and recalling (5.13), we have therefore shown

$$\mathcal{C}_2(N,\sigma) = \mathcal{C}_4(N,\sigma) + O(e^{WN/2}) \tag{5.35}$$

for $W = 0.05$, an implied constant depending only on σ, and

$$\mathcal{C}_4(N,\sigma) := \frac{1}{2N^{3/2}}\mathrm{Re}\int_{2.01}^{2.49} \exp\big(-N\cdot p(z)\big)q_{\mathcal{C}}(z)\exp\big(v_{\mathcal{C}}(z;N,\sigma)\big)\,dz. \tag{5.36}$$

5.3. A Path Through the Saddle-point

To apply the saddle-point method, Theorem 1.8, to $\mathcal{C}_4(N,\sigma)$ we first locate the unique solution to $p'(z) = 0$ for $3/2 < \mathrm{Re}(z) < 5/2$ as

$$z_1 := 2 + \frac{\log\big(1-w(0,-2)\big)}{2\pi i} \approx 2.20541 + 0.345648i$$

by Theorem 1.9. Then we replace the path of integration $[2.01, 2.49]$ in (5.36) with one passing through z_1.

Let $v = \mathrm{Im}(z_1)/\mathrm{Re}(z_1) \approx 0.156728$ and $c = 1 + iv$. The path we take through the saddle-point z_1 is $\mathcal{Q} := \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3$, the polygonal path between the points 2.01, $2.01c$, $2.49c$ and 2.49 as shown in Figure 5.

For Theorem 1.8 we require the next result.

Theorem 5.8. *For the path $\mathcal{Q}$ above, passing through the saddle point z_1, we have $\mathrm{Re}(p(z) - p(z_1)) > 0$ for $z \in \mathcal{Q}$ except at $z = z_1$.*

Theorem 5.8 seems apparent from Figure 6. We prove it by approximating $\mathrm{Re}(p(z))$ and its derivatives by the first terms in their series expansions and reducing the issue to a finite computation. This method was used in [9, Sect. 5.2] and we repeat the results from there. To take into account that we are using an approximation to z_1, we give proofs valid in a range $0.15 \leqslant v \leqslant 0.16$.

Generalizing to $p_d(z)$, we examine $\mathrm{Re}(p_d(z))$ for z on the ray $z = ct$ for $c = 1+iv$ with $v > 0$. We also write

$$c = \rho e^{i\theta} \qquad (0 < \rho,\ 0 < \theta < \pi/2).$$

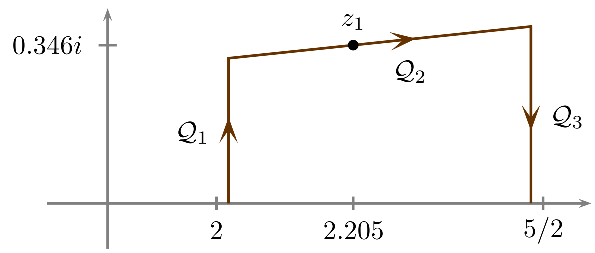

Figure 5: The path $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3$ through z_1

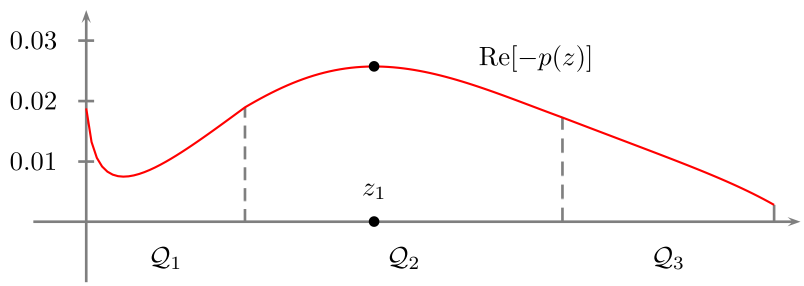

Figure 6: Graph of $\mathrm{Re}[-p(z)]$ for $z \in \mathcal{Q}$

For the second derivative we have

$$\frac{d^2}{dt^2}\mathrm{Re}[p_d(ct)] = R_2(L;t) + R_2^*(L;t)$$

for

$$R_2(L;t) := -\frac{\pi(24d+1)\sin\theta}{6\rho t^3} + \sum_{m=1}^{L-1}\Big(A_m(t)\cos(2\pi m t) + B_m(t)\sin(2\pi m t)\Big),$$

$$A_m(t) := e^{-2\pi m v t}\left(\frac{2}{mt^2} + \sin\theta\left(\frac{2\pi\rho}{t} + \frac{1}{m^2\pi\rho t^3}\right)\right),$$

$$B_m(t) := e^{-2\pi m v t}\cos\theta\left(\frac{2\pi\rho}{t} - \frac{1}{m^2\pi\rho t^3}\right)$$

and

$$|R_2^*(L;t)| \leqslant E_2(L;t) := \frac{e^{-2\pi L v t}}{1-e^{-2\pi v t}}\left(\frac{1}{\pi\rho L^2 t^3} + \frac{2}{Lt^2} + \frac{2\pi\rho}{t}\right).$$

We see that $E_2(L;t)$ is a decreasing function of L and t. We have $A_m(t)$ a positive and decreasing function of t. Also $B_m(t)$ is a positive and decreasing function of t when $t > \frac{\sqrt{3}}{\sqrt{2}\pi\rho m}$. The above formulas for $R_2(L;t)$ use (1.17), which is valid since

$|e^{2\pi iz}| \leqslant 1$ when $\mathrm{Im}(z) \geqslant 0$. For a ray $z = ct$ with $\mathrm{Im}(c) < 0$, the functional equation (4.1) must be applied first and then similar formulas are found.

Let $v_1 = 0.15$ and $v_2 = 0.16$. Writing $\rho_1 e^{i\theta_1} = 1 + iv_1$ and $\rho_2 e^{i\theta_2} = 1 + iv_2$ we have

$$1 < \rho_1 \leqslant \rho \leqslant \rho_2, \quad 0 < \theta_1 \leqslant \theta \leqslant \theta_2 < \pi/2.$$

For v in the interval $[v_1, v_2]$, we may bound $A_m(t)$, $B_m(t)$ and $E_2(L;t)$ from above and below by replacing v, ρ and θ appropriately by v_j, ρ_j and θ_j, $j = 1, 2$. For example

$$0 < A_m^-(t) \leqslant A_m(t) \leqslant A_m^+(t) \qquad (v \in [v_1, v_2])$$

with

$$A_m^-(t) := e^{-2\pi m v_2 t}\left(\frac{2}{mt^2} + \sin\theta_1\left(\frac{2\pi\rho_1}{t} + \frac{1}{m^2\pi\rho_2 t^3}\right)\right),$$
$$A_m^+(t) := e^{-2\pi m v_1 t}\left(\frac{2}{mt^2} + \sin\theta_2\left(\frac{2\pi\rho_2}{t} + \frac{1}{m^2\pi\rho_1 t^3}\right)\right)$$

and similarly write $0 < B_m^-(t) \leqslant B_m(t) \leqslant B_m^+(t)$ and $0 < E_2^-(L;t) \leqslant E_2(L;t) \leqslant E_2^+(L;t)$.

Lemma 5.9. *Let* $c = 1 + iv$ *with* $0.15 \leqslant v \leqslant 0.16$. *Then* $\frac{d^2}{dt^2}\mathrm{Re}[p(ct)] > 0$ *for* $t \in [2, 2.35]$.

Proof. Break up $[2, 2.35]$ into n equal segments $[x_{j-1}, x_j]$. Then

$$\frac{d^2}{dt^2}\mathrm{Re}[p(ct)] \geqslant \min_{1\leqslant j\leqslant n}\left(\left(\min_{t\in[x_{j-1},x_j]} R_2(L;t)\right) - E_2^+(L;x_{j-1})\right). \tag{5.37}$$

Let $t = x_{j,m}^*$ correspond to the minimum value of $\cos(2\pi mt)$ for $t \in [x_{j-1}, x_j]$ (so that $x_{j,m}^*$ equals x_{j-1}, x_j or a local minimum $k/2m$ for k odd). Similarly, let $t = x_{j,m}^{**}$ correspond to the minimum value of $\sin(2\pi mt)$ for $t \in [x_{j-1}, x_j]$. Then

$$\min_{t\in[x_{j-1},x_j]} R_2(L;t) \geqslant -\frac{\pi\sin\theta_2}{6\rho_1 x_{j-1}^3} + \sum_{m=1}^{L-1}\Big(A_m^-(x_j)\cos(2\pi m x_{j,m}^*) + B_m^-(x_j)\sin(2\pi m x_{j,m}^{**})\Big) \tag{5.38}$$

where we must replace $A_m^-(x_j)$ in (5.38) by $A_m^+(x_{j-1})$ if $\cos(2\pi m x_{j,m}^*) < 0$ and replace $B_m^-(x_j)$ in (5.38) by $B_m^+(x_{j-1})$ if $\sin(2\pi m x_{j,m}^{**}) < 0$.

A computation using (5.37) and (5.38) with $n = 10$ and $L = 3$ for example shows $\frac{d^2}{dt^2}\mathrm{Re}[p(ct)] > 0.09$. □

We may analyze the first derivative in a similar way. We have

$$\frac{d}{dt}\mathrm{Re}[p_d(ct)] = R_1(L;t) + R_1^*(L;t)$$

for

$$R_1(L;t) := \frac{\pi(24d+1)\sin\theta}{12\rho t^2} + \sum_{m=1}^{L-1}\Big(-C_m(t)\cos(2\pi mt) + D_m(t)\sin(2\pi mt)\Big),$$
$$C_m(t) := e^{-2\pi mvt}\left(\frac{1}{mt} + \frac{\sin\theta}{m^2 2\pi\rho t^2}\right), \qquad D_m(t) := e^{-2\pi mvt}\frac{\cos\theta}{m^2 2\pi\rho t^2}$$

and

$$|R_1^*(L;t)| \leqslant E_1(L;t) := \frac{e^{-2\pi Lvt}}{1-e^{-2\pi vt}}\left(\frac{1}{2\pi\rho L^2 t^2} + \frac{1}{Lt}\right).$$

We see that $E_1(L;t)$ is a decreasing function of L and t. Also $C_m(t)$ and $D_m(t)$ are positive and decreasing functions of t.

Lemma 5.10. *Let* $c = 1 + iv$ *with* $0.15 \leqslant v \leqslant 0.16$. *Then* $\frac{d}{dt}\mathrm{Re}[p(ct)] > 0$ *for* $t \in [2.35, 2.5]$.

Proof. Break $[2.35, 2.5]$ into n equal segments and, as in the proof of Lemma 5.9, bound $\frac{d}{dt}\mathrm{Re}[p(ct)]$ from below on each piece. Taking $n = 10$ and $L = 3$ shows $\frac{d}{dt}\mathrm{Re}[p(ct)] > 0.03$ for example. □

Corollary 5.11. *Let* $c = 1+iv$ *with* $0.15 \leqslant v \leqslant 0.16$. *There is a unique solution to* $\frac{d}{dt}\mathrm{Re}[p(ct)] = 0$ *for* $t \in [2, 2.5]$ *that we label as* t_0. *We then have* $\mathrm{Re}[p(ct)-p(ct_0)] > 0$ *for* $t \in [2, 2.5]$ *except at* $t = t_0$.

Proof. Check that $\frac{d}{dt}\mathrm{Re}[p(ct)] < 0$ when $t = 2$ and $\frac{d}{dt}\mathrm{Re}[p(ct)] > 0$ when $t = 2.35$. By Lemma 5.9 we see that $\frac{d}{dt}\mathrm{Re}[p(ct)]$ is strictly increasing for $t \in [2, 2.35]$. It necessarily has a unique zero that we label t_0. By Lemma 5.10, $\frac{d}{dt}\mathrm{Re}[p(ct)]$ remains > 0 for $t \in [2.35, 2.5]$. Hence $\mathrm{Re}[p(ct) - p(ct_0)]$ is strictly decreasing on $[2, t_0)$ and strictly increasing on $(t_0, 2.5]$ as required. □

Proposition 5.12. *For* $0.15 \leqslant v \leqslant 0.16$ *we have* $\mathrm{Re}[-p(z)] < 0.024$ *for* $z \in \mathcal{Q}_1 \cup \mathcal{Q}_3$.

Proof. We have x fixed as 2.01 on $\mathcal{Q}_1$ and 2.49 on $\mathcal{Q}_3$. Write

$$\mathrm{Re}[-p(z)] = \frac{f(y)+g(y)}{2\pi|z|^2}$$

for

$$f(y) := y\left(\mathrm{Li}_2(1) - \mathrm{Re}(\mathrm{Li}_2(e^{2\pi iz}))\right), \quad g(y) = x\mathrm{Im}(\mathrm{Li}_2(e^{2\pi iz})).$$

If $x = 2.01$ or 2.49 it follows from Lemma 4.1 that $g(y)$ is positive and decreasing. Similarly, it follows from Lemma 4.2 that $f(y)$ is always positive and increasing for $y > 0$.

For $z \in \mathcal{Q}_1$, so that $x = 2.01$ and $0 \leqslant y \leqslant Y := 2.01 \times 0.16 = 0.3216$,

$$\operatorname{Re}[-p(z)] \leqslant \begin{cases} (f(Y/3)+g(0))/(2\pi 2.01^2) \approx 0.0232 & y \in [0, Y/3] \\ (f(Y)+g(Y/3))/(2\pi(2.01^2+(Y/3)^2) \approx 0.0226 & y \in [Y/3, Y]. \end{cases}$$

For $z \in \mathcal{Q}_3$, so that $x = 2.49$ and $0 \leqslant y \leqslant Y := 2.49 \times 0.16 = 0.3984$,

$$\operatorname{Re}[-p(z)] \leqslant (f(Y)+g(0))/(2\pi 2.49^2) \approx 0.021, \qquad y \in [0, Y]. \qquad \square$$

Proof of Theorem 5.8. Let v be given by $\operatorname{Im}(z_1)/\operatorname{Re}(z_1)$. Then

$$\frac{d}{dt}\operatorname{Re}[p(ct)]\bigg|_{t=\operatorname{Re}(z_1)} = \operatorname{Re}[cp'(c\operatorname{Re}(z_1))] = \operatorname{Re}[cp'(z_1)] = 0.$$

It follows from Corollary 5.11 that $\operatorname{Re}[p(z) - p(z_1)] > 0$ for $z \in \mathcal{Q}_2$ and $z \neq z_1$. We also note that $\operatorname{Re}[-p(z_1)] \approx 0.0256706$.

For $z \in \mathcal{Q}_1 \cup \mathcal{Q}_3$, Proposition 5.12 implies $\operatorname{Re}[p(z) - p(z_1)] > -0.024 + 0.0256 > 0$. $\square$

5.4. Applying the Saddle-point Method

For $j \in \mathbb{Z}_{\geqslant 0}$ put

$$u_{\sigma,j}(z) := \sum_{m_1+3m_2+5m_3+\cdots=j} \frac{(2\pi i\sigma z + g_1(z))^{m_1}}{m_1!}\frac{g_2(z)^{m_2}}{m_2!}\cdots\frac{g_j(z)^{m_j}}{m_j!}, \tag{5.39}$$

with $u_{\sigma,0} = 1$. Recalling the definition of $g_\ell(z)$ in (5.4), we see that $u_{\sigma,j}(z)$ is holomorphic for $z \notin \mathbb{Z}$. The proof of the next proposition uses Corollary 5.3; see [9, Sect. 5.3].

Proposition 5.13. *For $2.01 \leqslant \operatorname{Re}(z) \leqslant 2.49$ and $|\operatorname{Im}(z)| \leqslant 1$, say, there is a holomorphic function $\zeta_d(z; N, \sigma)$ of z so that*

$$\exp\bigl(v_{\mathcal{C}}(z; N, \sigma)\bigr) = \sum_{j=0}^{d-1} \frac{u_{\sigma,j}(z)}{N^j} + \zeta_d(z; N, \sigma) \quad \text{for} \quad \zeta_d(z; N, \sigma) = O\left(\frac{1}{N^d}\right)$$

with an implied constant depending only on σ and d where $1 \leqslant d \leqslant 2L - 1$ and $L = \lfloor 0.006\pi e \cdot N/2 \rfloor$.

We now have everything in place to get the asymptotic expansion of $\mathcal{C}_2(N, \sigma)$.

Theorem 5.14. *With $c_0 = -z_1 e^{-\pi i z_1}/2$ and explicit $c_1(\sigma)$, $c_2(\sigma), \ldots$ depending on $\sigma \in \mathbb{Z}$ we have*

$$\mathcal{C}_2(N, \sigma) = \operatorname{Re}\left[\frac{w(0,-2)^{-N}}{N^2}\left(c_0 + \frac{c_1(\sigma)}{N} + \cdots + \frac{c_{m-1}(\sigma)}{N^{m-1}}\right)\right] + O\left(\frac{|w(0,-2)|^{-N}}{N^{m+2}}\right) \tag{5.40}$$

for an implied constant depending only on σ and m.

Proof. Recall from (1.33) that $e^{p(z_1)} = w(0,-2)$. Proposition 5.13 implies

$$\mathcal{C}_4(N,\sigma) = \mathrm{Re}\Bigg[\sum_{j=0}^{d-1} \frac{1}{2N^{3/2+j}} \int_{\mathcal{Q}} e^{-N\cdot p(z)} \cdot q_{\mathcal{C}}(z) \cdot u_{\sigma,j}(z)\, dz + \frac{1}{2N^{3/2}} \int_{\mathcal{Q}} e^{-N\cdot p(z)} \cdot q_{\mathcal{C}}(z) \cdot \zeta_d(z;N,\sigma)\, dz\Bigg] \quad (5.41)$$

where the last term in (5.41) is

$$\ll \frac{1}{N^{3/2}} \int_{\mathcal{Q}} \left|e^{-N\cdot p(z)}\right| \cdot 1 \cdot \frac{1}{N^d}\, dz \ll \frac{1}{N^{d+3/2}} e^{-N\mathrm{Re}(p(z_1))} = \frac{|w(0,-2)|^{-N}}{N^{d+3/2}}$$

by Theorem 5.8, (5.20) and Proposition 5.13. Applying Theorem 1.8 to each integral in the first part of (5.41) we obtain

$$\int_{\mathcal{Q}} e^{-N\cdot p(z)} \cdot q_{\mathcal{C}}(z) \cdot u_{\sigma,j}(z)\, dz = 2e^{-Np(z_1)} \left(\sum_{s=0}^{S-1} \Gamma(s+1/2) \frac{a_{2s}(q_{\mathcal{C}} \cdot u_{\sigma,j})}{N^{s+1/2}} + O\left(\frac{1}{N^{S+1/2}}\right)\right). \quad (5.42)$$

The error term in (5.42) corresponds to an error of size $O(|w(0,-2)|^{-N}/N^{S+j+2})$ for $\mathcal{C}_4(N,\sigma)$. We choose $S = d$ so that this error is less than $O(|w(0,-2)|^{-N}/N^{d+3/2})$ for all $j \geqslant 0$. Therefore

$$\mathcal{C}_4(N,\sigma) = \mathrm{Re}\left[\sum_{j=0}^{d-1} \frac{1}{N^{j+3/2}} e^{-N\cdot p(z_1)} \sum_{s=0}^{d-1} \frac{\Gamma(s+1/2) a_{2s}(q_{\mathcal{C}} \cdot u_{\sigma,j})}{N^{s+1/2}}\right] + O\left(\frac{|w(0,-2)|^{-N}}{N^{d+3/2}}\right)$$

and we may rewrite the sums inside the square brackets as

$$w(0,-2)^{-N} \sum_{t=0}^{2d-2} \frac{1}{N^{t+2}} \sum_{s=\max(0,t-d+1)}^{\min(t,d-1)} \Gamma(s+1/2) a_{2s}(q_{\mathcal{C}} \cdot u_{\sigma,t-s}).$$

Hence

$$\mathcal{C}_4(N,\sigma) = \mathrm{Re}\left[w(0,-2)^{-N} \sum_{t=0}^{d-2} \frac{1}{N^{t+2}} \sum_{s=0}^{t} \Gamma(s+1/2) a_{2s}(q_{\mathcal{C}} \cdot u_{\sigma,t-s})\right] + O\left(\frac{|w(0,-2)|^{-N}}{N^{d+1}}\right).$$

Recalling (5.35) and with

$$c_t(\sigma) := \sum_{s=0}^{t} \Gamma(s+1/2) a_{2s}(q_{\mathcal{C}} \cdot u_{\sigma,t-s}), \tag{5.43}$$

we obtain (5.40) in the statement of the theorem.

Use the formula (1.30) for a_0 to get

$$c_0(\sigma) = \Gamma(1/2) a_0(q_{\mathcal{C}} \cdot u_{\sigma,0}) = \sqrt{\pi} \frac{\omega}{2(\omega^2 p_0)^{1/2}} q_0$$

which is independent of σ. The terms p_0 and q_0 are defined in (1.27), (1.28). Using the identity

$$p''(z) = -\frac{1}{z}\left(2p'(z) + \frac{2\pi i \cdot e^{2\pi i z}}{1 - e^{2\pi i z}}\right) \tag{5.44}$$

we obtain

$$p_0 = p''(z_1)/2 = \frac{-\pi i e^{2\pi i z_1}}{z_1 w(0,-2)}, \qquad q_0^2 = q_{\mathcal{C}}(z_1)^2 = \frac{-i z_1}{w(0,-2)}. \tag{5.45}$$

Therefore

$$c_0^2 = \frac{\pi q_0^2}{4 p_0} = \frac{z_1^2}{4 e^{2\pi i z_1}}.$$

We may take $\omega = z_1$ since the path $\mathcal{Q}_2$ is a segment of the ray from the origin through z_1. A numerical check then gives us the correct square root:

$$c_0 = \sqrt{\pi} \frac{\omega}{2(\omega^2 p_0)^{1/2}} q_0 = -\frac{z_1}{2 e^{\pi i z_1}}. \qquad \square$$

For example, Table 1 compares both sides of (5.40) in Theorem 5.14 with $\sigma = 1$ and some different values of m and N. For other values of σ we get similar agreement.

N	$m=1$	$m=2$	$m=3$	$m=4$	$\mathcal{C}_2(N,1)$
800	293.204	301.757	303.016	303.119	303.112
1000	−263123.	−261461.	−261486.	−261493.	−261493.

Table 1: Theorem 5.14's approximations to $\mathcal{C}_2(N,1)$.

6. The Asymptotic Behavior of $\mathcal{C}_2^*(N,\sigma)$

We find the asymptotic expansion of

$$\mathcal{C}_2^*(N,\sigma) = 2\mathrm{Re} \sum_{k \text{ odd } : \ 2N/k \in [3,4)} Q_{2k\sigma}(N),$$

the second component of $\mathcal{C}_1(N,\sigma)$, in this section.

6.1. Approximating the Sine Product

From (5.2), $Q_{2k\sigma}(N)$ contains the sine product $\prod_m^{-1}(2/k)$ for $m = N - k$ and $k/2 < m < k$. The next result expresses this product in terms of a new variable a.

Proposition 6.1. *Let k be an odd positive integer. Write $m = a + (k-1)/2$ for $1 \leqslant a \leqslant (k-1)/2$. Then*

$$\prod_m^{-1}(2/k) = \frac{(-1)^a}{\sqrt{k}} \frac{\prod_{2a}^{-1}(1/k)}{\prod_a^{-1}(2/k)} \qquad \textit{for} \quad k/2 < m < k.$$

Proof. The formula

$$\prod_{k-1}^{-1}(h/k) = (-1)^{(h-1)(k-1)/2}\frac{1}{k}$$

from [9, Sect. 2.2] implies $\prod_{k-1}^{-1}(2/k) = (-1)^{(k-1)/2}/k$ and therefore, by symmetry,

$$\prod_{(k-1)/2}^{-1}(2/k) = \frac{1}{\sqrt{k}}.$$

Hence

$$\begin{aligned}
\prod_m^{-1}(2/k) &= \prod_{(k-1)/2}^{-1}(2/k) \prod_{j=1}^{a} \frac{1}{2\sin(\pi(j+(k-1)/2)2/k)} \\
&= \frac{1}{\sqrt{k}} \prod_{j=1}^{a} \frac{1}{2\sin(\pi(2j-1)/k + \pi)} \\
&= \frac{(-1)^a}{\sqrt{k}} \prod_{j=1}^{a} \frac{1}{2\sin(\pi(2j-1)/k)}
\end{aligned}$$

and the result follows. □

In this subsection we define $\hat{z} = \hat{z}(N,k) := 2(N+1/2)/k$ because of (6.3) below. The next result is the sine product approximation we need here.

Theorem 6.2. *Fix $W > 0$. Let Δ be in the range $0.0048 \leqslant \Delta \leqslant 0.0079$ and set $\alpha = \Delta\pi e$. Suppose δ and δ' satisfy*

$$\frac{\Delta}{1-\Delta} < \delta \leqslant \frac{1}{e}, \ 0 < \delta' \leqslant \frac{1}{e} \qquad \textit{and} \qquad \delta \log 1/\delta, \quad \delta' \log 1/\delta' \leqslant W.$$

Then for all $N \geqslant 3 \cdot R_\Delta$ we have

$$\prod_{N-k}^{-1}(2/k) = O\left(e^{WN/3}\right) \qquad \textit{for} \qquad \hat{z} \in [3, \ 3+\delta] \cup [7/2 - \delta', \ 4) \tag{6.1}$$

and also for $\hat{z} \in (3+\delta,\ 7/2-\delta')$

$$\prod\nolimits_{N-k}^{-1}(2/k) = \frac{(-1)^{N+(k+1)/2}}{\sqrt{2k}} \exp\left(\frac{N+1/2}{2\pi\hat{z}} \operatorname{Cl}_2(2\pi\hat{z})\right)$$
$$\times \exp\left(\sum_{\ell=1}^{L-1} \frac{g_\ell(\hat{z})}{(2(N+1/2))^{2\ell-1}} - \sum_{\ell=1}^{L^*-1} \frac{g_\ell(\hat{z})}{(N+1/2)^{2\ell-1}}\right) + O\left(e^{WN/3}\right) \quad (6.2)$$

with $L = \lfloor \alpha \cdot 2N/3 \rfloor$ *and* $L^* = \lfloor \alpha \cdot N/3 \rfloor$. *The implied constants in* (6.1), (6.2) *are absolute.*

Proof. We have $2N/k \in [3,4)$ so that $N/2 < k \leqslant 2N/3$. For $m = N-k$ this corresponds to $k/2 < m < k$. In terms of a this means

$$1 \leqslant a < k/2, \qquad 2a/k = \hat{z} - 3. \quad (6.3)$$

For $L = 1$, Proposition 4.5 implies

$$\prod\nolimits_{2a}^{-1}(1/k) = \left(\frac{1}{2k\sin(2\pi a/k)}\right)^{1/2} \exp\left(\frac{k}{2\pi} \operatorname{Cl}_2(4\pi a/k)\right) \exp\left(-T_1(2a,1/k)\right),$$
$$\prod\nolimits_{a}^{-1}(2/k) = \left(\frac{1}{k\sin(2\pi a/k)}\right)^{1/2} \exp\left(\frac{k}{4\pi} \operatorname{Cl}_2(4\pi a/k)\right) \exp\left(-T_1(a,2/k)\right)$$

so that

$$\frac{\prod_{2a}^{-1}(1/k)}{\prod_{a}^{-1}(2/k)} = \frac{1}{2^{1/2}} \exp\left(\frac{k}{4\pi} \operatorname{Cl}_2(4\pi a/k)\right) \exp\left(-T_1(2a,1/k) + T_1(a,2/k)\right)$$
$$= \left(\frac{k}{2}\sin(2\pi a/k)\right)^{1/2} \exp\left(-T_1(2a,1/k) + 2T_1(a,2/k)\right) \cdot \prod\nolimits_{a}^{-1}(2/k).$$

Therefore, employing Lemma 4.6,

$$\frac{\prod_{2a}^{-1}(1/k)}{\prod_{a}^{-1}(2/k)} \ll k^{1/2} \prod\nolimits_{a}^{-1}(2/k) \qquad (1 \leqslant a < k/2) \quad (6.4)$$

with an absolute implied constant. A similar argument proves

$$\frac{\prod_{2a}^{-1}(1/k)}{\prod_{a}^{-1}(2/k)} \ll k^{1/4} \left(\prod\nolimits_{2a}^{-1}(1/k)\right)^{1/2} \qquad (1 \leqslant a < k/2). \quad (6.5)$$

Then using Proposition 4.7 to bound $\prod_{a}^{-1}(2/k)$ on the right of (6.4) and noting that $k \leqslant 2N/3$ proves (6.1).

For positive integers L_1, L_2, Proposition 4.5 implies

$$\begin{aligned}\frac{\prod_{2a}^{-1}(1/k)}{\prod_{a}^{-1}(2/k)} &= \frac{1}{\sqrt{2}} \exp\left(\frac{k}{4\pi}\,\mathrm{Cl}_2(4\pi a/k)\right) \\ &\quad \times \exp\left(-\sum_{\ell=1}^{L_1-1} \frac{B_{2\ell}}{(2\ell)!}\left(\frac{\pi}{k}\right)^{2\ell-1} \cot^{(2\ell-2)}\left(\frac{2a\pi}{k}\right)\right) \\ &\quad \times \exp\left(\sum_{\ell=1}^{L_2-1} \frac{B_{2\ell}}{(2\ell)!}\left(\frac{2\pi}{k}\right)^{2\ell-1} \cot^{(2\ell-2)}\left(\frac{2a\pi}{k}\right)\right) \\ &\quad \times \exp\left(-T_{L_1}(2a,1/k) + T_{L_2}(a,2/k)\right). \quad (6.6)\end{aligned}$$

Use Proposition 4.8 with $h = 2$, $m = a$ and $s = 2N/3$ to show that we have, for $\Delta N/3 \leqslant a \leqslant k/4$,

$$\prod\nolimits_{a}^{-1}(2/k) T_{L_2}(a, 2/k) \ll e^{WN/3} \quad (6.7)$$

$$T_{L_2}(a, 2/k) \ll 1 \quad (6.8)$$

with absolute implied constants, $L_2 := \lfloor \pi e\Delta \cdot N/3 \rfloor$ and $N \geqslant 3 \cdot R_\Delta$. The above inequality (6.7) is valid with $\prod_{a}^{-1}(2/k)$ replaced by $\prod_{2a}^{-1}(1/k)/\prod_{a}^{-1}(2/k)$ using (6.4):

$$\left(\prod\nolimits_{2a}^{-1}(1/k)\Big/\prod\nolimits_{a}^{-1}(2/k)\right) T_{L_2}(a, 2/k) \ll N^{1/2} e^{WN/3}. \quad (6.9)$$

Use Proposition 4.8 with $h = 1$, $m = 2a$ and $s = 2N/3$ to show that, also for $\Delta N/3 \leqslant a \leqslant k/4$,

$$\prod\nolimits_{2a}^{-1}(1/k) T_{L_1}(2a, 1/k) \ll e^{2WN/3} \quad (6.10)$$

$$T_{L_1}(2a, 1/k) \ll 1 \quad (6.11)$$

with absolute implied constants, $L_1 := \lfloor \pi e\Delta \cdot 2N/3 \rfloor$ and $N \geqslant 3 \cdot R_\Delta/2$. Taking square roots of both sides of the inequality (6.10) and using (6.5) and that we have $|T_{L_1}(2a, 1/k)| \ll |T_{L_1}(2a, 1/k)|^{1/2}$ shows

$$\left(\prod\nolimits_{2a}^{-1}(1/k)\Big/\prod\nolimits_{a}^{-1}(2/k)\right) T_{L_1}(2a, 1/k) \ll N^{1/4} e^{WN/3}. \quad (6.12)$$

With the inequalities (6.7) - (6.12) established, the arguments of Proposition 4.10 now go through, applied to (6.6). This allows us to remove the factor in (6.6) of $\exp\left(-T_{L_1}(2a, 1/k) + T_{L_2}(a, 2/k)\right)$ at the expense of adding an $O(e^{WN/3})$ error. The interval $\Delta N/3 \leqslant a \leqslant k/4$ corresponds to

$$3 + \frac{\Delta}{1 - \Delta/3} \leqslant \hat{z} \leqslant \frac{7}{2}$$

so we require

$$\frac{\Delta}{1 - \Delta/3} < \delta. \quad (6.13)$$

The inequality (6.13) is equivalent to $1/\Delta - 1/\delta \geqslant 1/3$. Since our assumption $\Delta/(1-\Delta) < \delta$ is equivalent to $1/\Delta - 1/\delta > 1$, we have that (6.13) is true. This completes the proof of (6.2). □

We rewrite (5.2) as

$$Q_{2k\sigma}(N) = \frac{e^{-\pi i/4}}{k^2} \exp\left((N+1/2)\frac{-\pi i}{2}(\hat{z} - 5 + 2/\hat{z})\right) \times \exp\left(\frac{\pi i(16\sigma+1)\hat{z}}{8(N+1/2)}\right) \prod\nolimits_{N-k}^{-1}(2/k)$$

and combine with (6.2) from Theorem 6.2 as follows. With (4.2) for $m = 3$ we obtain

$$\mathrm{Cl}_2(2\pi z) = -i\,\mathrm{Li}_2(e^{2\pi i z}) + i\pi^2(z^2 - 7z + 73/6) \qquad (3 < z < 4).$$

Hence

$$\frac{\mathrm{Cl}_2(2\pi z)}{2\pi z} - \frac{\pi i}{2}(z - 5 + 2/z) = -\pi i + \frac{1}{2\pi i z}\Big[\mathrm{Li}_2(e^{2\pi i z}) - \mathrm{Li}_2(1) - 10\pi^2\Big].$$

Define the following functions

$$\begin{aligned} r^*_{\mathcal{C}}(z) &:= \frac{1}{2\pi i z}\Big[\mathrm{Li}_2(e^{2\pi i z}) - \mathrm{Li}_2(1) - 10\pi^2\Big], \\ q^*_{\mathcal{C}}(z) &:= e^{-3\pi i/4}\sqrt{z}, \\ v^*_{\mathcal{C}}(z; N, \sigma) &:= \frac{\pi i(16\sigma+1)z}{8(N+1/2)} + \sum_{\ell=1}^{L-1} \frac{g_\ell(z)}{(2(N+1/2))^{2\ell-1}} - \sum_{\ell=1}^{L^*-1} \frac{g_\ell(z)}{(N+1/2)^{2\ell-1}} \end{aligned}$$

for $L = \lfloor \alpha \cdot 2N/3 \rfloor$ and $L^* = \lfloor \alpha \cdot N/3 \rfloor$. Set

$$\mathcal{C}^*_3(N,\sigma) := \frac{1}{(N+1/2)^{1/2}} \times \mathrm{Re} \sum_{k \text{ odd } : \ \hat{z} \in (3+\delta, 7/2-\delta')} \frac{(-1)^{(k+1)/2}}{k^2} \exp\big((N+1/2) r^*_{\mathcal{C}}(\hat{z})\big) q^*_{\mathcal{C}}(\hat{z}) \exp\big(v^*_{\mathcal{C}}(\hat{z}; N, \sigma)\big).$$

It follows from Theorem 6.2 that

$$\mathcal{C}^*_2(N,\sigma) = \mathcal{C}^*_3(N,\sigma) + O(e^{WN/3}). \qquad (6.14)$$

6.2. Expressing $\mathcal{C}^*_3(N,\sigma)$ as an Integral

Similarly to Proposition 5.2 we have

Proposition 6.3. *Suppose* $5/2 \leqslant \mathrm{Re}(z) \leqslant 7/2$ *and* $|z-3| \geqslant \varepsilon > 0$ *and assume* $\max\{1+\frac{1}{\varepsilon},\ 16\} < \frac{\pi e}{\alpha}$. *Then for* $d \geqslant 2$,

$$\sum_{\ell=d}^{L-1} \frac{g_\ell(z)}{(2(N+1/2))^{2\ell-1}} - \sum_{\ell=d}^{L^*-1} \frac{g_\ell(z)}{(N+1/2)^{2\ell-1}} \ll \frac{1}{N^{2d-1}} e^{-\pi|y|} \tag{6.15}$$

where $L = \lfloor \alpha \cdot 2N/3 \rfloor$, $L^* = \lfloor \alpha \cdot N/3 \rfloor$ *and the implied constant depends only on* ε, α *and* d.

Fixing the choice of constants in (5.17) and with $\varepsilon = 0.0061$ and

$$g_{\mathcal{C},\ell}(z) := g_\ell(z)(2^{-(2\ell-1)} - 1) \tag{6.16}$$

we obtain:

Corollary 6.4. *With* $\delta, \delta' \in [0.0061, 0.01]$ *and* $z \in \mathbb{C}$ *such that* $3 + \delta \leqslant \mathrm{Re}(z) \leqslant 7/2 - \delta'$ *we have*

$$v^*_{\mathcal{C}}(z; N, \sigma) = \frac{\pi i(16\sigma+1)z}{8(N+1/2)} + \sum_{\ell=1}^{d-1} \frac{g_{\mathcal{C},\ell}(z)}{(N+1/2)^{2\ell-1}} + O\left(\frac{1}{N^{2d-1}}\right)$$

for $2 \leqslant d \leqslant L^* = \lfloor 0.006\pi e \cdot N/3 \rfloor$ *and an implied constant depending only on* d.

Next,

$$\begin{aligned} r^*_{\mathcal{C}}(z) + \frac{2\pi i}{z}(j-1/2) &= \frac{1}{2\pi i z}\Big[-\mathrm{Li}_2(1) + \mathrm{Li}_2(e^{2\pi i z}) - 4\pi^2(j+2)\Big], \\ r^*_{\mathcal{C}}(z) + \frac{2\pi i}{z}(j+1/2) &= \frac{1}{2\pi i z}\Big[\mathrm{Li}_2(1) - \mathrm{Li}_2(e^{-2\pi i z}) - 4\pi^2(j-3)\Big] - \pi i(2z-7) \end{aligned} \tag{6.17}$$

where (6.17) follows from (4.1) when $3 < \mathrm{Re}(z) < 4$. Then with a similar proof to Theorem 5.4 we have

Theorem 6.5. *The functions* $r^*_{\mathcal{C}}(z)$, $q^*_{\mathcal{C}}(z)$ *and* $v^*_{\mathcal{C}}(z; N, \sigma)$ *are holomorphic for* $3 < \mathrm{Re}(z) < 7/2$. *In this strip, for* $j \in \mathbb{R}$,

$$\mathrm{Re}\left(r^*_{\mathcal{C}}(z) + \frac{2\pi i}{z}(j-1/2)\right) \leqslant \frac{1}{2\pi|z|^2}\left(x\,\mathrm{Cl}_2(2\pi x) + \pi^2|y|\left[\frac{1}{3} + 4(j+2)\right]\right) \tag{6.18}$$

when $y \geqslant 0$ *and*

$$\mathrm{Re}\left(r^*_{\mathcal{C}}(z) + \frac{2\pi i}{z}(j+1/2)\right) \leqslant \frac{1}{2\pi|z|^2}\left(x\,\mathrm{Cl}_2(2\pi x) + \pi^2|y|\left[\frac{1}{3} - 4(j+3/2)\right]\right) \tag{6.19}$$

when $y \leqslant 0$. *Also, in the box with* $3 + \delta \leqslant \mathrm{Re}(z) \leqslant 7/2 - \delta'$ *and* $-1 \leqslant \mathrm{Im}(z) \leqslant 1$,

$$q^*_{\mathcal{C}}(z), \quad \exp\big(v^*_{\mathcal{C}}(z; N, \sigma)\big) \ll 1 \tag{6.20}$$

for an implied constant depending only on $\sigma \in \mathbb{R}$.

By the calculus of residues,

$$\sum_{a\leqslant k\leqslant b,\ k\text{ odd}} (-1)^{(k+1)/2}\varphi(k) = \frac{1}{2}\int_C \frac{\varphi(z)}{2i\cos(\pi z/2)}\,dz$$

for $\varphi(z)$ a holomorphic function and C a positively oriented closed contour surrounding the interval $[a,b]$ and not surrounding any integers outside this interval. Hence

$$\sum_{a\leqslant k\leqslant b,\ k\text{ odd}} \frac{(-1)^{(k+1)/2}}{k^2}\varphi(2(N{+}1/2)/k) = \frac{-1}{4(N+1/2)}\int \frac{\varphi(z)}{2i\cos(\pi(N+1/2)/z)}\,dz, \tag{6.21}$$

for C now surrounding $\{2(N+1/2)/k \mid a\leqslant k\leqslant b\}$ with $a>0$. Therefore

$$\mathcal{C}_3^*(N,\sigma) = \frac{-1}{4(N+1/2)^{3/2}} \times \operatorname{Re}\int_{C_2} \exp\big((N+1/2)r_{\mathcal{C}}^*(z)\big)\frac{q_{\mathcal{C}}^*(z)}{2i\cos(\pi(N+1/2)/z)}\exp\big(v_{\mathcal{C}}^*(z;N,\sigma)\big)\,dz \tag{6.22}$$

where C_2 is the positively oriented rectangle with horizontal sides C_2^+, C_2^- having imaginary parts $1/N^2$, $-1/N^2$ and vertical sides $C_{2,L}$, $C_{2,R}$ having real parts $3+\delta$ and $7/2-\delta'$ respectively, as shown in Figure 4.

Arguing as in Proposition 5.5 proves the contribution to (6.22) from integrating over the vertical sides $C_{2,L}$, $C_{2,R}$ is $O(e^{0.016N})$. We have

$$\frac{1}{2i\cos(\pi(N+1/2)/z)} = -i\times\begin{cases}\sum_{j\leqslant 0}(-1)^j\exp\left(\frac{2\pi i}{z}(N+1/2)(j-1/2)\right) & \operatorname{Im}z>0\\ \sum_{j\geqslant 0}(-1)^j\exp\left(\frac{2\pi i}{z}(N+1/2)(j+1/2)\right) & \operatorname{Im}z<0.\end{cases} \tag{6.23}$$

Therefore

$$\begin{aligned}&-4(N+1/2)^{3/2}\mathcal{C}_3^*(N,\sigma) =\\ &\sum_{j\leqslant 0}(-1)^j\operatorname{Im}\int_{C_2^+}\exp\left((N+1/2)\left[r_{\mathcal{C}}^*(z)+\frac{2\pi i}{z}(j-1/2)\right]\right)q_{\mathcal{C}}^*(z)\exp\big(v_{\mathcal{C}}^*(z;N,\sigma)\big)dz\\ &+\sum_{j\geqslant 0}(-1)^j\operatorname{Im}\int_{C_2^-}\exp\left((N+1/2)\left[r_{\mathcal{C}}^*(z)+\frac{2\pi i}{z}(j+1/2)\right]\right)q_{\mathcal{C}}^*(z)\exp\big(v_{\mathcal{C}}^*(z;N,\sigma)\big)dz\\ &\qquad+O(e^{0.016N}).\end{aligned}$$

A similar proof to Proposition 5.6's, employing Theorem 6.5, shows that the total size of all but the $j=-1$, -2 terms above is $O(e^{0.013N})$. Let $d=j+2$ and we see

$p_d(z) = -(r^*_{\mathcal{C}}(z) + 2\pi i(j-1/2)/z)$ so that

$$4(N+1/2)^{3/2}\mathcal{C}^*_3(N,\sigma)$$
$$= \sum_{d=0,1} (-1)^d \mathrm{Im} \int_{3.01}^{3.49} \exp\bigl(-(N+1/2)p_d(z)\bigr) q^*_{\mathcal{C}}(z) \exp\bigl(v^*_{\mathcal{C}}(z;N,\sigma)\bigr)\, dz + O(e^{0.016N}). \tag{6.24}$$

6.3. Paths Through the Saddle-points

We treat the $d = 0$ case of (6.24) first. The unique solution to $p'(z) = 0$ for $5/2 < \mathrm{Re}(z) < 7/2$ is

$$z_2 := 3 + \frac{\log\bigl(1 - w(0,-3)\bigr)}{2\pi i} \approx 3.21625 + 0.402898i$$

by Theorem 1.9. Let $v = \mathrm{Im}(z_2)/\mathrm{Re}(z_2) \approx 0.125269$ and $c = 1 + iv$. The path we take through the saddle point z_2 is $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, the polygonal path between the points 3.01, $3.01c$, $3.49c$ and 3.49.

A similar proof to that of Theorem 5.8 shows that $\mathrm{Re}[p(z) - p(z_2)] > 0$ for $z \in \mathcal{R}$ except at $z = z_2$, as seen in Figure 7. Hence

$$\mathrm{Re}[-p(z)] \leqslant \mathrm{Re}[-p(z_2)] \approx 0.013764 \qquad (z \in \mathcal{R})$$

and it follows that the term corresponding to $d = 0$ in (6.24) is $O(e^{0.014N})$.

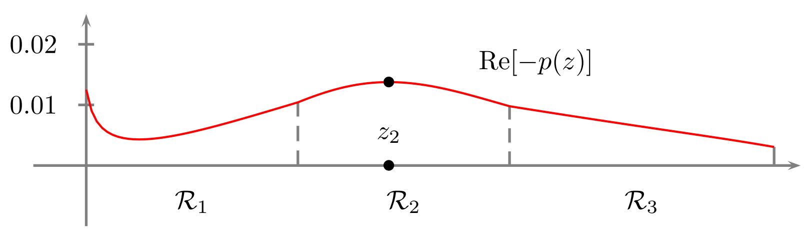

Figure 7: Graph of $\mathrm{Re}[-p(z)]$ for $z \in \mathcal{R}$

Define

$$\mathcal{C}^*_4(N,\sigma) := \frac{-1}{4(N+1/2)^{3/2}} \mathrm{Im} \int_{3.01}^{3.49} \exp\bigl(-(N+1/2)p_1(z)\bigr) q^*_{\mathcal{C}}(z) \exp\bigl(v^*_{\mathcal{C}}(z;N,\sigma)\bigr)\, dz \tag{6.25}$$

and we now know from (6.14), (6.24) and the above that

$$\mathcal{C}^*_2(N,\sigma) = \mathcal{C}^*_4(N,\sigma) + O(e^{WN/3}). \tag{6.26}$$

The unique solution to $p_1'(z) = 0$ for $5/2 < \mathrm{Re}(z) < 7/2$ is

$$z_3 := 3 + \frac{\log\bigl(1 - w(1,-3)\bigr)}{2\pi i} \approx 3.08382 - 0.0833451i$$

by Theorem 1.9. Let $v = \mathrm{Im}(z_3)/\mathrm{Re}(z_3) \approx -0.027027$ and $c = 1 + iv$. The path we take through the saddle point z_3 is $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, the polygonal path between the points 3.01, 3.01c, 3.49c and 3.49. A similar proof to that of Theorem 5.8 shows that $\mathrm{Re}[p_1(z) - p_1(z_3)] > 0$ for $z \in \mathcal{S}$ except at $z = z_3$. This is seen in Figure 8.

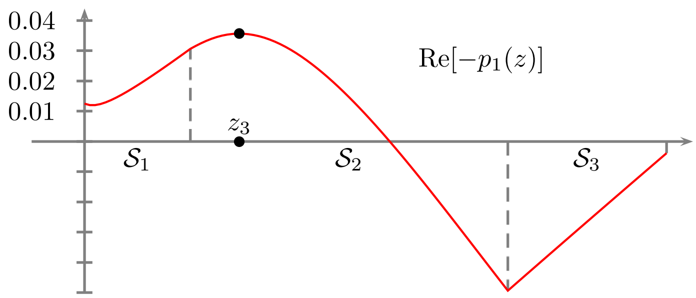

Figure 8: Graph of $\mathrm{Re}[-p_1(z)]$ for $z \in \mathcal{S}$

6.4. Applying the Saddle-point Method

Recall (6.16) and for $j \in \mathbb{Z}_{\geqslant 0}$ put

$$u^*_{\sigma,j}(z) := \sum_{m_1+3m_2+5m_3+\cdots=j} \frac{(\pi i(16\sigma+1)z/8 + g_{\mathcal{C},1}(z))^{m_1}}{m_1!} \frac{g_{\mathcal{C},2}(z)^{m_2}}{m_2!} \cdots \frac{g_{\mathcal{C},j}(z)^{m_j}}{m_j!},$$

with $u^*_{\sigma,0} = 1$. Similarly to Proposition 5.13 we have

Proposition 6.6. *For $3.01 \leqslant \mathrm{Re}(z) \leqslant 3.49$ and $|\mathrm{Im}(z)| \leqslant 1$, say, there is a holomorphic function $\zeta^*_d(z; N, \sigma)$ of z so that*

$$\exp\bigl(v^*_{\mathcal{C}}(z; N, \sigma)\bigr) = \sum_{j=0}^{d-1} \frac{u^*_{\sigma,j}(z)}{(N+1/2)^j} + \zeta^*_d(z; N, \sigma) \quad \text{for} \quad \zeta^*_d(z; N, \sigma) = O\left(\frac{1}{N^d}\right)$$

with an implied constant depending only on σ and d where $1 \leqslant d \leqslant 2L - 1$ and $L = \lfloor 0.006\pi e \cdot N/2 \rfloor$.

Theorem 6.7. *With $c^*_0 = -z_3 e^{-\pi i z_3}/4$ and explicit $c^*_1(\sigma)$, $c^*_2(\sigma), \dots$ depending on $\sigma \in \mathbb{Z}$ we have*

$$\mathcal{C}^*_2(N, \sigma) = \mathrm{Re}\left[\frac{w(1,-3)^{-N}}{N^2}\left(c^*_0 + \frac{c^*_1(\sigma)}{N} + \cdots + \frac{c^*_{m-1}(\sigma)}{N^{m-1}}\right)\right] + O\left(\frac{|w(1,-3)|^{-N}}{N^{m+2}}\right) \tag{6.27}$$

for an implied constant depending only on σ and m.

Proof. As in Theorem 5.14, applying the saddle-point method to (6.25), with the path of integration moved to $\mathcal{S}$, yields

$$\mathcal{C}_4^*(N,\sigma) = \mathrm{Re}\left[e^{-(N+1/2)\cdot p_1(z_3)} \sum_{t=0}^{d-2} \frac{1}{(N+1/2)^{t+2}} \sum_{s=0}^{t} \frac{\Gamma(s+1/2)}{2} a_{2s}(iq_{\mathcal{C}}^* \cdot u_{\sigma,t-s}^*)\right] + O\left(\frac{|w(1,-3)|^{-N}}{N^{d+1}}\right).$$

From (1.33) we know that $e^{p_1(z_3)} = w(1,-3)$. Hence, set

$$c_t^{**}(\sigma) := e^{-p_1(z_3)/2} \sum_{s=0}^{t} \Gamma(s+1/2) a_{2s}(iq_{\mathcal{C}}^* \cdot u_{\sigma,t-s}^*)/2. \tag{6.28}$$

We want to convert the above series in $1/(N+1/2)$ to one in $1/N$. With the Binomial Theorem we have $(1+z)^{-j} = \sum_{r=0}^{\infty} \binom{-j}{r} z^r$ for $|z| < 1$. Also, by Taylor's Theorem,

$$\frac{1}{(1+z)^j} = \sum_{r=0}^{m-1} \binom{-j}{r} z^r + O(z^m) \qquad (|z| < 1). \tag{6.29}$$

With $z = 1/(2N)$ above we find

$$\frac{\alpha_j}{(N+1/2)^{j+2}} = \sum_{r=0}^{\infty} \binom{-j-2}{r} \frac{2^{-r} \cdot \alpha_j}{N^{j+2+r}}$$

for any α_js, and can write

$$\frac{\alpha_0}{(N+1/2)^2} + \frac{\alpha_1}{(N+1/2)^3} + \cdots = \frac{\beta_0}{N^2} + \frac{\beta_1}{N^3} + \cdots$$

with

$$\beta_t = \sum_{j+r=t} \binom{-j-2}{r} 2^{-r} \cdot \alpha_j = \sum_{j=0}^{t} \binom{-j-2}{t-j} 2^{j-t} \cdot \alpha_j = \sum_{j=0}^{t} (-2)^{j-t} \binom{t+1}{j+1} \alpha_j.$$

So we set

$$c_t^*(\sigma) := \sum_{j=0}^{t} (-2)^{j-t} \binom{t+1}{j+1} c_j^{**}(\sigma)$$

and with (6.26) we obtain (6.27) in the statement of the theorem. Note that the omitted terms satisfy

$$\sum_{t=m}^{\infty} \frac{c_t^*(\sigma)}{N^t} = O\left(\frac{1}{N^m}\right)$$

by (6.29) and can be incorporated into the error term of (6.27).

A similar computation to that of c_0 in the proof of Theorem 5.14 shows that

$$\big(c_0^*(\sigma)\big)^2 = z_3^2 e^{-2\pi i z_3}/16$$

and a numerical check then indicates that the correct square root has a minus sign. □

For example, Table 2 compares both sides of (6.27) in Theorem 6.7 for some different values of m and N. This is for $\sigma = 1$ and the results for other values of σ are similar.

N	$m=1$	$m=2$	$m=4$	$\mathcal{C}_2^*(N,1)$
800	1.43938×10^6	1.39381×10^6	1.39341×10^6	1.39341×10^6
1000	1.7278×10^9	1.74062×10^9	1.74028×10^9	1.74028×10^9

Table 2: Theorem 6.7's approximations to $\mathcal{C}_2^*(N,1)$.

A consequence of Theorem 5.14 is that

$$\mathcal{C}_2(N,\sigma) = O(e^{U_{\mathcal{C}} N}/N^2) \qquad \text{for} \qquad U_{\mathcal{C}} := -\log|w(0,-2)| \approx 0.0256706. \tag{6.30}$$

Since $-\log|w(1,-3)| \approx 0.0356795$ we see that $\mathcal{C}_2(N,\sigma)$ is much smaller than $\mathcal{C}_2^*(N,\sigma)$ (despite appearances in Figure 3) and is bounded by the error term in (6.27). Therefore, Theorem 1.5 on the asymptotic expansion of $\mathcal{C}_1(N,\sigma) = \mathcal{C}_2(N,\sigma) + \mathcal{C}_2^*(N,\sigma)$ follows from Theorems 5.14 and 6.7.

7. The Sum $\mathcal{D}_1(N,\sigma)$

Let $\sigma \in \mathbb{Z}$. In this section we prove Theorem 1.6, giving the asymptotic expansion as $N \to \infty$ of

$$\mathcal{D}_1(N,\sigma) := \sum_{h/k \in \mathcal{D}(N)} Q_{hk\sigma}(N) = 2\mathrm{Re} \sum_{\frac{N}{2} < k \leqslant N,\ k \text{ odd}} Q_{(\frac{k-1}{2})k\sigma}(N).$$

With k odd, setting $h = (k-1)/2$ in Proposition 4.4 yields

$$Q_{(\frac{k-1}{2})k\sigma}(N) = \frac{1}{k^2} \exp\left(\frac{\pi i}{4}\left[\frac{N^2+N-4\sigma}{k}\right]\right) \times \exp\left(\frac{-\pi i}{4}\Big[(N-k)(N-k+1)-3k-3\Big]\right) \prod\nolimits_{N-k}^{-1}((k-1)/2k). \tag{7.1}$$

7.1. $\mathcal{D}_1(N,\sigma)$ for N Odd

If N is odd then $N-k$ is even and $(N-k)(N-k+1) \equiv (k-N) \bmod 8$. Hence (7.1) becomes

$$Q_{\left(\frac{k-1}{2}\right)k\sigma}(N) = \frac{1}{k^2}\exp\left(N\left[\frac{\pi i}{4}\left(\frac{N}{k}+1+\frac{2k}{N}\right)\right]\right) \times \exp\left(\frac{\pi i}{4}\left(\frac{N}{k}+3\right)\right)\exp\left(\frac{1}{N}\left[-\pi i\sigma\frac{N}{k}\right]\right)\prod_{N-k}^{-1}((k-1)/2k). \quad (7.2)$$

We next get $\prod_{N-k}^{-1}((k-1)/2k)$ into the right form to apply Proposition 4.5.

Proposition 7.1. *For k odd and m even with $0 \leqslant m < k$ we have*

$$\prod_{m}^{-1}((k-1)/2k) = \frac{\prod_{m}^{-1}(1/k)}{\prod_{m}^{-1}(1/2k)} \times \frac{\prod_{m/2}^{-2}(1/k)}{\prod_{m/2}^{-1}(2/k)}. \quad (7.3)$$

Proof. Since

$$\sin(\pi j(k-1)/2k) = \begin{cases} (-1)^{j/2+1}\sin(\pi j/2k) & j \text{ even} \\ (-1)^{(j-1)/2}\cos(\pi j/2k) & j \text{ odd} \end{cases}$$

we have

$$\prod_{m}^{-1}((k-1)/2k) = \prod_{\substack{1\leqslant j\leqslant m \\ j \text{ even}}} \frac{(-1)^{j/2+1}}{2\sin(\pi j/2k)} \prod_{\substack{1\leqslant j\leqslant m \\ j \text{ odd}}} \frac{(-1)^{(j-1)/2}}{2\cos(\pi j/2k)}. \quad (7.4)$$

Hence

$$\begin{aligned}\prod_{m}^{-1}((k-1)/2k) &= \prod_{m/2}^{-1}(1/k) \prod_{\substack{1\leqslant j\leqslant m \\ j \text{ odd}}} \frac{1}{2\cos(\pi j/2k)} \\ &= \prod_{m/2}^{-1}(1/k) \prod_{1\leqslant j\leqslant m} \frac{1}{2\cos(\pi j/2k)} \Bigg/ \prod_{1\leqslant j\leqslant m/2} \frac{1}{2\cos(\pi j/k)}.\end{aligned} \quad (7.5)$$

Use the identity $2\sin 2\theta = 2\sin\theta \cdot 2\cos\theta$ to convert the cosines in (7.5) back to sines and complete the proof. □

Recall the definition of $g_\ell(z)$ in (5.4), define

$$g^*_\ell(z) := -\frac{B_{2\ell}}{(2\ell)!}(\pi z/2)^{2\ell-1}\cot^{(2\ell-2)}(\pi(z-1)/2) \quad (7.6)$$

and set $\hat{z} = \hat{z}(N,k) := N/k$. The sine product approximation we need is as follows.

Theorem 7.2. *Fix $W > 0$. Let Δ be in the range $0.0048 \leqslant \Delta \leqslant 0.0079$ and set $\alpha = \Delta\pi e$. Suppose δ and δ' satisfy*

$$\frac{\Delta}{1-\Delta} < \delta \leqslant \frac{1}{e}, \ 0 < \delta' \leqslant \frac{1}{e} \quad \text{and} \quad \delta \log 1/\delta, \ \ \delta' \log 1/\delta' \leqslant W.$$

Then for all N odd $\geqslant 2 \cdot R_\Delta$ we have

$$\textstyle\prod_{N-k}^{-1}((k-1)/2k) = O\left(e^{WN/2}\right) \quad \text{for} \quad \hat{z} \in [1, \ 1+\delta] \cup [3/2-\delta', \ 2) \tag{7.7}$$

and

$$\begin{aligned} &\textstyle\prod_{N-k}^{-1}((k-1)/2k) \\ &\quad = \exp\left(N\frac{\mathrm{Cl}_2(2\pi\hat{z})}{4\pi\hat{z}}\right)\left(\frac{\hat{z}}{2N\sin(\pi(\hat{z}-1)/2)}\right)^{1/2}\exp\left(\sum_{\ell=1}^{L-1}\frac{g_\ell(\hat{z})-g_\ell^*(\hat{z})}{N^{2\ell-1}}\right) \\ &\qquad \times \exp\left(\sum_{\ell=1}^{L^*-1}\frac{2g_\ell^*(\hat{z})-g_\ell(\hat{z})}{(N/2)^{2\ell-1}}\right) + O\left(e^{WN/2}\right) \quad \text{for} \quad \hat{z} \in (1+\delta, \ 3/2-\delta') \end{aligned} \tag{7.8}$$

with $L = \lfloor \alpha \cdot N \rfloor$ and $L^* = \lfloor \alpha \cdot N/2 \rfloor$. The implied constants in (7.7), (7.8) are absolute.

Proof. Applying Proposition 4.5 to each of the factors on the right of (7.3) shows

$$\begin{aligned} \textstyle\prod_m^{-1}((k-1)/2k) &= \left(\frac{1}{2k\sin(\pi m/(2k))}\right)^{1/2}\exp\left(\frac{k}{4\pi}\,\mathrm{Cl}_2(2\pi m/k)\right) \\ &\quad \times \exp\left(-\sum_{\ell=1}^{L_1-1}\frac{B_{2\ell}}{(2\ell)!}\left(\frac{\pi}{k}\right)^{2\ell-1}\cot^{(2\ell-2)}\left(\frac{\pi m}{k}\right)\right)\exp\left(-T_{L_1}(m,1/k)\right) \\ &\quad \times \exp\left(-2\sum_{\ell=1}^{L_2-1}\frac{B_{2\ell}}{(2\ell)!}\left(\frac{\pi}{k}\right)^{2\ell-1}\cot^{(2\ell-2)}\left(\frac{\pi m}{2k}\right)\right)\exp\left(-2T_{L_2}(m/2,1/k)\right) \\ &\quad \times \exp\left(\sum_{\ell=1}^{L_3-1}\frac{B_{2\ell}}{(2\ell)!}\left(\frac{2\pi}{k}\right)^{2\ell-1}\cot^{(2\ell-2)}\left(\frac{\pi m}{k}\right)\right)\exp\left(T_{L_3}(m/2,2/k)\right) \\ &\quad \times \exp\left(\sum_{\ell=1}^{L_4-1}\frac{B_{2\ell}}{(2\ell)!}\left(\frac{\pi}{2k}\right)^{2\ell-1}\cot^{(2\ell-2)}\left(\frac{\pi m}{2k}\right)\right)\exp\left(T_{L_4}(m,1/(2k))\right) \end{aligned} \tag{7.9}$$

for $1 \leqslant m < k$ and positive integers L_1, L_2, L_3, L_4.

First we set each L_i to 1 in (7.9) to see

$$\begin{aligned} \textstyle\prod_m^{-1}((k-1)/2k) &= \left(\frac{1}{2k\sin(\pi m/(2k))}\right)^{1/2}\exp\left(\frac{k}{4\pi}\,\mathrm{Cl}_2(2\pi m/k)\right) \\ &\quad \times \exp\left(-T_1(m,1/k) - 2T_1(m/2,1/k) + T_1(m/2,2/k) + T_1(m,1/(2k))\right). \end{aligned} \tag{7.10}$$

Comparing (7.10) with the expansion of $\prod_{m/2}^{-1}(2/k)$ from Proposition 4.5 then shows

$$\begin{aligned}\prod\nolimits_m^{-1}((k-1)/2k) &= (\cos(\pi m/(2k)))^{1/2}\\ &\quad\times \exp\left(-T_1(m,1/k)-2T_1(m/2,1/k)+2T_1(m/2,2/k)+T_1(m,1/(2k))\right)\\ &\quad\times \prod\nolimits_{m/2}^{-1}(2/k). \quad (7.11)\end{aligned}$$

It follows from (7.11) and Lemma 4.6 that for $0 \leqslant m < k$,

$$\prod\nolimits_m^{-1}((k-1)/2k) \ll \prod\nolimits_{m/2}^{-1}(2/k) \quad (7.12)$$

with an absolute implied constant. Similarly, by comparing (7.10) with the expansion of $\prod_m^{-1}(1/k)$ from Proposition 4.5,

$$\prod\nolimits_m^{-1}((k-1)/2k) \ll \Big(\prod\nolimits_m^{-1}(1/k)\Big)^{1/2}. \quad (7.13)$$

Using Proposition 4.7 to bound $\prod_{m/2}^{-1}(2/k)$ on the right of (7.12) and noting that $k \leqslant N$ proves (7.7).

To prove (7.8) we wish to apply the argument of Proposition 4.10 to (7.9). This requires finding L_1, L_2, L_3 and L_4 so that, for $m = N-k$,

$$\begin{aligned}&\prod\nolimits_m^{-1}((k-1)/2k)\\ &\times\Big(|T_{L_1}(m,1/k)|+|T_{L_2}(m/2,1/k)|+|T_{L_3}(m/2,2/k)|+|T_{L_4}(m,1/(2k))|\Big) \ll e^{WN/2}\end{aligned} \quad (7.14)$$

and

$$|T_{L_1}(m,1/k)|+|T_{L_2}(m/2,1/k)|+|T_{L_3}(m/2,2/k)|+|T_{L_4}(m,1/(2k))| \ll 1. \quad (7.15)$$

We examine the four terms T_{L_i} in (7.14) and (7.15) separately:

- **The term** $T_{L_3}(m/2,2/k)$**.** Use Proposition 4.8 with $h=2$ and $s=N$ to show that, for $\Delta N/2 \leqslant m/2 \leqslant k/4$,

$$\Big|\prod\nolimits_{m/2}^{-1}(2/k)\cdot T_{L_3}(m/2,2/k)\Big| \ll e^{WN/2} \quad (7.16)$$

$$|T_{L_3}(m/2,2/k)| \ll 1 \quad (7.17)$$

with absolute implied constants, $L_3 := \lfloor \pi e\Delta\cdot N/2\rfloor$ and $N \geqslant 2\cdot R_\Delta$. Inequality (7.16) is valid with $\prod_{m/2}^{-1}(2/k)$ replaced by $\prod_m^{-1}((k-1)/2k)$ using (7.12):

$$\Big|\prod\nolimits_m^{-1}((k-1)/2k)\cdot T_{L_3}(m/2,2/k)\Big| \ll e^{WN/2}. \quad (7.18)$$

- **The term $T_{L_2}(m/2, 1/k)$.** To prove

$$\left|\prod_m^{-1}((k-1)/2k) \cdot T_{L_2}(m/2, 1/k)\right| \ll e^{WN/2} \tag{7.19}$$

$$|T_{L_2}(m/2, 1/k)| \ll 1 \tag{7.20}$$

for $\Delta N/2 \leqslant m/2 \leqslant k/4$, choose $L_2 = L_3$ and note that (7.16) and (7.17) are valid with $2/k$ replaced by $1/k$ using Corollary 4.9.

- **The term $T_{L_1}(m, 1/k)$.** Use Proposition 4.8 with $h = 1$ and $s = N$ to show that, also for $\Delta N \leqslant m \leqslant k/2$,

$$\left|\prod_m^{-1}(1/k) \cdot T_{L_1}(m, 1/k)\right| \ll e^{WN} \tag{7.21}$$

$$|T_{L_1}(m, 1/k)| \ll 1 \tag{7.22}$$

with absolute implied constants, $L_1 := \lfloor \pi e \Delta \cdot N \rfloor$ and $N \geqslant R_\Delta$. Taking square roots of both sides of (7.21) and using (7.13) shows

$$\left|\prod_m^{-1}((k-1)/2k) \cdot T_{L_1}(m, 1/k)\right| \ll e^{WN/2}. \tag{7.23}$$

- **The term $T_{L_4}(m, 1/(2k))$.** To prove

$$\left|\prod_m^{-1}((k-1)/2k) \cdot T_{L_4}(m, 1/(2k))\right| \ll e^{WN/2} \tag{7.24}$$

$$|T_{L_4}(m, 1/(2k))| \ll 1 \tag{7.25}$$

for $\Delta N \leqslant m \leqslant k/2$, choose $L_4 = L_1$ and note that (7.21) and (7.22) are valid with $1/k$ replaced by $1/(2k)$ using Corollary 4.9.

The inequalities (7.17) - (7.25) establish (7.14), (7.15) and the arguments of Proposition 4.10 now go through, applied to (7.9). This allows us to remove the $\exp(T_{L_i})$ factors in (7.9) at the expense of adding an $O(e^{WN/2})$ error. Write L for L_1, L_4 and L^* for L_2, L_3. The interval $\Delta N \leqslant m \leqslant k/2$ corresponds to

$$1 + \frac{\Delta}{1-\Delta} \leqslant \hat{z} \leqslant 3/2.$$

This completes the proof of (7.8). □

It simplifies things to work with the conjugate of (7.2):

$$\overline{Q_{\left(\frac{k-1}{2}\right)k\sigma}(N)} = \frac{1}{k^2} \exp\left(N\left[\frac{-\pi i}{4}\left(\hat{z} + 1 + \frac{2}{\hat{z}}\right)\right]\right)$$
$$\times \exp\left(\frac{-\pi i}{4}(\hat{z}+3)\right) \exp\left(\frac{\pi i \sigma \hat{z}}{N}\right) \prod_{N-k}^{-1}((k-1)/2k). \tag{7.26}$$

From (4.2) we have

$$\mathrm{Cl}_2(2\pi z) = -i\,\mathrm{Li}_2(e^{2\pi iz}) + i\pi^2(z^2 - 3z + 13/6) \qquad (1 < z < 2)$$

so that

$$\frac{\mathrm{Cl}_2(2\pi z)}{4\pi z} - \frac{\pi i}{4}\left(z + 1 + \frac{2}{z}\right) = -\pi i + \frac{1}{4\pi iz}\Big[\mathrm{Li}_2(e^{2\pi iz}) - \mathrm{Li}_2(1)\Big].$$

Set

$$r_{\mathcal{D}}(z) := \frac{1}{4\pi iz}\Big[\mathrm{Li}_2(e^{2\pi iz}) - \mathrm{Li}_2(1)\Big] \tag{7.27}$$

$$q_{\mathcal{D}}(z) := \left(\frac{z}{2\sin(\pi(z-1)/2)}\right)^{1/2} \exp\left(-\frac{\pi i}{4}(z+3)\right)$$

$$v_{\mathcal{D}}(z; N, \sigma) := \frac{\pi i\sigma z}{N} + \sum_{\ell=1}^{L-1} \frac{g_\ell(z) - g_\ell^*(z)}{N^{2\ell-1}} + \sum_{\ell=1}^{L^*-1} \frac{2g_\ell^*(z) - g_\ell(z)}{(N/2)^{2\ell-1}} \tag{7.28}$$

for $L = \lfloor \alpha \cdot N \rfloor$ and $L^* = \lfloor \alpha \cdot N/2 \rfloor$. With N odd, define

$$\mathcal{D}_2(N, \sigma) := \frac{-2}{N^{1/2}}\mathrm{Re} \sum_{k \text{ odd } : \ \hat{z} \in (1+\delta,\ 3/2-\delta')} \frac{1}{k^2} \exp\big(N \cdot r_{\mathcal{D}}(\hat{z})\big) q_{\mathcal{D}}(\hat{z}) \exp\big(v_{\mathcal{D}}(\hat{z}; N, \sigma)\big). \tag{7.29}$$

It follows from (7.26) and Theorem 7.2 that for $\sigma \in \mathbb{Z}$ and an absolute implied constant

$$\mathcal{D}_1(N, \sigma) = \mathcal{D}_2(N, \sigma) + O(e^{WN/2}) \qquad (N \text{ odd}). \tag{7.30}$$

7.2. Expressing $\mathcal{D}_2(N, \sigma)$ as an Integral for N Odd

Similarly to Proposition 5.2 we have

Proposition 7.3. *Suppose* $1/2 \leqslant \mathrm{Re}(z) \leqslant 3/2$ *and* $|z - 1| \geqslant \varepsilon > 0$ *and assume* $\max\{1 + \frac{1}{\varepsilon},\ 16\} < \frac{\pi e}{\alpha}$. *Then*

$$\sum_{\ell=d}^{L-1} \frac{g_\ell(z) - g_\ell^*(z)}{N^{2\ell-1}} + \sum_{\ell=d}^{L^*-1} \frac{2g_\ell^*(z) - g_\ell(z)}{(N/2)^{2\ell-1}} \ll \frac{1}{N^{2d-1}} e^{-\pi|y|/2} \tag{7.31}$$

for $d \geqslant 2$ *where* $L = \lfloor \alpha \cdot N \rfloor$, $L^* = \lfloor \alpha \cdot N/2 \rfloor$ *and the implied constant depends only on* ε, α *and* d.

Fixing the choice of constants in (5.17) and with $\varepsilon = 0.0061$ and

$$g_{\mathcal{D},\ell}(z) := g_\ell(z) - g_\ell^*(z) + 2^{2\ell-1}\big(2g_\ell^*(z) - g_\ell(z)\big) \tag{7.32}$$

we obtain:

Corollary 7.4. *With $\delta, \delta' \in [0.0061, 0.01]$ and $z \in \mathbb{C}$ such that $1+\delta < \mathrm{Re}(z) < 3/2-\delta'$ we have*

$$v_{\mathcal{D}}(z;N,\sigma) = \frac{\pi i \sigma z}{N} + \sum_{\ell=1}^{d-1} \frac{g_{\mathcal{D},\ell}(z)}{N^{2\ell-1}} + O\left(\frac{1}{N^{2d-1}}\right)$$

for $2 \leqslant d \leqslant L^* = \lfloor 0.006\pi e \cdot N/2 \rfloor$ and an implied constant depending only on d.

Similarly to Theorem 5.4 we have

Theorem 7.5. *The functions $r_{\mathcal{D}}(z)$, $q_{\mathcal{D}}(z)$ and $v_{\mathcal{D}}(z;N,\sigma)$ are holomorphic for $1 < \mathrm{Re}(z) < 3/2$. In this strip*

$$\mathrm{Re}\left(r_{\mathcal{D}}(z) + \frac{\pi i j}{z}\right) \leqslant \frac{1}{4\pi |z|^2}\left(x\,\mathrm{Cl}_2(2\pi x) + \pi^2 |y| \left[\frac{1}{3} + 4j\right]\right) \qquad (y \geqslant 0) \tag{7.33}$$

$$\mathrm{Re}\left(r_{\mathcal{D}}(z) + \frac{\pi i j}{z}\right) \leqslant \frac{1}{4\pi |z|^2}\left(x\,\mathrm{Cl}_2(2\pi x) + \pi^2 |y| \left[\frac{1}{3} - 4(j-1/2)\right]\right) \qquad (y \leqslant 0). \tag{7.34}$$

for $j \in \mathbb{R}$. Also, in the box with $1+\delta \leqslant \mathrm{Re}(z) \leqslant 3/2-\delta'$ and $-1 \leqslant \mathrm{Im}(z) \leqslant 1$,

$$q_{\mathcal{D}}(z), \quad \exp\bigl(v_{\mathcal{D}}(z;N,\sigma)\bigr) \ll 1 \tag{7.35}$$

for an implied constant depending only on $\sigma \in \mathbb{R}$.

Let C be the positively oriented rectangle with horizontal sides C^+, C^- having imaginary parts $1/N^2$, $-1/N^2$ and vertical sides C_L, C_R having real parts $1+\delta$ and $3/2-\delta'$ respectively, as used in [9, Sect. 4.4]. Recalling (5.22), (5.29) and arguing as in Proposition 5.5, we find

$$\begin{aligned}
&\mathcal{D}_2(N,\sigma) \\
&= \frac{(-2)}{N^{1/2}} \frac{(-1)}{2N} \mathrm{Re} \int_C \exp\bigl(N \cdot r_{\mathcal{D}}(z)\bigr) \frac{q_{\mathcal{D}}(z)}{2i \tan(\pi(N/z-1)/2)} \exp\bigl(v_{\mathcal{D}}(z;N,\sigma)\bigr)\, dz \\
&= \frac{1}{N^{3/2}} \mathrm{Re}\Biggl[{\sum_{j \leqslant 0}}' (-1)^j \int_{C^+} \exp\bigl(N[r_{\mathcal{D}}(z) + \pi i j/z]\bigr) q_{\mathcal{D}}(z) \exp\bigl(v_{\mathcal{D}}(z;N,\sigma)\bigr)\, dz \\
&\qquad - {\sum_{j \geqslant 0}}' (-1)^j \int_{C^-} \exp\bigl(N[r_{\mathcal{D}}(z) + \pi i j/z]\bigr) q_{\mathcal{D}}(z) \exp\bigl(v_{\mathcal{D}}(z;N,\sigma)\bigr)\, dz \Biggr] \\
&\qquad\qquad + O(e^{WN/2}).
\end{aligned}$$

With Theorem 7.5, and reasoning as in Proposition 5.6, we see that the two $j=0$ terms above dominate and $\mathcal{D}_2(N,\sigma) = \mathcal{D}_3(N,\sigma) + O(e^{WN/2})$ for

$$\mathcal{D}_3(N,\sigma) := \frac{-1}{N^{3/2}} \mathrm{Re} \int_{1.01}^{1.49} \exp\bigl(-N \cdot p(z)/2\bigr) q_{\mathcal{D}}(z) \exp\bigl(v_{\mathcal{D}}(z;N,\sigma)\bigr)\, dz \qquad (N \text{ odd}) \tag{7.36}$$

since $r_{\mathcal{D}}(z) = -p(z)/2$.

7.3. $\mathcal{D}_1(N-1,\sigma)$ for N Odd

Let N be odd. If v is even then $v-k$ is odd and $(v-k)(v-k+1) \equiv (v-k+1) \bmod 8$. Hence, with $v = N-1$, the conjugate of (7.1) becomes

$$\overline{Q_{(\frac{k-1}{2})k\sigma}(N-1)} = \frac{1}{k^2}\exp\left(N\left[\frac{-\pi i}{4}\left(\hat{z}-1+\frac{4}{\hat{z}}\right)\right]\right) \times \exp\left(\frac{\pi i}{4}(\hat{z}-3)\right)\exp\left(\frac{\pi i\sigma\hat{z}}{N}\right)\prod\nolimits_{N-1-k}^{-1}((k-1)/2k). \quad (7.37)$$

For m even, (7.4) implies

$$\prod\nolimits_{m-1}^{-1}((k-1)/(2k)) = 2(-1)^{m/2+1}\sin(\pi m/(2k))\prod_{\substack{1\leqslant j\leqslant m\\ j \text{ even}}}\frac{(-1)^{j/2+1}}{2\sin(\pi j/2k)}\prod_{\substack{1\leqslant j\leqslant m\\ j \text{ odd}}}\frac{(-1)^{(j-1)/2}}{2\cos(\pi j/2k)} = 2(-1)^{m/2+1}\sin(\pi m/(2k))\cdot\prod\nolimits_{m}^{-1}((k-1)/(2k)).$$

It follows that for N odd we have

$$\prod\nolimits_{N-1-k}^{-1}((k-1)/(2k)) = 2(-1)^{(N-k)/2+1}\sin(\pi(N/k-1)/2)\cdot\prod\nolimits_{N-k}^{-1}((k-1)/(2k)) \quad (7.38)$$

and can use our results from the last subsection. Recall $r_{\mathcal{D}}(z)$ and $v_{\mathcal{D}}(z;N,\sigma)$ from (7.27), (7.28) and set

$$q^*_{\mathcal{D}}(z) := 2\sin(\pi(z-1)/2)\left(\frac{z}{2\sin(\pi(z-1)/2)}\right)^{1/2}\exp\left(\frac{\pi i}{4}(z-1)\right).$$

With N odd, define

$$\mathcal{D}_2(N-1,\sigma) := \frac{-2}{N^{1/2}} \times \mathrm{Re}\sum_{k \text{ odd }:\ \hat{z}\in(1+\delta,\ 3/2-\delta')}\frac{(-1)^{(k+1)/2}}{k^2}\exp\left(N\left[r_{\mathcal{D}}(\hat{z})-\frac{\pi i}{2\hat{z}}\right]\right)q^*_{\mathcal{D}}(\hat{z})\exp\bigl(v_{\mathcal{D}}(\hat{z};N,\sigma)\bigr).$$

It follows from (7.37), (7.38) and Theorem 7.2 that

$$\mathcal{D}_1(N-1,\sigma) = \mathcal{D}_2(N-1,\sigma) + O(e^{WN/2}) \qquad (N \text{ odd}).$$

The next result is mostly a restatement of Theorem 7.5.

Theorem 7.6. *The functions $r_{\mathcal{D}}(z) - \frac{\pi i}{2z}$, $q^*_{\mathcal{D}}(z)$ and $v_{\mathcal{D}}(z;N,\sigma)$ are holomorphic for $1 < \mathrm{Re}(z) < 3/2$. In this strip, for $j \in \mathbb{R}$,*

$$\mathrm{Re}\left(r_{\mathcal{D}}\left(z\right) - \frac{\pi i}{2z} + \frac{\pi i(j-1/2)}{z}\right) \leqslant \frac{1}{4\pi|z|^2}\left(x\,\mathrm{Cl}_2(2\pi x) + \pi^2|y|\left[\frac{1}{3} + 4(j-1)\right]\right) \tag{7.39}$$

when $y \geqslant 0$ and

$$\mathrm{Re}\left(r_{\mathcal{D}}\left(z\right) - \frac{\pi i}{2z} + \frac{\pi i(j+1/2)}{z}\right) \leqslant \frac{1}{4\pi|z|^2}\left(x\,\mathrm{Cl}_2(2\pi x) + \pi^2|y|\left[\frac{1}{3} - 4(j-1/2)\right]\right) \tag{7.40}$$

when $y \leqslant 0$. Also, in the box with $1+\delta \leqslant \mathrm{Re}(z) \leqslant 3/2 - \delta'$ and $-1 \leqslant \mathrm{Im}(z) \leqslant 1$,

$$q^*_{\mathcal{D}}(z), \quad \exp\bigl(v_{\mathcal{D}}(z;N,\sigma)\bigr) \ll 1 \tag{7.41}$$

for an implied constant depending only on $\sigma \in \mathbb{R}$.

With the rectangle C from the last subsection and recalling (6.21), (6.23)

$$\begin{aligned}
&\mathcal{D}_2(N-1,\sigma) = \\
&\frac{(-2)}{N^{1/2}}\frac{(-1)}{2N}\mathrm{Re}\int_C \exp\left(N\left[r_{\mathcal{D}}(z) - \frac{\pi i}{2z}\right]\right)\frac{q^*_{\mathcal{D}}(z)}{2i\cos(\pi N/(2z))}\exp\bigl(v_{\mathcal{D}}(z;N,\sigma)\bigr)\,dz = \\
&\frac{-1}{N^{3/2}}\mathrm{Re}\left[i\sum_{j\leqslant 0}(-1)^j\int_{C^+}\exp\left(N\left[r_{\mathcal{D}}\left(z\right) - \frac{\pi i}{2z} + \frac{\pi i(j-1/2)}{z}\right]\right)q^*_{\mathcal{D}}(z)\exp\bigl(v_{\mathcal{D}}(z;N,\sigma)\bigr)\,dz\right. \\
&\qquad \left. + i\sum_{j\geqslant 0}(-1)^j\int_{C^-}\exp\left(N\left[r_{\mathcal{D}}\left(z\right) - \frac{\pi i}{2z} + \frac{\pi i(j+1/2)}{z}\right]\right)q^*_{\mathcal{D}}(z)\exp\bigl(v_{\mathcal{D}}(z;N,\sigma)\bigr)\,dz\right] \\
&\qquad + O(e^{WN/2}).
\end{aligned}$$

With (7.39), (7.40) we see the $j=0$ term on C^- dominates and in this way $\mathcal{D}_2(N-1,\sigma) = \mathcal{D}_3(N-1,\sigma) + O(e^{WN/2})$ where we define, for N odd,

$$\mathcal{D}_3(N-1,\sigma) := \frac{-1}{N^{3/2}}\mathrm{Re}\int_{1.01}^{1.49}\exp\bigl(-N\cdot p(z)/2\bigr)iq^*_{\mathcal{D}}(z)\exp\bigl(v_{\mathcal{D}}(z;N,\sigma)\bigr)\,dz. \tag{7.42}$$

Thus, with the definitions (7.36) and (7.42) we have shown that for all N

$$\mathcal{D}_1(N,\sigma) = \mathcal{D}_3(N,\sigma) + O(e^{WN/2}). \tag{7.43}$$

7.4. The Asymptotic Behavior of $\mathcal{D}_1(N,\sigma)$

Recall (7.32) and for $j \in \mathbb{Z}_{\geqslant 0}$ put

$$u_{\mathcal{D},\sigma,j}(z) := \sum_{m_1+3m_2+5m_3+\cdots=j}\frac{(\pi i\sigma z + g_{\mathcal{D},1}(z))^{m_1}}{m_1!}\frac{g_{\mathcal{D},2}(z)^{m_2}}{m_2!}\cdots\frac{g_{\mathcal{D},j}(z)^{m_j}}{m_j!},$$

with $u_{\mathcal{D},\sigma,0} = 1$. The proof of the next proposition is similar to Proposition 5.13's and uses Corollary 7.4.

Proposition 7.7. *For* $1.01 \leqslant \operatorname{Re}(z) \leqslant 1.49$ *and* $|\operatorname{Im}(z)| \leqslant 1$*, say, there is a holomorphic function* $\zeta_{\mathcal{D},d}(z; N, \sigma)$ *of* z *so that*

$$\exp\bigl(v_{\mathcal{D}}(z; N, \sigma)\bigr) = \sum_{j=0}^{d-1} \frac{u_{\mathcal{D},\sigma,j}(z)}{N^j} + \zeta_{\mathcal{D},d}(z; N, \sigma) \quad \textit{for} \quad \zeta_{\mathcal{D},d}(z; N, \sigma) = O\left(\frac{1}{N^d}\right)$$

with an implied constant depending only on σ *and* d *where* $1 \leqslant d \leqslant 2L^* - 1$ *and* $L^* = \lfloor 0.006\pi e \cdot N/2 \rfloor$*.*

We restate Theorem 1.6 here. Recall that $z_0 = 1 + \log(1 - w_0)/(2\pi i)$ where w_0 is the dilogarithm zero $w(0, -1)$.

Theorem 1.6. Let $\overline{N}$ denote N modulo 2. With

$$d_0\bigl(\overline{N}\bigr) = z_0 \sqrt{2e^{-\pi i z_0}\bigl(e^{-\pi i z_0} + (-1)^N\bigr)} \tag{7.44}$$

and explicit $d_1\bigl(\sigma, \overline{N}\bigr)$, $d_2\bigl(\sigma, \overline{N}\bigr), \dots$ depending on $\sigma \in \mathbb{Z}$ and $\overline{N}$, we have

$$\mathcal{D}_1(N, \sigma) = \operatorname{Re}\left[\frac{w_0^{-N/2}}{N^2}\left(d_0\bigl(\overline{N}\bigr) + \frac{d_1\bigl(\sigma, \overline{N}\bigr)}{N} + \cdots + \frac{d_{m-1}\bigl(\sigma, \overline{N}\bigr)}{N^{m-1}}\right)\right] + O\left(\frac{|w_0|^{-N/2}}{N^{m+2}}\right) \tag{7.45}$$

for an implied constant depending only on σ and m.

Proof. Let $v = \operatorname{Im}(z_0)/\operatorname{Re}(z_0) \approx 0.216279$ and $c = 1 + iv$. We replace the path of integration $[1.01, 1.49]$ in (7.36) and (7.42) with the path $\mathcal{P}$ through z_0 made up of the lines joining 1.01, $1.01c$, $1.49c$ and 1.49. This path is used in [9, Sect. 5.2] and it is proved there that $\operatorname{Re}(p(z) - p(z_0)) > 0$ for $z \in \mathcal{P}$ except at $z = z_0$.

For N odd, applying the saddle-point method to (7.36), as in Theorem 5.14, gives

$$\mathcal{D}_3(N, \sigma) = \operatorname{Re}\left[e^{-Np(z_0)/2} \sum_{t=0}^{d-2} \frac{-2}{N^{t+2}} \sum_{s=0}^{t} \Gamma(s + 1/2) a_{2s}(q_{\mathcal{D}} \cdot u_{\mathcal{D},\sigma,t-s})\right] + O\left(\frac{|w_0|^{-N/2}}{N^{d+1}}\right).$$

Therefore we set

$$d_t\bigl(\sigma, \overline{N}\bigr) := -2 \sum_{s=0}^{t} \Gamma(s + 1/2) a_{2s}(q_{\mathcal{D}} \cdot u_{\mathcal{D},\sigma,t-s}) \qquad (N \text{ odd}). \tag{7.46}$$

Since $\sqrt{w_0} = e^{p(z_0)/2}$ and (7.43) is true, we obtain (7.45) in the statement of the theorem in this odd case.

For N even, (7.42) implies

$$\mathcal{D}_3(N,\sigma)$$
$$= \frac{-1}{(N+1)^{3/2}} \mathrm{Re} \int_{1.01}^{1.49} \exp\bigl(-(N+1)\cdot p(z)/2\bigr) i q^*_{\mathcal{D}}(z) \exp\bigl(v_{\mathcal{D}}(z; N+1, \sigma)\bigr)\, dz$$

and applying the saddle-point method yields

$$\mathcal{D}_3(N,\sigma) = \mathrm{Re}\left[e^{-(N+1)p(z_0)/2} \sum_{t=0}^{d-2} \frac{-2}{(N+1)^{t+2}} \sum_{s=0}^{t} \Gamma(s+1/2) a_{2s}(i q^*_{\mathcal{D}} \cdot u_{\mathcal{D},\sigma,t-s}) \right] + O\left(\frac{|w_0|^{-N/2}}{N^{d+1}} \right).$$

Define

$$d^*_t(\sigma) := -2e^{-p(z_0)/2} \sum_{s=0}^{t} \Gamma(s+1/2) a_{2s}(i q^*_{\mathcal{D}} \cdot u_{\mathcal{D},\sigma,t-s}) \tag{7.47}$$

and we want to convert the above series in $1/(N+1)$ to one in $1/N$. The method to do this is given in the proof of Theorem 6.7. Let

$$d_t\bigl(\sigma, \overline{N}\bigr) := \sum_{j=0}^{t} (-1)^{t-j} \binom{t+1}{j+1} d^*_j(\sigma) \qquad (N \text{ even}) \tag{7.48}$$

and with (7.43) we obtain (7.45) in the statement of the theorem in this even case.

To calculate $d_0\bigl(\sigma, \overline{N}\bigr)$, we begin with N odd and see from (7.46) and (1.30) that

$$d_0\bigl(\sigma, \overline{N}\bigr) = -2\sqrt{\pi} a_0(q_{\mathcal{D}} \cdot 1) = -2\sqrt{\pi} \frac{\omega}{2(\omega^2 p_0/2)^{1/2}} q_0$$

for $q_0 = q_{\mathcal{D}}(z_0)$, $p_0 = p''(z_0)/2$ and the direction $\omega = z_0$. Short computations (see (5.44)) provide

$$p_0 = \frac{\pi i}{z_0(1 - e^{-2\pi i z_0})}, \qquad q_0^2 = \frac{-i z_0 e^{-\pi i z_0}}{e^{-\pi i z_0} + 1}$$

so that

$$d_0\bigl(\sigma, \overline{N}\bigr)^2 = 2 z_0^2 e^{-\pi i z_0}\bigl(e^{-\pi i z_0} - 1\bigr) \qquad (N \text{ odd})$$

and (7.44) follows in this case. The N even case is similar: from (7.47), (7.48) and (1.30)

$$d_0\bigl(\sigma, \overline{N}\bigr) = -2e^{-p(z_0)/2} \sqrt{\pi} a_0(i q^*_{\mathcal{D}} \cdot 1) = -2 w_0^{-1/2} \sqrt{\pi} \frac{\omega}{2(\omega^2 p_0/2)^{1/2}} i q^*_0$$

for $q_0^* = q_{\mathcal{D}}^*(z_0)$. We see that $(q_0^*)^2 = iz_0(e^{\pi i z_0}+1)$ and so

$$d_0\big(\sigma,\overline{N}\big)^2 = 2z_0^2 e^{-\pi i z_0}\big(e^{-\pi i z_0}+1\big) \qquad (N \text{ even})$$

and (7.44) follows in this case also. □

Table 3 gives an example of the accuracy of (7.45) in Theorem 1.6.

N	$m=1$	$m=2$	$m=4$	$\mathcal{D}_1(N,1)$
1000	-1.7713×10^9	-1.7785×10^9	-1.77778×10^9	-1.77778×10^9
1001	-2.10996×10^9	-2.11483×10^9	-2.11418×10^9	-2.11418×10^9

Table 3: Theorem 1.6's approximations to $\mathcal{D}_1(N,1)$.

8. The Sum $\mathcal{E}_1(N,\sigma)$

Let $\sigma\in\mathbb{Z}$. In this section we prove Theorem 1.7, giving the asymptotic expansion as $N\to\infty$ of

$$\mathcal{E}_1(N,\sigma) := \sum_{h/k\in\mathcal{E}(N)} Q_{hk\sigma}(N) = 2\mathrm{Re}\sum_{\frac{N}{3}<k\leqslant\frac{N}{2}} Q_{1k\sigma}(N).$$

8.1. Higher-order Poles

Recall from (1.6) that

$$Q_{hk\sigma}(N) := 2\pi i \operatorname*{Res}_{z=h/k} \frac{e^{2\pi i\sigma z}}{(1-e^{2\pi i z})(1-e^{2\pi i 2z})\cdots(1-e^{2\pi i N z})}$$

and the expression on the right above has a pole at $z=h/k$ of order $s=\lfloor N/k\rfloor$. We calculated $Q_{hk\sigma}(N)$ in the case of a simple pole ($s=1$ or equivalently $N/2<k\leqslant N$) in Proposition 4.4 and require the double pole case ($s=2$ or $N/3<k\leqslant N/2$) in this section. In general, we have

$$e^{2\pi i\sigma z} = e^{2\pi i\sigma h/k}\sum_{r=0}^{\infty}\frac{(2\pi i\sigma)^r}{r!}(z-h/k)^r$$

and for $m\in\mathbb{Z}_{\neq 0}$ write

$$\frac{1}{1-e^{2\pi i m z}} = \sum_{r=0}^{\infty}\frac{\beta_r(m,h/k)}{r!}(z-h/k)^r \times \begin{cases}(z-h/k)^{-1} & \text{if } \ k\mid m\\ 1 & \text{if } \ k\nmid m.\end{cases}$$

Therefore, for any k,

$$Q_{hk\sigma}(N) = 2\pi i \cdot e^{2\pi i\sigma h/k} \times \sum_{r_0+r_1+\cdots+r_N=s-1} (2\pi i\sigma)^{r_0} \frac{\beta_{r_1}(1,h/k)\beta_{r_2}(2,h/k)\cdots\beta_{r_N}(N,h/k)}{r_0!r_1!\cdots r_N!} \tag{8.1}$$

where

$$\beta_r(m,h/k) = -(2\pi i m)^{r-1}B_r \qquad (k \mid m),$$
$$\beta_r(m,h/k) = \left.\frac{d^r}{dz^r}\frac{1}{1-e^{2\pi i m z}}\right|_{z=h/k} \qquad (k \nmid m). \tag{8.2}$$

Formula (8.2) implies for example,

$$\begin{aligned}
\beta_0(m,h/k) &= \frac{1}{1-e^{2\pi i m h/k}} \\
\beta_1(m,h/k) &= -2\pi i m\beta_0(m,h/k)\Big(1-\beta_0(m,h/k)\Big) \\
\beta_2(m,h/k) &= (2\pi i m)^2\beta_0(m,h/k)\Big(1-3\beta_0(m,h/k)+2\beta_0(m,h/k)^2\Big)
\end{aligned}$$

for $k \nmid m$.

8.2. Second-order Poles

For $N/3 < k \leqslant N/2$ (and $s=2$), formula (8.1) shows that

$$Q_{hk\sigma}(N) = 2\pi i \cdot e^{2\pi i\sigma h/k}\left[2\pi i\sigma + \frac{\beta_1(1,h/k)}{\beta_0(1,h/k)} + \cdots + \frac{\beta_1(N,h/k)}{\beta_0(N,h/k)}\right]\prod_{j=1}^{N}\beta_0(j,h/k)$$

and hence, recalling the root of unity identity after (3.12),

$$Q_{hk\sigma}(N) = \frac{-e^{2\pi i\sigma h/k}}{2k^4}\left(\frac{N(N+1)-3k-2\sigma}{2} - \sum_{1\leqslant m\leqslant N,\ k\nmid m}\frac{m}{1-e^{2\pi i m h/k}}\right) \times \prod_{j=1}^{N-2k}\frac{1}{1-e^{2\pi i h j/k}}. \tag{8.3}$$

For the case we need, $h=1$,

$$\begin{aligned}
\sum_{1\leqslant m\leqslant N,\ k\nmid m}\frac{m}{e^{2\pi i m/k}-1} &= \sum_{1\leqslant m\leqslant N,\ k\nmid m}\frac{m\cdot e^{-\pi i m/k}}{e^{\pi i m/k}-e^{-\pi i m/k}} \\
&= \frac{1}{2i}\sum_{1\leqslant m\leqslant N,\ k\nmid m}\frac{m(\cos(-\pi m/k)+i\sin(-\pi m/k))}{\sin(\pi m/k)} \\
&= \frac{-1}{2i}\sum_{1\leqslant m\leqslant N,\ k\nmid m} im + \frac{1}{2i}\sum_{1\leqslant m\leqslant N,\ k\nmid m} m\cot(\pi m/k).
\end{aligned}$$

Therefore

$$Q_{1k\sigma}(N) = \frac{1}{2k^2}\phi(N,k,\sigma)\exp\left(N\left[\frac{-i\pi}{2}\left(\frac{N}{k}-1+2\frac{k}{N}\right)\right]\right) \times \exp\left(\frac{-i\pi}{2}\frac{N}{k}\right)\exp\left(\frac{1}{N}\left[2i\pi\sigma\frac{N}{k}\right]\right)\textstyle\prod_{N-2k}^{-1}(1/k) \quad (8.4)$$

for

$$\phi(N,k,\sigma) := \frac{1}{4k^2}(N^2+N-4\sigma) + \frac{1}{2\pi i k}\sum_{1\leqslant j\leqslant N,\ k\nmid j}\frac{\pi j}{k}\cot\left(\frac{\pi j}{k}\right). \quad (8.5)$$

Also note that

$$|Q_{1k\sigma}(N)| = \frac{1}{2k^2}\left|\phi(N,k,\sigma)\cdot\textstyle\prod_{N-2k}^{-1}(1/k)\right|. \quad (8.6)$$

Lemma 8.1. *For $N/3 < k \leqslant N/2$ and an implied constant depending only on σ*

$$\phi(N,k,\sigma) = O(N).$$

Proof. Verify that $1/|\sin(\pi j/k)| < 2k/\pi$ for $k \nmid j$ (as in [9, Sect. 3.3]). Therefore

$$|\cot(\pi j/k)| < 2k/\pi \qquad (k \nmid j)$$

and the lemma follows. □

Set $\hat{z} = \hat{z}(N,k) := N/k$. Applications of Propositions 4.7 and 4.10, with $m = N-2k$ and $s = N/2$, prove the following.

Theorem 8.2. *Fix $W > 0$. Let Δ be in the range $0.0048 \leqslant \Delta \leqslant 0.0079$ and set $\alpha = \Delta\pi e$. Suppose δ and δ' satisfy*

$$\frac{\Delta}{1-\Delta} < \delta \leqslant \frac{1}{e},\ 0 < \delta' \leqslant \frac{1}{e} \quad \text{and} \quad \delta\log 1/\delta,\ \ \delta'\log 1/\delta' \leqslant W.$$

Then for all $N \geqslant 2\cdot R_\Delta$ we have

$$\textstyle\prod_{N-2k}^{-1}(1/k) = O\left(e^{WN/2}\right) \quad \text{for} \quad \hat{z} \in [2,\ 2+\delta] \cup [5/2-\delta',\ 3) \quad (8.7)$$

and

$$\textstyle\prod_{N-2k}^{-1}(1/k) = \frac{1}{N^{1/2}}\exp\left(N\frac{\mathrm{Cl}_2(2\pi\hat{z})}{2\pi\hat{z}}\right)\left(\frac{\hat{z}}{2\sin(\pi(\hat{z}-2))}\right)^{1/2} \times \exp\left(\sum_{\ell=1}^{L-1}\frac{g_\ell(\hat{z})}{N^{2\ell-1}}\right) + O\left(e^{WN/2}\right) \quad \text{for} \quad \hat{z} \in (2+\delta,\ 5/2-\delta') \quad (8.8)$$

with $L = \lfloor \alpha\cdot N/2\rfloor$. The implied constants in (8.7), (8.8) are absolute.

8.3. Estimating $\phi(N, k, \sigma)$

With Lemma 8.1 and (8.7), we see that

$$\mathcal{E}_1(N,\sigma) = 2\mathrm{Re} \sum_{k \,:\, \hat{z}\in(2+\delta,\ 5/2-\delta')} Q_{1k\sigma}(N) + O(Ne^{WN/2}) \tag{8.9}$$

and so we may restrict our attention to indices k corresponding to this range. Let

$$f(x) := x\cot(x),$$

a smooth function of $x \in \mathbb{R}$ except at $x = \pm\pi, \pm 2\pi \dots$ and with $f(0) = 1$. Note the identities

$$f(-x) = f(x), \quad f(\pi + x) = f(x) + \pi\cot(x), \quad f(\pi - x) = f(x) - \pi\cot(x)$$

for example. Let $m = N - 2k$ as before, so that $0 \leqslant m < k$. With (8.9) we may assume

$$\delta k < m < k/2 - \delta' k,$$

and in particular, $m \neq 0$. For $m < k/2$, the sum we need from (8.5) is

$$\sum_{1\leqslant j\leqslant N,\ k\nmid j} f\left(\frac{\pi j}{k}\right) = \sum_{m<j<k-m} \left(f\left(\frac{\pi j}{k}\right) + f\left(\frac{\pi(k+j)}{k}\right)\right) +$$
$$\sum_{1\leqslant j\leqslant m} \left(f\left(\frac{\pi j}{k}\right) + f\left(\frac{\pi(k-j)}{k}\right) + f\left(\frac{\pi(k+j)}{k}\right) + f\left(\frac{\pi(2k-j)}{k}\right) + f\left(\frac{\pi(2k+j)}{k}\right)\right)$$
$$= 5\sum_{1\leqslant j\leqslant m} f\left(\frac{\pi j}{k}\right) + 2\sum_{m<j<k-m} f\left(\frac{\pi j}{k}\right). \tag{8.10}$$

With $\rho(z) := \log\big((\sin z)/z\big)$, we have

$$f(x) = 1 + x\rho'(x)$$

and for $d \in \mathbb{Z}_{\geqslant 1}$

$$f^{(d)}(x) = x\cot^{(d)}(x) + d\cot^{(d-1)}(x) \tag{8.11}$$
$$= x\rho^{(d+1)}(x) + d\rho^{(d)}(x). \tag{8.12}$$

Since $\rho^{(d)}(0)$ equals 0 for d odd, (and equals $-2^d|B_d|/d$ for d even), we see

$$f^{(d)}(0) = 0 \qquad (d \text{ odd}). \tag{8.13}$$

Also note the relation

$$f^{(d)}(\pi - x) = (-1)^d\left(f^{(d)}(x) - \pi\cot^{(d)}(x)\right). \tag{8.14}$$

Applying Euler-Maclaurin summation to (8.10), as in [12, Chap. 2] or [7, p. 285], and simplifying with (8.13), (8.14) produces

$$\sum_{1\leqslant j\leqslant N,\ k\nmid j} f\left(\frac{\pi j}{k}\right) = 5\int_0^m f\left(\frac{\pi x}{k}\right)\,dx + 2\int_m^{k-m} f\left(\frac{\pi x}{k}\right)\,dx - \frac{5}{2} + \frac{1}{2}f\left(\frac{\pi m}{k}\right) + \pi\cot\left(\frac{\pi m}{k}\right)$$
$$+\sum_{\ell=1}^{L-1}\frac{B_{2\ell}}{(2\ell)!}\left(\frac{\pi}{k}\right)^{2\ell-1}\left\{f^{(2\ell-1)}\left(\frac{\pi m}{k}\right) + 2\pi\cot^{(2\ell-1)}\left(\frac{\pi m}{k}\right)\right\} + \varepsilon_L(m,1/k) \quad (8.15)$$

for

$$\varepsilon_L(m,1/k) := \left(\frac{\pi}{k}\right)^{2L}\left[5\int_0^m +2\int_m^{k-m}\right]\frac{B_{2L}-B_{2L}(x-\lfloor x\rfloor)}{(2L)!}f^{(2L)}\left(\frac{\pi x}{k}\right)\,dx. \quad (8.16)$$

With the evaluation

$$\int_0^t x\cot(x)\,dx = \frac{1}{2}\operatorname{Cl}_2(2t) - t\operatorname{Cl}_2'(2t)$$

we find

$$5\int_0^m f\left(\frac{\pi x}{k}\right)\,dx + 2\int_m^{k-m} f\left(\frac{\pi x}{k}\right)\,dx = \frac{k}{2\pi}\operatorname{Cl}_2(2\pi m/k) - N\operatorname{Cl}_2'(2\pi m/k).$$

Using (8.11) also, (8.15) becomes

$$\sum_{1\leqslant j\leqslant N,\ k\nmid j} f\left(\frac{\pi j}{k}\right) = \frac{k}{2\pi}\operatorname{Cl}_2(2\pi m/k) - N\operatorname{Cl}_2'(2\pi m/k) - \frac{5}{2} + \frac{\pi N}{2k}\cot\left(\frac{\pi m}{k}\right)$$
$$+\sum_{\ell=1}^{L-1}\frac{B_{2\ell}}{(2\ell)!}\left(\frac{\pi}{k}\right)^{2\ell-1}\left\{\frac{\pi N}{k}\cot^{(2\ell-1)}\left(\frac{\pi m}{k}\right) + (2\ell-1)\cot^{(2\ell-2)}\left(\frac{\pi m}{k}\right)\right\}+\varepsilon_L(m,1/k). \quad (8.17)$$

Define

$$\tilde{g}_\ell(z) := \frac{B_{2\ell}}{(2\ell)!}(\pi z)^{2\ell-1}\left\{\pi z\cot^{(2\ell-1)}(\pi z) + (2\ell-1)\cot^{(2\ell-2)}(\pi z)\right\}.$$

With (8.5) and (8.17) we have demonstrated that

$$\phi(N,k,\sigma) = \left[\frac{\operatorname{Cl}_2(2\pi\hat{z})}{4\pi^2 i} - \frac{\hat{z}\operatorname{Cl}_2'(2\pi\hat{z})}{2\pi i} + \frac{\hat{z}^2}{4}\right] + \frac{1}{N}\left[\frac{\hat{z}^2\cot(\pi\hat{z})}{4i} + \frac{\hat{z}^2}{4} - \frac{5\hat{z}}{4\pi i}\right]$$
$$- \frac{\sigma\hat{z}^2}{N^2} + \frac{\hat{z}}{2\pi i}\sum_{\ell=1}^{L-1}\frac{\tilde{g}_\ell(\hat{z})}{N^{2\ell}} + \frac{\varepsilon_L(m,1/k)}{2\pi i k} \quad (8.18)$$

which we write as

$$\phi(N,k,\sigma) = \sum_{\ell=0}^{2L-1} \frac{\phi_{\sigma,\ell}(\hat{z})}{N^\ell} + \frac{\varepsilon_L(m,1/k)}{2\pi i k},$$

though only $\phi_{\sigma,2}(\hat{z})$ depends on σ.

Proposition 8.3. *For $1 \leqslant m \leqslant k/2$ we have*

$$\frac{|\varepsilon_L(m,1/k)|}{2\pi k} \leqslant 2\pi^2(2L-1)\left(\frac{2L-1}{2\pi e m}\right)^{2L-1}.$$

Proof. The arguments here are similar to those in [9, Sect. 3]. Use the inequalities

$$|B_{2n} - B_{2n}(x - \lfloor x \rfloor)| \leqslant 2|B_{2n}|, \qquad \frac{|B_{2n}|}{(2n)!} \leqslant \frac{\pi^2}{3(2\pi)^{2n}}$$

from [7, Thm 1.1, p. 283] and [12, (9.6)] to see that

$$|\varepsilon_L(m,1/k)| \leqslant \frac{2\pi^2}{3(2\pi)^{2L}}\left(\frac{\pi}{k}\right)^{2L}\left[5\int_0^m + 2\int_m^{k-m}\right]\left|f^{(2L)}\left(\frac{\pi x}{k}\right)\right| dx. \tag{8.19}$$

By [12, (11.1)]

$$-\rho'(w) = \sum_{r=1}^{\infty} \frac{2^{2r}|B_{2r}|}{(2r)!} w^{2r-1} \qquad (|w| < \pi) \tag{8.20}$$

so that

$$\rho^{(d)}(x) \leqslant 0 \quad \text{ for all } \quad x \in [0,\pi),\ d \in \mathbb{Z}_{\geqslant 0}.$$

Hence (8.12) implies

$$f^{(d)}(x) \leqslant 0 \quad \text{ for all } \quad x \in [0,\pi),\ d \in \mathbb{Z}_{\geqslant 1}$$

and $\left|f^{(2L)}(\pi x/k)\right| = -f^{(2L)}(\pi x/k)$ in (8.19). On integrating and applying (8.13), (8.14) we obtain

$$|\varepsilon_L(m,1/k)| \leqslant -\frac{\pi}{3}\left(\frac{1}{2k}\right)^{2L-1}\left(f^{(2L-1)}\left(\frac{\pi m}{k}\right) + 2\pi \cot^{(2L-1)}\left(\frac{\pi m}{k}\right)\right) =$$

$$\frac{\pi}{3}\left(\frac{1}{2k}\right)^{2L-1}\left(2\pi(2L-1)!\left(\frac{k}{\pi m}\right)^{2L} - \frac{\pi N}{k}\rho^{(2L)}\left(\frac{\pi m}{k}\right) - (2L-1)\rho^{(2L-1)}\left(\frac{\pi m}{k}\right)\right)$$

with the last line coming from (8.12) and the further identity

$$\cot^{(d)}(x) = \frac{(-1)^d d!}{x^{d+1}} + \rho^{(d+1)}(x).$$

Use

$$\left|\rho^{(d+1)}(x)\right| \leqslant \frac{2\pi d!}{3}\left(\frac{2}{\pi}\right)^d \qquad (|x| \leqslant \pi/2,\ d \in \mathbb{Z}_{\geqslant 0})$$

from (8.20), and $(n-1)! < 3\,(n/e)^n$ from Stirling's formula, to complete the proof. □

8.4. Approximating $\mathcal{E}_1(N,\sigma)$

With (4.2) for $m=2$ we find, for $2<z<3$,

$$\frac{\mathrm{Cl}_2(2\pi z)}{2\pi z}-\frac{i\pi}{2}(z-1+2/z)=-2\pi i+\frac{1}{2\pi i z}\Big[\mathrm{Li}_2(e^{2\pi i z})-\mathrm{Li}_2(1)-4\pi^2\Big].$$

Put

$$r_{\mathcal{E}}(z):=\frac{1}{2\pi i z}\Big[\mathrm{Li}_2(e^{2\pi i z})-\mathrm{Li}_2(1)-4\pi^2\Big] \tag{8.21}$$

$$q_{\mathcal{E}}(z;N,\sigma):=\left(\frac{z}{2\sin(\pi z)}\right)^{1/2}\exp\left(\frac{-\pi i z}{2}\right)\times\sum_{\ell=0}^{2L-1}\frac{\phi_{\sigma,\ell}(z)}{N^{\ell}} \tag{8.22}$$

$$v_{\mathcal{E}}(z;N,\sigma):=\frac{2\pi i\sigma z}{N}+\sum_{\ell=1}^{L-1}\frac{g_\ell(z)}{N^{2\ell-1}} \tag{8.23}$$

for $L:=\lfloor\alpha\cdot N/2\rfloor$ in (8.22) and (8.23). Also set

$$\mathcal{E}_2(N,\sigma):=\frac{1}{N^{1/2}}\mathrm{Re}\sum_{k\ :\ \hat{z}\in(2+\delta,\ 5/2-\delta')}\frac{1}{k^2}\exp\big(N\cdot r_{\mathcal{E}}(\hat{z})\big)q_{\mathcal{E}}(\hat{z};N,\sigma)\exp\big(v_{\mathcal{E}}(\hat{z};N,\sigma)\big).$$

The terms summed for $\mathcal{E}_2(N,\sigma)$ above differ from the terms in $\mathcal{E}_1(N,\sigma)$ only in the removal of the error terms from the approximations of $\prod_{N-2k}^{-1}(1/k)$ and $\phi(N,k,\sigma)$. The next proposition lets us control what happens on removing the error term for $\phi(N,k,\sigma)$.

Proposition 8.4. *Suppose* Δ *and* W *satisfy* $0.0048\leqslant\Delta\leqslant 0.0079$ *and* $\Delta\log 1/\Delta\leqslant W$. *For the integers* k, s *and* m *we require*

$$1<k\leqslant s,\quad R_\Delta\leqslant s,\quad \Delta s\leqslant m\leqslant k/2.$$

Then for $L:=\lfloor\pi e\Delta\cdot s\rfloor$ *we have*

$$\begin{aligned}\textstyle\prod_m^{-1}(1/k)\frac{\varepsilon_L(m,1/k)}{2\pi i k}&=O(se^{sW})\\ \frac{\varepsilon_L(m,1/k)}{2\pi i k}&=O(s).\end{aligned}$$

Proof. We may copy the proof of Proposition 4.8 in [9, Sect. 3.4]. The bound used for $T_L(m,h/k)$ in that result is $\left(\frac{2L-1}{2\pi e m}\right)^{2L-1}$. The corresponding bound for $\varepsilon_L(m,1/k)/(2\pi i k)$ in Proposition 8.3 is bigger by a factor $2L-1\ll s$. $\square$

Choosing $s=N/2$ and $m=N-2k$ in Proposition 8.4 shows

$$\textstyle\prod_{N-2k}^{-1}(1/k)\frac{\varepsilon_L(m,1/k)}{2\pi i k}=O(Ne^{WN/2}) \tag{8.24}$$

$$\frac{\varepsilon_L(m,1/k)}{2\pi i k}=O(N) \tag{8.25}$$

for $N\geqslant 2\cdot R_\Delta$, $L=\lfloor\alpha\cdot N/2\rfloor$ and $2+\Delta/(1-\Delta/2)\leqslant\hat{z}\leqslant 5/2$.

Proposition 8.5. *For an implied constant depending only on* σ

$$\mathcal{E}_1(N,\sigma) = \mathcal{E}_2(N,\sigma) + O(Ne^{WN/2}).$$

Proof. Starting with (8.9), write

$$\sum_{k\ :\ \hat{z}\in(2+\delta,\ 5/2-\delta')} Q_{1k\sigma}(N)$$
$$= \sum_{k\ :\ \hat{z}\in(2+\delta,\ 5/2-\delta')} \frac{Q_{1k\sigma}(N)}{\phi(N,k,\sigma)} \left(\sum_{\ell=0}^{2L-1} \frac{\phi_{\sigma,\ell}(\hat{z})}{N^\ell} + \frac{\varepsilon_L(m,1/k)}{2\pi i k} \right)$$

where

$$\frac{Q_{1k\sigma}(N)}{\phi(N,k,\sigma)} = \frac{1}{2k^2} \exp\left(N \frac{-i\pi(\hat{z}-1+2/\hat{z})}{2} - \frac{\pi i \hat{z}}{2} + \frac{2\pi i \sigma \hat{z}}{N} \right) \prod\nolimits_{N-2k}^{-1}(1/k)$$

by (8.4). We have

$$\sum_{k\ :\ \hat{z}\in(2+\delta,\ 5/2-\delta')} \frac{Q_{1k\sigma}(N)}{\phi(N,k,\sigma)} \frac{\varepsilon_L(m,1/k)}{2\pi i k}$$
$$\ll \sum_{k\ :\ \hat{z}\in(2+\delta,\ 5/2-\delta')} \frac{1}{k^2} \left| \prod\nolimits_{N-2k}^{-1}(1/k) \frac{\varepsilon_L(m,1/k)}{2\pi i k} \right| \ll Ne^{WN/2}$$

using (8.24) and that

$$\frac{\Delta}{1-\Delta/2} < \frac{\Delta}{1-\Delta} < \delta$$

so the bound (8.24) is valid for $\hat{z} \in (2+\delta,\ 5/2-\delta')$. Therefore,

$$\mathcal{E}_1(N,\sigma) = 2\mathrm{Re} \sum_{k\ :\ \hat{z}\in(2+\delta,\ 5/2-\delta')} \frac{Q_{1k\sigma}(N)}{\phi(N,k,\sigma)} \left(\sum_{\ell=0}^{2L-1} \frac{\phi_{\sigma,\ell}(\hat{z})}{N^\ell} \right) + O(Ne^{WN/2}). \quad (8.26)$$

Next note that

$$\left| \sum_{\ell=0}^{2L-1} \frac{\phi_{\sigma,\ell}(\hat{z})}{N^\ell} \right| \leqslant |\phi(N,k,\sigma)| + \left| \frac{\varepsilon_L(m,1/k)}{2\pi i k} \right| \ll N \quad (8.27)$$

by Lemma 8.1 and (8.25). With (8.27) we see that replacing $\prod_{N-2k}^{-1}(1/k)$ in (8.26) by the main term on the right of (8.8) changes $\mathcal{E}_1(N,\sigma)$ by at most $O(Ne^{WN/2})$, as required. □

Comparing (8.21)-(8.23) and (5.9)-(5.11) gives the relations

$$r_{\mathcal{E}}(z) = r_{\mathcal{C}}(z), \qquad q_{\mathcal{E}}(z;N,\sigma) = q_{\mathcal{C}}(z) \sum_{\ell=0}^{2L-1} \frac{\phi_{\sigma,\ell}(z)}{N^\ell}, \qquad v_{\mathcal{E}}(z;N,\sigma) = v_{\mathcal{C}}(z;N,\sigma)$$

so that we may reuse our work from Section 5. We fix the choice of constants as in (5.17).

Lemma 8.6. *The function* $q_{\mathcal{E}}(z;N,\sigma)$ *is holomorphic for* $2 < \mathrm{Re}(z) < 5/2$*. In the box with* $2+\delta \leqslant \mathrm{Re}(z) \leqslant 5/2-\delta'$ *and* $-1 \leqslant \mathrm{Im}(z) \leqslant 1$*,*

$$q_{\mathcal{E}}(z;N,\sigma) \ll 1 \tag{8.28}$$

for an implied constant depending only on $\sigma \in \mathbb{R}$*.*

Proof. The first issue is that $\phi_{\sigma,0}(z)$ has only been defined in (8.18) for $z \in \mathbb{R}$. Use (4.2) and its derivative with $m=2$ to show

$$\phi_{\sigma,0}(z) = \frac{1}{4\pi^2}\left[\mathrm{Li}_2(1) - \mathrm{Li}_2(e^{2\pi i z}) + 6\pi^2 - 2\pi i z \log(1-e^{2\pi i z})\right] \tag{8.29}$$

giving the analytic continuation of $\phi_{\sigma,0}(z)$ to all z with $2 < \mathrm{Re}(z) < 5/2$. It follows, as in Theorem 5.4, that $q_{\mathcal{E}}(z;N,\sigma)$ is holomorphic in z as required. The bound (8.28) follows from

$$\frac{\tilde{g}_\ell(z)}{N^{2\ell-1}} \ll N\big(F_{N,\varepsilon}(2\ell-1) + F_{N,\varepsilon}(2\ell)\big)e^{-\pi|y|},$$

with $F_{N,\varepsilon}$ defined in (5.16), as in Proposition 5.2 and Corollary 5.3. □

With the rectangle C_1 from Figure 4 we find

$$\mathcal{E}_2(N,\sigma) = \frac{-1}{N^{3/2}}\mathrm{Re}\int_{C_1} \exp\big(N\cdot r_{\mathcal{C}}(z)\big)\frac{q_{\mathcal{E}}(z;N,\sigma)}{2i\tan\big(\pi N/z\big)}\exp\big(v_{\mathcal{C}}(z;N,\sigma)\big)\,dz$$

where

$$\frac{1}{2i\tan(\pi N/z)} = \begin{cases} 1/2 + \sum_{j\leqslant -1} e^{2\pi i j N/z} & \text{if } \mathrm{Im} z > 0 \\ -1/2 - \sum_{j\geqslant 1} e^{2\pi i j N/z} & \text{if } \mathrm{Im} z < 0. \end{cases}$$

The arguments of Propositions 5.5, 5.6 and 5.7 now go through almost unchanged:

$$\begin{aligned}\mathcal{E}_2(N,\sigma) &\\ = \frac{-1}{N^{3/2}}\mathrm{Re}\Bigg[&\sum_{j\leqslant 0}{}' \int_{C_1^+} \exp\big(N[r_{\mathcal{C}}(z)+2\pi i j/z]\big)q_{\mathcal{E}}(z;N,\sigma)\exp\big(v_{\mathcal{C}}(z;N,\sigma)\big)\,dz \\ - \sum_{j\geqslant 0}{}' &\int_{C_1^-} \exp\big(N[r_{\mathcal{C}}(z)+2\pi i j/z]\big)q_{\mathcal{E}}(z;N,\sigma)\exp\big(v_{\mathcal{C}}(z;N,\sigma)\big)\,dz\Bigg] + O(e^{WN/2}),\end{aligned}$$

the term with $j=-1$ is the largest and

$$\mathcal{E}_2(N,\sigma) = \mathcal{E}_3(N,\sigma) + O(e^{WN/2}) \tag{8.30}$$

for $W = 0.05$, an implied constant depending only on σ, and

$$\mathcal{E}_3(N,\sigma) := \frac{1}{N^{3/2}}\mathrm{Re}\int_{2.01}^{2.49} \exp\big(-N\cdot p(z)\big)q_{\mathcal{E}}(z;N,\sigma)\exp\big(v_{\mathcal{C}}(z;N,\sigma)\big)\,dz. \tag{8.31}$$

8.5. The Asymptotic Behavior of $\mathcal{E}_1(N,\sigma)$

Arguing as in Lemma 8.6 shows the next result.

Proposition 8.7. *For* $2.01 \leqslant \operatorname{Re}(z) \leqslant 2.49$ *and* $|\operatorname{Im}(z)| \leqslant 1$*, say, there is a holomorphic function* $\xi_r(z;N,\sigma)$ *of* z *so that*

$$q_{\mathcal{E}}(z;N,\sigma) = q_{\mathcal{C}}(z)\sum_{k=0}^{r-1}\frac{\phi_{\sigma,k}(z)}{N^k} + \xi_r(z;N,\sigma) \quad \text{for} \quad \xi_r(z;N,\sigma) = O\left(\frac{1}{N^r}\right)$$

with an implied constant depending only on σ *and* r *where* $1 \leqslant r \leqslant 2L-1$ *and* $L = \lfloor 0.006\pi e \cdot N/2 \rfloor$*.*

We restate Theorem 1.7:

Theorem 1.7. With $e_0 = -3z_1 e^{-\pi i z_1}/2$ and explicit $e_1(\sigma)$, $e_2(\sigma), \dots$ depending on $\sigma \in \mathbb{Z}$ we have

$$\mathcal{E}_1(N,\sigma) = \operatorname{Re}\left[\frac{w(0,-2)^{-N}}{N^2}\left(e_0 + \frac{e_1(\sigma)}{N} + \cdots + \frac{e_{m-1}(\sigma)}{N^{m-1}}\right)\right] + O\left(\frac{|w(0,-2)|^{-N}}{N^{m+2}}\right) \tag{8.32}$$

for an implied constant depending only on σ and m.

Proof. With Propositions 5.13 and 8.7, write

$$q_{\mathcal{E}}(z;N,\sigma)\exp\bigl(v_{\mathcal{C}}(z;N,\sigma)\bigr) = q_{\mathcal{C}}(z)\left(\sum_{k=0}^{r-1}\frac{\phi_{\sigma,k}(z)}{N^k}\right)\left(\sum_{j=0}^{d-1}\frac{u_{\sigma,j}(z)}{N^j}\right)$$
$$+ q_{\mathcal{E}}(z;N,\sigma)\zeta_d(z;N,\sigma) + \xi_r(z;N,\sigma)\exp\bigl(v_{\mathcal{C}}(z;N,\sigma)\bigr) - \xi_r(z;N,\sigma)\zeta_d(z;N,\sigma).$$

Then putting this into (8.31) and moving the line of integration to $\mathcal{Q}$ (see Figure 5) gives

$$\mathcal{E}_3(N,\sigma) = \frac{1}{N^{3/2}}\operatorname{Re}\int_{\mathcal{Q}}\exp\bigl(-N\cdot p(z)\bigr)q_{\mathcal{C}}(z)\left(\sum_{k=0}^{r-1}\frac{\phi_{\sigma,k}(z)}{N^k}\right)\left(\sum_{j=0}^{d-1}\frac{u_{\sigma,j}(z)}{N^j}\right)dz$$
$$+ O\left(\frac{|w(0,-2)|^{-N}}{N^{3/2}}\left(\frac{1}{N^d} + \frac{1}{N^r} + \frac{1}{N^{d+r}}\right)\right). \tag{8.33}$$

The integral in (8.33) is

$$\sum_{k=0}^{r-1}\sum_{j=0}^{d-1}\frac{1}{N^{3/2+k+j}}\int_{\mathcal{Q}}\exp\bigl(-N\cdot p(z)\bigr)q_{\mathcal{C}}(z)\phi_{\sigma,k}(z)u_{\sigma,j}(z)\,dz$$

and applying the saddle-point method, Theorem 1.8, gives

$$\sum_{k=0}^{r-1}\sum_{j=0}^{d-1}\frac{2e^{-N\cdot p(z_1)}}{N^{3/2+k+j}}\left(\sum_{s=0}^{S-1}\Gamma(s+1/2)\frac{a_{2s}(q_{\mathcal{C}}\cdot\phi_{\sigma,k}\cdot u_{\sigma,j})}{N^{s+1/2}} + O\left(\frac{1}{N^{S+1/2}}\right)\right).$$

Letting $S = r = d$ we obtain, as in the proof of Theorem 5.14,

$$\mathcal{E}_3(N,\sigma) = \mathrm{Re}\left[e^{-N\cdot p(z_1)} \sum_{t=0}^{d-2} \frac{2}{N^{t+2}} \sum_{s=0}^{t} \sum_{k=0}^{t-s} \Gamma(s+1/2) a_{2s}(q_{\mathcal{C}} \cdot \phi_{\sigma,k} \cdot u_{\sigma,t-s-k})\right] + O\left(\frac{|w(0,-2)|^{-N}}{N^{d+1}}\right). \quad (8.34)$$

Hence, recalling Proposition 8.5, (8.30) and with

$$e_t(\sigma) := 2 \sum_{s=0}^{t} \sum_{k=0}^{t-s} \Gamma(s+1/2) a_{2s}(q_{\mathcal{C}} \cdot \phi_{\sigma,k} \cdot u_{\sigma,t-s-k}), \quad (8.35)$$

we obtain (8.32) in the statement of the theorem.

Computing $e_0(\sigma)$ with (8.35) gives

$$e_0(\sigma) = 2\sqrt{\pi} a_0(q_{\mathcal{C}} \cdot \phi_{\sigma,0} \cdot 1) = 2\sqrt{\pi} \frac{\omega}{2(\omega^2 p_0)^{1/2}} q_{\mathcal{C}}(z_1)\phi_{\sigma,0}(z_1).$$

With the identity

$$2\pi i z^2 p'(z) = \mathrm{Li}_2\left(e^{2\pi i z}\right) - \mathrm{Li}_2(1) + 2\pi i z \log\left(1 - e^{2\pi i z}\right)$$

from [9, Sect. 2.3] we find that

$$\phi_{\sigma,0}(z_1) = \frac{6\pi^2 - 2\pi i z_1^2 p'(z_1)}{4\pi^2} = \frac{3}{2}.$$

Combine this with the calculations in (5.45) to get $e_0(\sigma)^2 = 9z_1^2 e^{-2\pi i z_1}/4$ and the formula for $e_0 = e_0(\sigma)$ in the statement of the theorem follows. □

For example, a comparison of both sides of (8.32) in Theorem 1.7 with $\sigma = 1$ and some different values of m and N is shown in Table 4.

N	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$\mathcal{E}_1(N,1)$
800	879.611	905.272	909.048	909.358	909.337
1000	−789369.	−784383.	−784458.	−784480.	−784480.

Table 4: Theorem 1.7's approximations to $\mathcal{E}_1(N,1)$.

Proof of Theorem 1.3. Recall the sets $\mathcal{B}(K,N)$, $\mathcal{C}(N)$, $\mathcal{D}(N)$ and $\mathcal{E}(N)$ from (3.33), (1.14), (1.15) and (1.16) respectively. Then

$$_N - \left(_{100} \cup \mathcal{A}(N)\right) = \mathcal{B}(101,N) \cup \mathcal{C}(N) \cup \mathcal{D}(N) \cup \mathcal{E}(N).$$

Summing $Q_{hk\sigma}(N)$ for $h/k \in \mathcal{B}(101, N)$ is $O(e^{WN})$ for any $W > \mathrm{Cl}_2(\pi/3)/(6\pi) \approx 0.0538$ by Theorem 3.5. Since

$$\begin{aligned}-\log|w(1,-3)| &\approx 0.0356795,\\ -\log|w(0,-1)|/2 &\approx 0.0340381,\\ -\log|w(0,-2)| &\approx 0.0256706\end{aligned}$$

we see from Theorems 1.5, 1.6 and 1.7 that the sums of $Q_{hk\sigma}(N)$ for $h/k \in \mathcal{C}(N)$, $\mathcal{D}(N)$ and $\mathcal{E}(N)$ are $O(e^{0.0357N})$, $O(e^{0.0341N})$ and $O(e^{0.0257N})$ respectively. This completes the proof. □

As a final remark, comparing Tables 4 and 1 we notice that $\mathcal{E}_1(N, 1)$ is almost exactly 3 times the size of $\mathcal{C}_2(N, 1)$ and that their asymptotic expansions also seem to match. This is true for other values of σ too. From Theorems 5.14 and 1.7 we have

$$3 \cdot c_t(\sigma) = e_t(\sigma) \tag{8.36}$$

for the first expansion coefficients at $t = 0$. Numerically, (8.36) seems to be true for all t, as we mentioned before in (1.25).

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