IMPARTIAL CHOCOLATE BAR GAMES WITH A PASS

Ryohei Miyadera
Kwansei Gakuin, Hyogo, Japan
runners@kwansei.ac.jp

Maakito Inoue
Kwansei Gakuin, Hyogo, Japan
0731inoue@gmail.com

Masanori Fukui
Hyogo University of Teacher Education, Hyogo, Japan
m16195c@hyogo-u.ac.jp

Received: 7/18/15, Accepted: 12/4/16, Published: 12/29/16

Abstract

This paper presents a study of chocolate bar games with a pass. Chocolate bar games are variants of the game Nim in which the goal is to leave your opponent with the single bitter part of the chocolate. The rectangular chocolate bar is a thinly disguised form of Nim. In this work, we investigate step chocolate bars of which the width is proportional to the distance from the bitter square. The mathematical structure of these step chocolate bar games is very different from that of Nim. It is well-known that, in classical Nim, the introduction of the pass alters the underlying structure of the game, thereby increasing its complexity considerably; however, in the chocolate bar games treat in this paper the pass move is found to have a relatively minimal impact. Step chocolate bar games without a pass have simple formulas for Grundy numbers. This is not so after the introduction of a pass move, but they still have simple formulas for previous player's positions. Therefore, the authors address a longstanding open question in combinatorial game theory, namely, the extent to which the introduction of a pass move into a game affects its behavior. The game we develop seems to be the first variant of Nim that is fully solvable when a pass is not allowed, and remains yet stable following the introduction of a pass move.

1. Introduction

This paper presents the results of our study in which we introduced a pass move into chocolate bar games that are variants of the classical game Nim.

The original chocolate bar game, see [2], involved a rectangular bar of chocolate with one bitter corner. Each player in turn breaks the bar in a straight line along the grooves and eats the piece without the bitter part. The player who breaks the chocolate bar and eats to leave his opponent with the single bitter block (black block) is the winner. Because the horizontal and vertical grooves are independent from one another, the chocolate bar in Fig. 1.1 is equivalent to classical Nim with a heap of three stones, a heap of three stones, and a heap of two stones. This means there are three grooves to the left of the bitter square, three grooves above, and two grooves to the right of the bitter square.

In this paper we consider step bars with other shapes as in Figures 1.2 through 1.4, where the gray blocks are sweet chocolate that can be eaten, and the black block is the bitter square that cannot be eaten. In these cases, a vertical break can reduce the number of horizontal breaks. We can still consider the game as being played with heaps, but now a single move may change more than one heap. One of the authors presented the previous results obtained for this research on these chocolate games in [1], whereas the current paper presents the subsequent research on chocolate games that include a pass. Since we study only step chocolate bars, we omit "step" in the following.

Example 1.1. Examples of chocolate bar games.

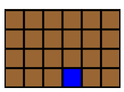


Figure 1.1.

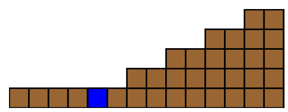


Figure 1.2.

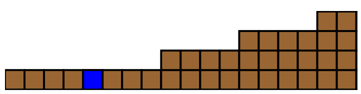


Figure 1.3.

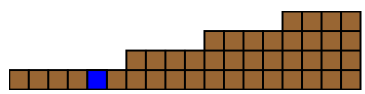


Figure 1.4.

An interesting, but very difficult, question in combinatorial game theory has been to determine what happens when standard game rules are modified so as to allow for a one-time pass, i.e., a pass move which may be used at most once in a game, and not from a terminal position. Once the pass has been used by either player, it is no longer available.

In the case of classical Nim, the introduction of the pass alters the mathematical structure of the game, thereby increasing its complexity considerably. In comparison, the pass move is found to have minimal impact on the chocolate bar games created by the authors. Chocolate bar games without a pass are treated in [1] and have simple formulas for Grundy numbers. Surprisingly, the P-positions remain computable after the introduction of the pass move.

Combinatorial games with a pass have been studied by some mathematicians (see [6] and [7]); however, the difficulties relating to the underlying structure of the game, and the theory of games with a pass have not yet been resolved. The effect of a pass on classical Nim remains an important open question that has defied traditional approaches, and the late mathematician David Gale even offered a monetary prize to the first person to develop a solution for a three-pile classical Nim with a pass.

Morrison, Friedman, and Landsberg studied two types of games in [4]: the three-pile classical Nim and three-row Chomp (see [3]). The former is a simple combinatorial game which has been fully solved (without the pass), whereas the latter (also without the pass) is an unsolved complex combinatorial game. The introduction of a pass has dramatically different effects on these two games; i.e., in the former, the pass radically changes the underlying structure and complexity of the game, whereas in the latter, no such dramatic changes have been found.

In [5] (p. 370) Friedman and Landsberg present a conjecture "Solvable combinatorial games are structurally unstable to perturbations, while generic, complex games will be structurally stable." One way in which to introduce such a perturbation, would be to allow a pass. These authors consider the difference in the responses of classical Nim and Chomp to a pass to be related to the solvability of the game and are of the opinion that the introduction of a pass move into solvable games tends to significantly modify their underlying mathematical structure, whereas games without an analytical solution would be intrinsically more robust and would therefore not be radically modified by the introduction of a pass.

The chocolate game in this paper is a counter-example to this conjecture, because it has a very simple formula for a Grundy number (Corollary 3.1). Hence, the game is fully solved, but contains a simple formula to determine the positions occupied by the previous player when a pass is introduced (Corollary 4.1).

2. Chocolate Games

Throughout this paper, we denote by $Z_{\geq 0}$ the set of non-negative integers. For completeness, we start this section with a quick review of the necessary game theory concepts; see [8] or [9] for more details.

As chocolate bar games are impartial games without draws there will only be two outcome classes.

Definition 2.1. (*i*) $\mathcal{N}$-positions, from which the next player can force a win, as long as he plays correctly at every stage.
(*ii*) $\mathcal{P}$-positions, from which the previous player (the player who will play after the next player) can force a win, as long as he plays correctly at every stage.

Definition 2.2. The *disjunctive sum* of two games, denoted by $G + H$, is a super-game, where a player may move either in G or in H but not in both.

Definition 2.3. For any position $\mathbf{p}$, there exists a set of positions that can be reached by making precisely one move from $\mathbf{p}$, which we will denote by $move(\mathbf{p})$.

Example 3.4 demonstrates the use of *move*.

Definition 2.4. (i) The *minimum excluded value* (*mex*) of a set, S, of non-negative integers is the smallest non-negative integer which is not in S.
(ii) Each position $\mathbf{p}$ of an impartial game has an associated Grundy number, which is denoted by $\mathcal{G}(\mathbf{p})$. The Grundy number is found recursively: $\mathcal{G}(\mathbf{p}) = mex\{\mathcal{G}(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\}$.

The power of the Sprague-Grundy theory for impartial games is contained in the next result.

Theorem 2.1. *Let G and H be impartial games, and let $\mathcal{G}_G$ and $\mathcal{G}_H$ be Grundy numbers of G and H, respectively. Then, the following hold:*
(i) *For any position $\mathbf{g}$ of G we have $\mathcal{G}_G(\mathbf{g}) = 0$ if and only if $\mathbf{g}$ is a P-position.*
(ii) *The Grundy number of a position $\{\mathbf{g}, \mathbf{h}\}$ in the game $G + H$ is $\mathcal{G}_G(\mathbf{g}) \oplus \mathcal{G}_H(\mathbf{h})$.*

For the proof of this theorem see [8].

Let x and y be non-negative integers, which we write in base 2 as $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0, 1\}$. We define the nim-sum $x \oplus y = \sum_{i=0}^{n} z_i 2^i$ where $z_i \equiv x_i + y_i \pmod 2$.

This paper focuses on the study of chocolate bars which grow regularly in height, as opposed to general bars, for which the strategies seem complicated.

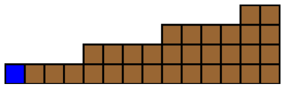
3. Chocolate Game of Which the Coordinates $\{x, y, z\}$ Satisfy the Inequality $y \leq \lfloor (z + s)/k \rfloor$ for a Fixed Natural Number s

In this section, we present some definitions and lemmas that we use in Section 4 and which have partially been published in [1]. In this section, we do not allow for a one-time pass.

Definition 3.1. Fix a natural number k and a non-negative integer s. For non-negative integers y and z such that $y \leq \lfloor \frac{z+s}{k} \rfloor$ the chocolate bar will consist of $z + 1$ columns where the 0th column is the bitter square and the height of the i-th column is $t(i) = \min(y, \lfloor \frac{i+s}{k} \rfloor) + 1$. We denote these by $CB(s, k, y, z)$.

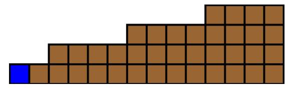
Throughout this paper, we assume that k is an even number.

Example 3.1. The following are examples of chocolate bar games $CB(s, k, y, z)$, in which the shape of the chocolate bar is determined by Definition 3.1. For example, the height of the ith column of the chocolate bar $CB(3, 4, 2, 11)$ in Figure 3.6 is determined by $t(i) = \min(y, \lfloor \frac{i+3}{4} \rfloor) + 1$. By using $\{t(i) : i = 0, 1, 2, ..., 11\} = \{1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3\}$, we can obtain the chocolate bar $CB(3, 4, 2, 11)$.



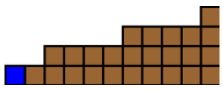
$CB(0, 4, 3, 13)$

Figure 3.1.



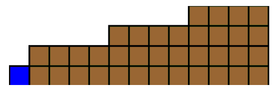
$CB(2, 4, 3, 13)$

Figure 3.2.



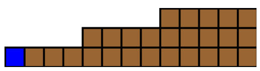
$CB(2, 4, 3, 10)$

Figure 3.3.



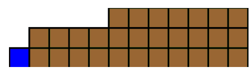
$CB(3, 4, 3, 12)$

Figure 3.4.



$CB(0, 4, 2, 12)$

Figure 3.5.

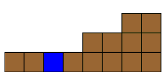


$CB(3, 4, 2, 11)$

Figure 3.6.

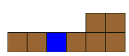
Throughout this paper, we have a disjunctive sum of a chocolate bar to the right of the bitter square and a single strip of chocolate to the left, as in Figures 1.2, 1.3, and 1.4. We denote the positions of Figures 1.2, 1.3, and 1.4 by $\{4\} + CB(0, 2, 4, 9)$, $\{4\} + CB(0, 4, 3, 13)$, and $\{4\} + CB(2, 4, 3, 13)$. When we use a fixed even number k and a fixed natural number s, we denote such a position by $\{x, y, z\}$, where x is the number of possible cuts in the strip, and y and z are the number of vertical and horizontal cuts in the bar, respectively. Figures 3.7, 3.8, 3.9, 3.10, and 3.11 are examples of coordinates of chocolates, when $k = 2$ and $s = 0$.

Example 3.2. Here, we have examples of the coordinates of the positions of chocolates.



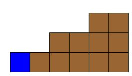
$\{2, 2, 5\}$

Figure 3.7.



$\{2, 1, 3\}$

Figure 3.8.

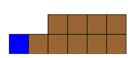


$\{0, 2, 5\}$

Figure 3.9.

$\{2, 0, 5\}$

Figure 3.10.



$\{0, 1, 5\}$

Figure 3.11.

We define $move_s(\{x, y, z\})$ for each position $\{x, y, z\}$ of which the coordinates satisfy $y \leq \lfloor (z + s)/k \rfloor$. The set $move_s(\{x, y, z\})$ consists of positions that can be reached directly from $\{x, y, z\}$.

Definition 3.2. For $x, y, z \in Z_{\geq 0}$, we let

$$M_1 = \{\{u, y, z\} : u < x\}, \tag{1}$$
$$M_2 = \{\{x, v, z\} : v < y\}, \tag{2}$$
$$M_3 = \{\{x, y, w\} : w < z \text{ and } y \leq \lfloor (w+s)/k \rfloor\}, \tag{3}$$

and

$$M_4 = \{\{x, \min(y, \lfloor (w+s)/k \rfloor), w\} : w < z\}, \tag{4}$$

where $u, v, w \in Z_{\geq 0}$.
We define

$$move_s(\{x, y, z\}) = M_1 \cup M_2 \cup M_3 \cup M_4. \tag{5}$$

Remark 3.1. Definition 3.2 is the same as Definition 3.1 in [1]. (In the present paper, we use $move_s$ instead of $moveh$ used in [1].)

Example 3.3. Let $k = 2$, $s = 0$ and $\{x, y, z\} = \{2, 2, 5\}$. Then,

$$M_1 = \{\{u, 2, 5\} : u < 2\} = \{\{1, 2, 5\}, \{0, 2, 5\}\}, \tag{6}$$
$$M_2 = \{\{2, v, 5\} : v < 2\} = \{\{2, 1, 5\}, \{2, 0, 5\}\}, \tag{7}$$
$$M_3 = \{\{2, 2, w\} : w < 5 \text{ and } 2 \leq \lfloor w/2 \rfloor\} = \{\{2, 2, 4\}\} \tag{8}$$

and

$$M_4 = \{\{2, \min(2, \lfloor w/2 \rfloor), w\} : w < 5\}$$
$$= \{\{2, 1, 3\}, \{2, 1, 2\}, \{2, 0, 1\}, \{2, 0, 0\}\}, \tag{9}$$

where $u, v, w \in Z_{\geq 0}$. M_1 and M_2 are the sets of positions that we get by reducing the first coordinate and the second coordinate respectively. M_3 is the set of positions that we get by reducing the third coordinate without reducing the second coordinate, and M_4 is the set of positions that we get by reducing the third coordinate and the second coordinate at the same time. Since $move_s(\{2, 2, 5\}) = M_1 \cup M_2 \cup M_3 \cup M_4$, the set $\{\{1, 2, 5\}, \{0, 2, 5\}, \{2, 1, 5\}, \{2, 0, 5\}, \{2, 2, 4\}, \{2, 1, 3\}, \{2, 1, 2\}, \{2, 0, 1\},$ $\{2, 0, 0\}\}$ consists of positions that can be reached directly from $\{2, 2, 5\}$.

Example 3.4. Let $k = 2$ and $s = 0$. We study the function $move_s(\{x, y, z\})$ by using the positions in Example 3.2 as examples. If we start from the position $\{x, y, z\} = \{2, 2, 5\}$ and reduce $z = 5$ to $w = 3$, then the y-coordinate (the second coordinate) will be $\min(2, \lfloor 3/2 \rfloor) = \min(2, 1) = 1$. Therefore, we have $\{2, 1, 3\} \in move_s(\{2, 2, 5\})$. It is easy to see that $\{2, 0, 5\} \in move_s(\{2, 2, 5\})$, $\{0, 1, 5\} \in move_s(\{0, 2, 5\})$ and $\{2, 0, 5\} \notin move_s(\{2, 1, 3\})$.

Definition 3.3. Let $A_{k,s} = \{\{x, y, z\} : x, y, z \in Z_{\geq 0}, y \leq \lfloor (z+s)/k \rfloor$ and $(x + s) \oplus y \oplus (z + s) = 0\}$ and $B_{k,s} = \{\{x, y, z\} : x, y, z \in Z_{\geq 0}, y \leq \lfloor (z+s)/k \rfloor$ and $(x + s) \oplus y \oplus (z + s) \neq 0\}$.

Remark 3.2. Definition 3.3 is the same as Definition 3.2 in [1]. (In the present paper, we use $A_{k,s}$ and $B_{k,s}$ instead of $A_{h,s}$ and $B_{h,s}$ that are used in [1].)

Lemma 3.1. *$move_s(\{x,y,z\}) \subset B_{k,s}$ for any $\{x,y,z\} \in A_{k,s}$.*

Lemma 3.1 is Lemma 3.6 of [1]. (In the present paper, we use $move_s$ instead of $moveh$ used in [1].)

Lemma 3.2. *We assume that a non-negative integer s satisfies the following two conditions:*
(i) $s = k2^t + m2^{t+1}$ for non-negative integers t, m such that $m = 0, 1, 2, \cdots, \frac{k}{2} - 1$.
(ii) $s = 1, 2, \cdots, k-1$.
Then, for each $\{x,y,z\} \in B_{k,s}$, we have $move_s(\{x,y,z\}) \cap A_{k,s} \neq \phi$.

Proof. This lemma is the same as Lemma 3.7 in [1]. Note that we use s in the present paper, whereas h is used in [1]. □

Theorem 3.1. *The Grundy number of $CB(s,k,y,z)$ is $(y \oplus (z+s)) - s$ if a non-negative integer s satisfies one of the following two conditions:*
(i) $s = k2^t + m2^{t+1}$ for non-negative integers t, m such that $m = 0, 1, 2, ..., \frac{k}{2} - 1$;
(ii) $s = 1, 2, ..., k-1$.

Proof. This theorem is the same as Theorem 3.2 in [1]. Note that we use s in the present paper, whereas h is used in [1]. □

When we have the disjunctive sum of the chocolate bar $CB(s,k,y,z)$ to the right of the bitter square and a single strip of chocolate to the left, and denote the position of this sum of chocolate by coordinates $\{x,y,z\}$, we have Corollary 3.1.

Corollary 3.1. *The Grundy number of position $\{x,y,z\}$ is $x \oplus ((y \oplus (z+s)) - s)$ if a non-negative integer s satisfies one of the following two conditions:*
(i) $s = k2^t + m2^{t+1}$ for non-negative integers t, m such that $m = 0, 1, 2, ..., \frac{k}{2} - 1$;
(ii) $s = 1, 2, ..., k-1$.

Proof. This corollary follows directly from Theorem 3.1 and Theorem 2.1. □

By Lemma 3.1 and Lemma 3.2, we obtain the following Lemma 3.3 and Lemma 3.4, respectively. Note that we use s such that s is odd and $0 < s < k$. Hence, we use condition (ii) of Lemma 3.2, but we do not use condition (i).

Lemma 3.3. *Let s be an odd number such that $0 < s < k$.*

$$\text{If } (x+s) \oplus y \oplus (z+s) = 0 \text{ and } y \leq \lfloor (z+s)/k \rfloor, \tag{10}$$

then the following hold:
(i) $(u+s) \oplus y \oplus (z+s) \neq 0$ for any $u \in Z_{\geq 0}$ such that $u < x$.

(ii) $(x+s) \oplus v \oplus (z+s) \neq 0$ *for any* $v \in Z_{\geq 0}$ *such that* $v < y$.
(iii) $(x+s) \oplus y \oplus (w+s) \neq 0$ *for any* $w \in Z_{\geq 0}$ *such that* $w < z$ *and* $y \leq \lfloor (w+s)/k \rfloor$.
(iv) $(x+s) \oplus v \oplus (w+s) \neq 0$ *for any* $v, w \in Z_{\geq 0}$ *such that* $v < y, w < z$ *and* $v = \lfloor (w+s)/k \rfloor$.

Proof. By condition (10) and Definition 3.3, we have $\{x, y, z\} \in A_{k,s}$. Hence, Lemma 3.1 implies

$$move_s(\{x, y, z\}) \subset B_{k,s}. \tag{11}$$

By Definition 3.2, $\{u, y, z\} \in move_s(\{x, y, z\})$ for any $u \in Z_{\geq 0}$ such that $u < x$. Hence, condition (11) implies $\{u, y, z\} \in B_{k,s}$, and we have $(u+s) \oplus y \oplus (z+s) \neq 0$. Therefore, we have statement *(i)* of this lemma. Similarly, we have statements *(ii)* and *(iii)* of this lemma. By Definition 3.2, $\{x, v, w\} \in move_s(\{x, y, z\})$ for any $v, w \in Z_{\geq 0}$ such that $v < y, w < z$ and $v = \lfloor (w+s)/k \rfloor$. Hence, condition (11) implies $\{x, v, w\} \in B_{k,s}$, and we have $(x+s) \oplus v \oplus (w+s) \neq 0$. Therefore, we have statement *(iv)* of this lemma. □

Lemma 3.4. *Let* s *be an odd number such that* $0 < s < k$.

$$\textit{If } (x+s) \oplus y \oplus (z+s) \neq 0 \textit{ and } y \leq \lfloor (z+s)/k \rfloor, \tag{12}$$

then at least one of the following statements is true.
(i) $(u+s) \oplus y \oplus (z+s) = 0$ *for some* $u \in Z_{\geq 0}$ *such that* $u < x$.
(ii) $(x+s) \oplus v \oplus (z+s) = 0$ *for some* $v \in Z_{\geq 0}$ *such that* $v < y$.
(iii) $(x+s) \oplus y \oplus (w+s) = 0$ *for some* $w \in Z_{\geq 0}$ *such that* $w < z$ *and* $y \leq \lfloor (w+s)/k \rfloor$.
(iv) $(x+s) \oplus v \oplus (w+s) = 0$ *for some* $v, w \in Z_{\geq 0}$ *such that* $v < y, w < z$ *and* $v = \lfloor (w+s)/k \rfloor$.

Proof. By condition (12) and Definition 3.3, we have $\{x, y, z\} \in B_{k,s}$. By Lemma 3.2, we have $move_s(\{x, y, z\}) \cap A_{k,s} \neq \phi$. By Definition 3.2 $move_s(\{x, y, z\})$ is a union of four sets (1), (2), (3), and (4), and any element of $move_s(\{x, y, z\}) \cap A_{k,s} \neq \phi$ belongs to one of these four sets. If an element of $move_s(\{x, y, z\}) \cap A_{k,s} \neq \phi$ belongs to (1), that is, the first of these four sets, then we express this element as $\{u, y, z\} \in move_s(\{x, y, z\}) \cap A_{k,s}$ for some $u \in Z_{\geq 0}$ such that $u < x$. Then, $(u+s) \oplus y \oplus (z+s) = 0$, and we have statement *(i)* of this lemma.

If an element of $move_s(\{x, y, z\}) \cap A_{k,s} \neq \phi$ belongs to (2) that is the second of these four sets, then we express this element as $\{x, v, z\} \in move_s(\{x, y, z\}) \cap A_{k,s}$ for some $v \in Z_{\geq 0}$ such that $v < y$. Then $(x+s) \oplus v \oplus (z+s) = 0$, and we have statement *(ii)* of this lemma.

In this way, we have at least one of the statements *(i)*, *(ii)*, *(iii)*, or *(iv)* of this lemma. □

Lemma 3.5. *Let s be an odd number such that $0 < s < k$.*

$$\text{If } y, z \in Z_{\geq 0} \text{ and } y \leq \lfloor (z+s)/k \rfloor, \tag{13}$$

then

$$y \oplus (z+s) \geq s.$$

In particular, $y \oplus (z+s) = s$ if and only if $y = z = 0$.

Proof. We consider three cases.

`Case (1)` If $y = 0$, we have $y \oplus (z+s) = z+s \geq s$. Then $y \oplus (z+s) = z+s = s$ if and only if $z = 0$.

`Case (2)` Suppose that $y = 1$. Then

$$z + s \geq k > s. \tag{14}$$

We consider Subcases (2.1) and (2.2).

`Subcase (2.1)` Suppose that z is odd. This means that $z+s$ is even. Hence, $y \oplus (z+s) = 1 \oplus (z+s) = z+s+1 > s$.

`Subcase (2.2)` Suppose that z is even. Then, the inequality in (14) implies $z \geq 1$. Hence, $z \geq 2$. Since $z+s$ is odd, $y \oplus (z+s) = 1 \oplus (z+s) = z+s-1 \geq 2+s-1 > s$.

`Case (3)` Suppose that $y \geq 2$. Let $k = \sum_{i=0}^{n} k_i 2^i$ and $y = \sum_{i=0}^{m} y_i 2^i$, where $k_i, y_i \in \{0,1\}$, n, m are natural numbers and $k_n = y_m = 1$. Because $s < k$ and $(z+s) \geq ky$, we have $z+s \geq 2^{n+m}$, $s < 2^{n+1}$, and $y < 2^{m+1}$. Then, we have $y \oplus (z+s) \geq 2^{n+m} > s$. □

Lemma 3.6. *Let s be an odd number such that $0 < s < k$. Suppose that $x, z > 0$,*

$$(x+s) \oplus y \oplus (z+s) \neq 0, 1 \tag{15}$$

and

$$y \leq \lfloor (z+s)/k \rfloor.$$

Then, at least one of the following statements is true.
(i) $(u+s) \oplus y \oplus (z+s) = 1$ for some $u \in Z_{\geq 0}$ such that $u < x$.
(ii) $(x+s) \oplus v \oplus (z+s) = 1$ for some $v \in Z_{\geq 0}$ such that $v < y$.
(iii) $(x+s) \oplus y \oplus (w+s) = 1$ for some $w \in Z_{\geq 0}$ such that $w < z$ and $y \leq \lfloor (w+s)/k \rfloor$.
(iv) $(x+s) \oplus v \oplus (w+s) = 1$ for some $v, w \in Z_{\geq 0}$ such that $v < y, w < z$ and $v = \lfloor (w+s)/k \rfloor$.

Proof. The inequality in (15) implies

$$(x+s) \oplus 1 \oplus y \oplus (z+s) \neq 0. \tag{16}$$

We consider two cases.

`Case (1)` First, we suppose that

$$(x+s) \oplus 1 < y \oplus (z+s). \tag{17}$$

We consider Subcase (1.1) and (1.2).

`Subcase (1.1)` Suppose that x is odd. As s is odd,

$$(x+s) \oplus 1 = x+s+1. \tag{18}$$

By (17) and (18), we have

$$x+s+1 < y \oplus (z+s). \tag{19}$$

Hence,

$$(x+s+1) \oplus y \oplus (z+s) \neq 0. \tag{20}$$

By the inequality in (19), there does not exist $u \in Z_{\geq 0}$ such that

$$u < x+1 \text{ and } (u+s) \oplus y \oplus (z+s) = 0. \tag{21}$$

Although we would like to apply Lemma 3.4 to the inequality in (20), condition (21) implies that we cannot use statement (*i*) of Lemma 3.4. By statements (*ii*), (*iii*), and (*iv*) of Lemma 3.4, we have the following Subsubcases (1.1.1),(1.1.2) and (1.1.3).

`Subsubcase (1.1.1)` Suppose that $(x+s+1) \oplus v \oplus (z+s) = 0$ for some $v \in Z_{\geq 0}$ such that $v < y$. Then, Equation (18) implies $(x+s) \oplus v \oplus (z+s) = 1$, and we have statement (*ii*) of this lemma.

`Subsubcase (1.1.2)` Suppose that $(x+s+1) \oplus y \oplus (w+s) = 0$ for some $w \in Z_{\geq 0}$ such that $w < z$ and $y \leq \lfloor (w+s)/k \rfloor$. Then, Equation (18) implies $(x+s) \oplus y \oplus (w+s) = 1$, and we have statement (*iii*) of this lemma.

`Subsubcase (1.1.3)` Suppose that $(x+s+1) \oplus v \oplus (w+s) = 0$ for some $v, w \in Z_{\geq 0}$ such that $v < y, w < z$ and $v = \lfloor (w+s)/k \rfloor$. Then, Equation (18) implies $(x+s) \oplus v \oplus (w+s) = 1$, and we have statement (*iv*) of this lemma.

`Subcase (1.2)` Suppose that x is even. Because s is odd, $(x+s) \oplus 1 = x+s-1$. Then, the inequality in (17) implies

$$x+s-1 < y \oplus (z+s). \tag{22}$$

Hence,

$$(x+s-1) \oplus y \oplus (z+s) \neq 0. \tag{23}$$

By the inequality in (22), there does not exist $u \in Z_{\geq 0}$ such that

$$u < x - 1 \text{ and } (u + s) \oplus y \oplus (z + s) = 0. \tag{24}$$

Although we would like to apply Lemma 3.4 to the inequality in (23), condition (24) implies that we cannot use statement (*i*) of Lemma 3.4. By statements (*ii*), (*iii*), and (*iv*) of Lemma 3.4 we prove statements (*ii*), (*iii*), and (*iv*) of this lemma with a method that is very similar to the one used in (1.1).

`Case (2)` Next, we suppose that

$$(x + s) \oplus 1 > y \oplus (z + s). \tag{25}$$

Because $z > 0$, Lemma 3.5 implies

$$y \oplus (z + s) > s. \tag{26}$$

We consider Subcases (2.1) and (2.2).

`Subcase (2.1)` If x is odd, we have $(x + s) \oplus 1 = x + s + 1$. Hence, the inequality in (25) implies $x + s + 1 > y \oplus (z + s)$. Therefore, by the inequality in (26) there exists $x' \in Z_{\geq 0}$ such that

$$0 < x' < x + 1 \tag{27}$$

and

$$(x' + s) \oplus y \oplus (z + s) = 0. \tag{28}$$

By condition (15), we have

$$x' \neq x. \tag{29}$$

We consider Subsubcases (2.1.1) and (2.1.2).

`Subsubcase (2.1.1)` Suppose x' is odd. As x is odd, the inequalities in (27) and (29) imply

$$x' \leq x - 2. \tag{30}$$

Because both x' and s are odd, $x' + s = (x' + s + 1) \oplus 1$. Then, by Equation (28), we have $((x'+s+1) \oplus 1) \oplus y \oplus (z+s) = 0$. Therefore, $((x'+1)+s) \oplus y \oplus (z+s) = 1$. The inequality in (30) implies $x' + 1 < x$, and hence we have statement (*i*) of this lemma.

`Subsubcase (2.1.2)` Suppose that x' is even. As s is odd, $x' + s = (x' + s - 1) \oplus 1$. Therefore, Equation (28) implies

$$(x' + s - 1) \oplus 1 \oplus y \oplus (z + s) = 0. \tag{31}$$

By Equation (31) and the inequality in (27), we have $((x' - 1) + s) \oplus y \oplus (z + s) = 1$ and $0 \leq x' - 1 < x$. Hence, we have statement (*i*) of this lemma.

`Subcase (2.2)` Suppose that x is even. As s is odd, we have $(x+s)\oplus 1 = x+s-1$. Then, by the inequality in (25), we have $x+s-1 > y \oplus (z+s)$. Therefore, by the inequality in (26), there exists $x' \in Z_{\geq 0}$ such that

$$0 < x' < x - 1 \tag{32}$$

and

$$(x' + s) \oplus y \oplus (z + s) = 0. \tag{33}$$

`Subsubcase (2.2.1)` Suppose that x' is odd. As s is odd, $x'+s = (x'+s+1)\oplus 1$. Then, by Equation (33), we have $((x' + s + 1) \oplus 1) \oplus y \oplus (z + s) = 0$; hence, $(x' + s + 1) \oplus y \oplus (z + s) = 1$. By the inequality in (32), we have $0 < x' + 1 < x$; hence, we have statement (i) of this lemma.

`Subsubcase (2.2.2)` Suppose that x' is even. As s is odd, $x'+s = (x'+s-1)\oplus 1$. Therefore, by Equation (33), we have

$$(x' + s - 1) \oplus 1 \oplus y \oplus (z + s) = 0. \tag{34}$$

By the inequality in (32) and Equation (34) we have $(x' - 1 + s) \oplus y \oplus (z + s) = 1$ and $0 \leq x' - 1 < x$, and we have statement (i) of this lemma. □

4. Chocolate Game with a Pass of Which the Coordinates $\{x, y, z\}$ Satisfy the Inequality $y \leq \lfloor (z + s)/k \rfloor$ for a Fixed Natural Number s

Throughout the remainder of this paper, we modify the standard rules of chocolate bar games so as to allow for a one-time pass, i.e., a pass move which may be used at most once in a game, and not from a terminal position. Once the pass has been used by either player, it is no longer available.

In this section, we denote the position of chocolate with four coordinates $\{x, y, z, p\}$, where x, y, z define the shape of the chocolate, and $p = 1$ if the pass is still available and $p = 0$ if not.

4.1. The Number s Is Odd and $0 < s < k$

Definition 4.1. (i) For a natural number s let $\tilde{P}_{s,1} = \{\{x, y, z, p\}; x, y, z \in Z_{\geq 0}, y \leq \lfloor \frac{z+s}{k} \rfloor, (x+s)\oplus y \oplus (z+s) = 1$ and $p = 1\}$ and $\tilde{P}_{s,0} = \{\{x, y, z, p\}; x, y, z \in Z_{\geq 0}, y \leq \lfloor \frac{z+s}{k} \rfloor, (x+s)\oplus y \oplus (z+s) = 0$ and $p = 0\}$, and we let $\tilde{P} = \tilde{P}_{s,1} \cup \tilde{P}_{s,0} \cup \{\{0, 0, 0, 1\}\}$.
(ii) Let $\tilde{N} = \{\{x, y, z, p\}; x, y, z \in Z_{\geq 0}, y \leq \lfloor \frac{z+s}{k} \rfloor$ and $p = 0, 1\} - \tilde{P}$.

Remark 4.1. By Definition 4.1, $\tilde{P} = \{\{x, y, z, p\}; x, y, z \in Z_{\geq 0}, y \leq \lfloor \frac{z+s}{k} \rfloor, (x + s) \oplus y \oplus (z + s) \oplus p = 0$ and $p = 0, 1\} \cup \{\{0, 0, 0, 1\}\}$.

By Remark 4.1 the definition of the set $\tilde{P}$ is simple.

Lemma 4.1. *Let s be an odd number such that $0 < s < k$. Suppose that $\{x, y, z, p\} \in \tilde{P}$ and $\{x, y, z\} \neq \{0, 0, 0\}$. Then, the following statements hold:*
(i) $\{x', y, z, p\} \notin \tilde{P}$ for any $x' \in Z_{\geq 0}$ such that $x' < x$.
(ii) $\{x, y', z, p\} \notin \tilde{P}$ for any $y' \in Z_{\geq 0}$ such that $y' < y$.
(iii) $\{x, y, z', p\} \notin \tilde{P}$ for any $z' \in Z_{\geq 0}$ such that $z' < z$.
(iv) $\{x, y', z', p\} \notin \tilde{P}$ for any $y', z' \in Z_{\geq 0}$ such that $y' < y, z' < z$ and $y' = \lfloor (z' + s)/k \rfloor$.
(v) $\{x, y, z, 0\} \notin \tilde{P}$ if $p = 1$.

Proof. If $\{x, y, z, p\} \in \tilde{P}$ and $\{x, y, z\} \neq \{0, 0, 0\}$, then we have the following Case (1) or Case (2).

`Case (1)` If $\{x, y, z, p\} \in \tilde{P}_{s,0}$, we have $p = 0$, $(x + s) \oplus y \oplus (z + s) = 0$ and $y \leq \lfloor \frac{z+s}{k} \rfloor$. Then, statements (i), (ii), (iii), and (iv) of this lemma follow directly from statements (i), (ii), (iii), and (iv) of Lemma 3.3.

`Case (2)` We suppose that $\{x, y, z, p\} \in \tilde{P}_{s,1}$. Then, $p = 1$, $y \leq \lfloor \frac{z+s}{k} \rfloor$ and

$$(x + s) \oplus y \oplus (z + s) = 1. \tag{35}$$

If one of the three coordinates of a position of a chocolate game is decreased, the nim-sum of the three coordinates changes. Therefore, statements (i), (ii), and (iii) of this lemma follow directly from this fact. Next, we prove statement (iv). By Equation (35), we have

$$((x + s) \oplus 1) \oplus y \oplus (z + s) = 0. \tag{36}$$

By Lemma 3.5, we have $y \oplus (z+s) \geq s$. Hence, $(x+s) \oplus 1 \geq s$. Then, $(x+s) \oplus 1 = u+s$ for some $u \in Z_{\geq 0}$. Therefore, Equation (36) implies

$$(u + s) \oplus y \oplus (z + s) = 0. \tag{37}$$

Then, we can apply Lemma 3.3 for Equation (37), and we have $(u+s) \oplus y' \oplus (z'+s) \neq 0$ for any $y', z' \in Z_{\geq 0}$ such that $y' < y, z' < z$ and $y' = \lfloor (z' + s)/k \rfloor$. Then, we have $((x + s) \oplus 1) \oplus y' \oplus (z' + s) \neq 0$ for any $y', z' \in Z_{\geq 0}$ such that $y' < y, z' < z$ and $y' = \lfloor (z' + s)/k \rfloor$. Hence, $(x + s) \oplus y' \oplus (z' + s) \neq 1$. Therefore, we have statement (iv) of this lemma. Equation (35) implies that $\{x, y, z, 0\} \notin \tilde{P}_{s,1}$; hence, we have statement (v) of this lemma. □

Lemma 4.2. *Let s be an odd number such that $0 < s < k$. Suppose that $\{x, y, z, p\} \notin \tilde{P}$ and $y \leq \lfloor (z + s)/k \rfloor$. Then one of the following statements $(i), (ii), (iii)$, or (iv) is true.*
(i) $\{x', y, z, p\} \in \tilde{P}$ for some $x' \in Z_{\geq 0}$ such that $x' < x$.
(ii) $\{x, y', z, p\} \in \tilde{P}$ for some $y' \in Z_{\geq 0}$ such that $y' < y$.
(iii) $\{x, y, z', p\} \in \tilde{P}$ for some $z' \in Z_{\geq 0}$ such that $z' < z$.

(iv) $\{x, y', z', p\} \in \tilde{P}$ *for some* $y', z' \in Z_{\geq 0}$ *such that* $y' < y, z' < z$ *and* $y' = \lfloor (z'+s)/k \rfloor$.
(v) $\{x, y, z, 0\} \in \tilde{P}$ *if* $p = 1$.

Proof. If $\{x, y, z, p\} \notin \tilde{P}$, then we have the following Case (1), Case (2), and Case (3).

`Case (1)` Suppose that $(x+s) \oplus y \oplus (z+s) \neq 0$, $p = 0$ and $y \leq \lfloor (z+s)/k \rfloor$. Then, by statements $(i), (ii), (iii)$, and (iv) of Lemma 3.4, one of the following subcases (1.1), (1.2), (1.3), or (1.4) is true.

`Subcase (1.1)` $(x'+s) \oplus y \oplus (z+s) = 0$ for some $x' \in Z_{\geq 0}$ such that $x' < x$. Then, $\{x', y, z, p\} \in \tilde{P}_{s,0}$, and we have statement (i) of this lemma.

`Subcase (1.2)` $(x+s) \oplus y' \oplus (z+s) = 0$ for some $y' \in Z_{\geq 0}$ such that $y' < y$. Then, $\{x, y', z, p\} \in \tilde{P}_{s,0}$, and we have statement (ii) of this lemma.

`Subcase (1.3)` $(x+s) \oplus y \oplus (z'+s) = 0$ for some $z' \in Z_{\geq 0}$ such that $z' < z$. Then, $\{x, y, z', p\} \in \tilde{P}_{s,0}$, and we have statement (iii) of this lemma.

`Subcase (1.4)` $(x+s) \oplus y' \oplus (z'+s) = 0$ for some $y', z' \in Z_{\geq 0}$ such that $y' < y, z' < z$ and $y' = \lfloor (z'+s)/k \rfloor$. Then, $\{x, y', z', p\} \in \tilde{P}_{s,0}$, and we have statement (iv) of this lemma.

`Case (2)` Suppose that $(x+s) \oplus y \oplus (z+s) \neq 0, 1$, $p = 1$ and $y \leq \lfloor (z+s)/k \rfloor$. Then we have the following subcases (2.1) and (2.2).

`Subcase (2.1)` Suppose that $x, z > 0$. Then, by statements $(i), (ii), (iii)$, and (iv) of Lemma 3.6, one of the following subsubcases (2.1.1), (2.1.2), (2.1.3), or (2.1.4) is true.

`Subsubcase (2.1.1)` $(x'+s) \oplus y \oplus (z+s) = 1$ for some $x' \in Z_{\geq 0}$ such that $x' < x$. Then, $\{x', y, z, p\} \in \tilde{P}_{s,1}$, and we have statement (i) of this lemma.

`Subsubcase (2.1.2)` $(x+s) \oplus y' \oplus (z+s) = 1$ for some $y' \in Z_{\geq 0}$ such that $y' < y$. Then, $\{x, y', z, p\} \in \tilde{P}_{s,1}$, and we have statement (ii) of this lemma.

`Subsubcase (2.1.3)` $(x+s) \oplus y \oplus (z'+s) = 1$ for some $z' \in Z_{\geq 0}$ such that $z' < z$. Then, $\{x, y, z', p\} \in \tilde{P}_{s,1}$, and we have statement (iii) of this lemma.

`Subsubcase (2.1.4)` $(x+s) \oplus y' \oplus (z'+s) = 1$ for some $y', z' \in Z_{\geq 0}$ such that $y' < y, z' < z$ and $y' = \lfloor (z'+s)/k \rfloor$. Then, $\{x, y', z', p\} \in \tilde{P}_{s,1}$; hence, we have statement (iv) of this lemma.

`Subcase (2.2)` Suppose that $x = 0$ or $z = 0$. Then, we have the following subsubcases (2.2.1) and (2.2.2).

`Subsubcase (2.2.1)` Suppose that $x = 0$. Let $z' = 0$. Then, $y' = \lfloor s/k \rfloor = 0$, and $\{x, y', z', 1\} = \{0, 0, 0, 1\} \in \tilde{P}$. We have statements (iii) or (iv) of this lemma. (We have statement (iii) when $y = 0$.)

`Subsubcase (2.2.2)` Suppose that $z = 0$. Then $y = \lfloor s/k \rfloor = 0$. Let $x' = 0$. Then $\{x', y, z, 1\} = \{0, 0, 0, 1\} \in \tilde{P}$. We have statement (i) of this lemma.

`Case (3)` Suppose that $(x + s) \oplus y \oplus (z + s) = 0$, $p = 1$ and $y \leq \lfloor (z + s)/k \rfloor$. Let $p' = 0$. Then, $\{x, y, z, p'\} \in \tilde{P}_{s,0}$; hence, we have statement (v) of this lemma. □

We define $move_{pass}(\{x, y, z, p\})$ for each position $\{x, y, z, p\}$ of which the coordinates satisfy $y \leq \lfloor (z + s)/k \rfloor$. The set $move_{pass}(\{x, y, z, p\})$ consists of positions that can be reached directly from $\{x, y, z, p\}$.

Definition 4.2. Let s be an odd number such that $0 < s < k$. For $x, y, z \in Z_{\geq 0}$ and $p = 0, 1$ we define $move_{pass}(\{x, y, z, p\}) = \{\{u, y, z, p\} : u < x\} \cup \{\{x, v, z, p\} : v < y\} \cup \{\{x, y, w, p\} : w < z \text{ and } y \leq \lfloor (w + s)/k \rfloor\} \cup \{\{x, \min(y, \lfloor (w + s)/k \rfloor), w, p\} : w < z\} \cup \{\{x, y, z, q\} : q < p\}$, where $u, v, w, q \in Z_{\geq 0}$.

Note that $\{\{x, y, z, q\} : q < p\} = \phi$ when $p = 0$.

Lemma 4.3. *We have $move_{pass}(\{x, y, z, p\}) \subset \tilde{N}$ for any $\{x, y, z, p\} \in \tilde{P}$.*

Lemma 4.3 follows directly from Lemma 4.1, Definition 4.2, and Definition 4.1.

Lemma 4.4. *If $\{x, y, z, p\} \in \tilde{N}$ we have $move_{pass}(\{x, y, z, p\}) \cap \tilde{P} \neq \phi$.*

Lemma 4.4 follows directly from Lemma 4.2, Definition 4.2, and Definition 4.1.

For chocolate games with a pass of which the coordinates $\{x, y, z\}$ satisfy the inequality $y \leq \lfloor (z + s)/k \rfloor$, we have the following theorem for $\mathcal{P}$-positions and $\mathcal{N}$-positions.

Theorem 4.1. *Let $\tilde{P}$ and $\tilde{N}$ be the sets defined in Definition 4.1. Then, $\tilde{P}$ is the set of $\mathcal{P}$-positions, and $\tilde{N}$ is the set of $\mathcal{N}$-positions.*

Proof. If we start the game from a position $\{x, y, z, p\} \in \tilde{P}$, then Lemma 4.3 indicates that any option we take leads to a position $\{x', y', z', p'\} \in \tilde{N}$. From this position $\{x', y', z', p'\}$, Lemma 4.4 implies that our opponent can choose a proper option that leads to a position in $\tilde{P}$. Note that any option reduces some of the numbers in the coordinates. In this way our opponent can always reach a position in $\tilde{P}$, and will finally win by reaching $\{0, 0, 0, 0\}$ or $\{0, 0, 0, 1\} \in \tilde{P}$. Therefore, $\tilde{P}$ is the set of $\mathcal{P}$-positions. If we start the game from a position $\{x, y, z, p\} \in \tilde{N}$, then Lemma 4.4 means that we can choose a proper option that leads to a position $\{x', y', z', p'\}$ in $\tilde{P}$. Lemma 4.3 indicates that any option from $\{x', y', z', p'\}$ taken by our opponent leads to a position $\in \tilde{N}$. In this way, we win the game by reaching $\{0, 0, 0, 0\}$ or $\{0, 0, 0, 1\}$. Therefore, $\tilde{N}$ is the set of $\mathcal{N}$-positions. □

Corollary 4.1. *A position $\{x, y, z, p\}$ is a $\mathcal{P}$-position if and only if $\{x, y, z, p\} = \{0, 0, 0, 1\}$ or $x \oplus y \oplus z \oplus p = 0$.*

Proof. This corollary follows directly from Theorem 4.1, Definition 4.1, and Remark 4.1. □

Remark 4.2. Corollary 4.1 provides a simple formula for the set of $\mathcal{P}$-positions, but the formulas for the Grundy numbers of these chocolates are yet to be discovered.

4.2. The Number s Is Even and $0 < s < k$

In this subsection, we assume that s is even and $0 < s < k$. When we do not allow for a pass move, condition (ii) of Corollary 3.1 implies that chocolate games of which the coordinates $\{x, y, z\}$ satisfy the inequality $y \leq \lfloor (z+s)/k \rfloor$ have a simple formula for Grundy numbers. When we allow for a one-time pass, do these chocolate games still have a simple formula for $\mathcal{P}$-positions? We illustrate this question with the following two examples. Example 4.1 presents a Mathematica program to find a list of $\mathcal{P}$-positions, and Example 4.2 presents a Combinatorial Game Suite (CGSuite) program to find a list of $\mathcal{P}$-positions. These examples show that there is no simple formula for $\mathcal{P}$-positions, and Conjecture 4.1 presents a slightly complicated formula for $\mathcal{P}$-positions.

Example 4.1. Here, let $k = 4$ and $s = 2$. In this example we consider a chocolate game of which the coordinates $\{x, y, z\}$ satisfy the inequality $y \leq \lfloor (z+2)/4 \rfloor$. We select $\mathcal{P}$-positions from a set of positions $\{\{x, y, z, p\} : x, y, z \in Z_{\geq 0}, y \leq \lfloor (z + 2)/4 \rfloor, 0 \leq x, y, z \leq 20$ and $p = 0, 1\}$ by using the following Mathematica program.

```
k = 4; ss = 20; s = 2; al =
Flatten[Table[{a, b, c, d}, {a, 0, ss}, {b, 0, ss}, {c, 0, ss}, {d,
0, 1}], 3];
allcases = Select[al, Floor[(#[[3]] + s)/k] >= #[[2]] &];
move[z_] := Block[{p}, p = z; If[p[[1]] + p[[2]] + p[[3]] > 0,
Union[Table[{t1, p[[2]], p[[3]], p[[4]]}, {t1, 0, p[[1]] - 1}],
Table[{p[[1]], t2, p[[3]], p[[4]]}, {t2, 0, p[[2]] - 1}],
Table[{p[[1]], Min[Floor[(t3 + s)/k], p[[2]]], t3, p[[4]]}, {t3,
0, p[[3]] - 1}],
Table[{p[[1]], p[[2]], p[[3]], t4}, {t4, 0, p[[4]] - 1}]],
Union[Table[{t1, p[[2]], p[[3]], p[[4]]}, {t1, 0, p[[1]] - 1}],
Table[{p[[1]], t2, p[[3]], p[[4]]}, {t2, 0, p[[2]] - 1}],
Table[{p[[1]], Min[Floor[(t3 + s)/k], p[[2]]], t3, p[[4]]}, {t3,
0, p[[3]] - 1}]]]];
Mex[L_] := Min[Complement[Range[0, Length[L]], L]];
Gr[pos_] := Gr[pos] = Mex[Map[Gr, move[pos]]];
pposition = Select[allcases, Gr[#] == 0 &];
d2 = Map[Reverse, Sort[Map[Reverse, pposition]]]
```

This produces the following output, which is a list of $\mathcal{P}$-positions.

```
{{0,0,0,0},{1,0,1,0},{2,0,2,0},{3,1,2,0},{3,0,3,0},{2,1,3,0},
{4,0,4,0},{5,1,4,0},{5,0,5,0},{4,1,5,0},{6,0,6,0},{7,1,6,0},
{8,2,6,0},{7,0,7,0},{6,1,7,0},{9,2,7,0},{8,0,8,0},{9,1,8,0},
{6,2,8,0},{9,0,9,0},{8,1,9,0},{7,2,9,0},{10,0,10,0},{11,1,10,0},
{12,2,10,0},{13,3,10,0},{11,0,11,0},{10,1,11,0},{13,2,11,0},
{12,3,11,0},{12,0,12,0},{13,1,12,0},{10,2,12,0},{11,3,12,0},
{13,0,13,0},{12,1,13,0},{11,2,13,0},{10,3,13,0},{14,0,14,0},
{15,1,14,0},{16,2,14,0},{17,3,14,0},{18,4,14,0},{15,0,15,0},
{14,1,15,0},{17,2,15,0},{16,3,15,0},{19,4,15,0},{16,0,16,0},
{17,1,16,0},{14,2,16,0},{15,3,16,0},{20,4,16,0},{17,0,17,0},
{16,1,17,0},{15,2,17,0},{14,3,17,0},{18,0,18,0},{19,1,18,0},
{20,2,18,0},{14,4,18,0},{15,5,18,0},{19,0,19,0},{18,1,19,0},
{20,3,19,0},{15,4,19,0},{14,5,19,0},{20,0,20,0},{18,2,20,0},
{19,3,20,0},{16,4,20,0},{17,5,20,0},{0,0,0,1},{2,0,1,1},{1,0,2,1},
{4,1,2,1},{4,0,3,1},{1,1,3,1},{3,0,4,1},{6,1,4,1},{6,0,5,1},
{3,1,5,1},{5,0,6,1},{8,1,6,1},{7,2,6,1},{8,0,7,1},{5,1,7,1},
{10,2,7,1},{7,0,8,1},{10,1,8,1},{5,2,8,1},{10,0,9,1},{7,1,9,1},
{8,2,9,1},{9,0,10,1},{12,1,10,1},{11,2,10,1},{14,3,10,1},
{12,0,11,1},{9,1,11,1},{14,2,11,1},{11,3,11,1},{11,0,12,1},
{14,1,12,1},{9,2,12,1},{12,3,12,1},{14,0,13,1},{11,1,13,1},
{12,2,13,1},{9,3,13,1},{13,0,14,1},{16,1,14,1},{15,2,14,1},
{18,3,14,1},{17,4,14,1},{16,0,15,1},{13,1,15,1},{18,2,15,1},
{15,3,15,1},{20,4,15,1},{15,0,16,1},{18,1,16,1},{13,2,16,1},
{16,3,16,1},{19,4,16,1},{18,0,17,1},{15,1,17,1},{16,2,17,1},
{13,3,17,1},{17,0,18,1},{20,1,18,1},{19,2,18,1},{13,4,18,1},
{16,5,18,1},{20,0,19,1},{17,1,19,1},{19,3,19,1},{16,4,19,1},
{13,5,19,1},{19,0,20,1},{17,2,20,1},{20,3,20,1},{15,4,20,1},
{18,5,20,1}}
```

Then, we calculate $(x+2) \oplus y \oplus (z+2) \oplus p$ for each $\{x, y, z, p\}$ of the above list.

```
dd = Map[BitXor[#[[1]] + s, #[[2]], #[[3]] + s, #[[4]]] &, d2]
```

This produces the following list.

```
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,1,6,6,2,2,6,2,14,14,2,14,2,2,2,14,6,2,6,14,
6,2,2,6,2,2,30,2,6,30,2,2,30,6,2,30,2,2,6,30,2,2,6,6,2,30,6,2,
2,2,6,30,2,2,6,2,2,30,6,2,2,30,2,2,6,2,2,30,2,6,2,2,6}
```

The only numbers in the list are $0, 1, 2, 6, 14$, and 30.

Example 4.2. In this example, we study a chocolate game of which the coordinates $\{x, y, z\}$ satisfy the inequality $y \leq \lfloor (z+2)/4 \rfloor$. In Example 4.1 we used Mathematica, and in this example we use CGSuite (CGSuite version1.1) to perform the same calculation. First, we open the following file by CGSuite.

```
class GrundyWithAPass extends ImpartialGame

var x,y,z,p,k,s;

method GrundyWithAPass(x,y,z,p,k,s)
end

override method Options(Player player)
result := [];

// x
for x1 from 0 to x-1 do
result.Add(GrundyWithAPass(x1,y,z,p,k,s));
end

// y
if y<=((z+s)/k).Floor then
for y1 from 0 to y-1 do
result.Add(GrundyWithAPass(x,y1,z,p,k,s));
end
else
for y1 from 0 to ((z+s)/k).Floor-1 do
result.Add(GrundyWithAPass(x,y1,z,p,k,s));
end
end

// z
for z1 from 0 to z-1 do
if y<=((z1+s)/k).Floor then
result.Add(GrundyWithAPass(x,y,z1,p,k,s));
else
result.Add(GrundyWithAPass(x,((z1+s)/k).Floor,z1,p,k,s));
end
end

// pass
if x!=0 or y!=0 or z!=0 and p==1 then
result.Add(GrundyWithAPass(x,y,z,0,k,s));
end

result.Remove(this);

if x==0 and y==0 and z==0 then
return {};
else
return result;
```

```
end
end

override property ToString.get
return "GrundyWithAPass(" + x.ToString + "," +
 y.ToString + "," +z.ToString + "," + p.ToString +
  "," + k.ToString + "," + s.ToString + ")";
end

end
```

Second, we type in the following command.

```
k:=4;
s:=2;
x:=20;
z:=20;
for p from 0 to 1 do
for z1 from 0 to z do
for y1 from 0 to ((z+s)/k).Floor do
for x1 from 0 to x do
if examples.GrundyWithAPass(x1,y1,z1,p1,k,s).CanonicalForm==0 then
Worksheet.Print(*(x1+s)+*y1+*(z1+s)+*p1);
end
end
end
end
end
```

In this way we obtain the following Table 4.1.

The list in Table 4.1 contains only the numbers $0, 1, 2, 6, 14$, and 30. Note that we denote 1 by $*$ here.

By Examples 4.1 and 4.2, we offer the following Conjecture 4.1.

Conjecture 4.1. Let s be even and $0 < s < k$. Then the set $\{(x+s) \oplus y \oplus (z+s) \oplus p : \{x, y, z, p\}$ is a $\mathcal{P}$-position$\}$ consists of the numbers $0, 1, 2, 6, 14, 30, ..., 2^n - 2, ...$

4.3. The Number $s = k2^t + m2^{t+1}$ for Non-Negative Integers t,m Such That $m = 0, 1, 2, ..., \frac{k}{2} - 1$

In this subsection, we assume that $s = k2^t + m2^{t+1}$ for non-negative integers t, m such that $m = 0, 1, 2, ..., \frac{k}{2} - 1$. When we do not allow for a pass move, Corollary 3.1 implies that chocolate games of which the coordinates $\{x, y, z\}$ satisfy the inequality $y \leq \lfloor (z+s)/k \rfloor$ have a simple formula for Grundy numbers. When we allow for a one-time pass, do these chocolate games still have a simple formula for $\mathcal{P}$-positions? We illustrate this question by using the following two examples. Example 4.3 presents a

0	0	$*2$
0	0	$*30$
0	0	$*2$
0	0	$*6$
0	0	$*30$
0	0	$*2$
0	0	$*2$
0	0	$*30$
0	0	$*6$
0	0	$*2$
0	0	$*30$
0	0	$*2$
0	0	$*2$
0	0	$*6$
0	0	$*30$
0	0	$*2$
0	0	$*2$
0	0	$*6$
0	0	$*6$
0	0	$*2$
0	0	$*30$
0	0	$*6$
0	0	$*2$
0	0	$*2$
0	$*$	$*2$
0	$*6$	$*6$
0	$*6$	$*30$
0	$*2$	$*2$
0	$*2$	$*2$
0	$*6$	$*6$
0	$*2$	$*2$
0	$*14$	$*2$
0	$*14$	$*30$
0	$*2$	$*6$
0	$*14$	$*2$
0	$*2$	$*2$
0	$*2$	$*30$
0	$*2$	$*2$
0	$*14$	$*2$
0	$*6$	$*6$
0	$*2$	$*2$
0	$*6$	$*2$
0	$*14$	$*30$
0	$*6$	$*2$
0	$*2$	$*6$
0	$*2$	$*2$
0	$*6$	$*2$
0	$*2$	$*6$

Table 4.1: Data by Combinatorial Game Suite

calculation by the computer algebra system Mathematica, and Example 4.4 presents a calculation by CGSuite.

Example 4.3. Here, let $k = 4$ and $s = 6$. In this example, we study a chocolate game of which the coordinates $\{x, y, z\}$ satisfy the inequality $y \leq \lfloor (z+6)/4 \rfloor$. We select $\mathcal{P}$-positions from a set of positions $\{\{x, y, z, p\} : x, y, z \in Z_{\geq 0}$ and $0 \leq x, y, z \leq 40\}$ by the following Mathematica program.

```
k = 4; ss = 40; s = 6; al =
Flatten[Table[{a, b, c, d}, {a, 0, ss}, {b, 0, ss}, {c, 0, ss}, {d,
0, 1}], 3];
allcases = Select[al, Floor[(#[[3]] + s)/k] >= #[[2]] &];
move[z_] := Block[{p}, p = z; If[p[[1]] + p[[2]] + p[[3]] > 0,
Union[Table[{t1, p[[2]], p[[3]], p[[4]]}, {t1, 0, p[[1]] - 1}],
```

```
Table[{p[[1]], t2, p[[3]], p[[4]]}, {t2, 0, p[[2]] - 1}],
Table[{p[[1]], Min[Floor[(t3 + s)/k], p[[2]]], t3, p[[4]]}, {t3,
0, p[[3]] - 1}],
Table[{p[[1]], p[[2]], p[[3]], t4}, {t4, 0, p[[4]] - 1}]],
Union[Table[{t1, p[[2]], p[[3]], p[[4]]}, {t1, 0, p[[1]] - 1}],
Table[{p[[1]], t2, p[[3]], p[[4]]}, {t2, 0, p[[2]] - 1}],
Table[{p[[1]], Min[Floor[(t3 + s)/k], p[[2]]], t3, p[[4]]}, {t3,
0, p[[3]] - 1}]]]];
Mex[L_] := Min[Complement[Range[0, Length[L]], L]];
Gr[pos_] := Gr[pos] = Mex[Map[Gr, move[pos]]];
pposition = Select[allcases, Gr[#] == 0 &];
d2 = Map[Reverse, Sort[Map[Reverse, pposition]]];
```

This produces the list "d2" which is a list of $\mathcal{P}$-positions. Next, we calculate $(x+2) \oplus y \oplus (z+2) \oplus p$ for each $\{x, y, z, p\}$ element in the list d2.

```
dd = Map[BitXor[#[[1]] + s, #[[2]], #[[3]] + s, #[[4]]] &, d2]
```

In this way we obtain the following list.

```
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,1,14,14,0,14,14,2,2,0,12,2,0,2,6,0,2,6,0,2,0,2,0,2,0,
2,0,2,0,30,0,2,0,30,0,2,0,6,2,0,2,0,0,2,0,2,0,2,6,0,2,0,6,6,0,
2,0,0,2,2,0,2,0,2,0,2,0,2,0,11,11,14,0,2,0,11,9,14,0,2,0,11,7,
3,2,0,2,0,11,11,3,2,0,2,0,3,9,11,6,0,2,0,3,7,11,6,0,2,0,3,7,3,
58,2,0,2,0,3,57,59,3,2,0,2,0,59,59,2,58,62,0,2,0,59,7,58,2,62,
0,2,0,59,57,4,2,12,2,0,2,0,59,59,2,2,14,2,0,2,0,3,3,59,59,12,6,
0,2,0,3,0,59,59,2,6,0,2,0,3,8,3,3,51,51,2,0,2,0,3,3,10,10,51,
51,2,0,2,0,11,11,11,3,14,5,14,0,2,0,11,0,11,11,10,14,14,0,2,0,
11,8,10,5,2,12,14,2,0,2,0,11,11,2,2,14,2,6,2,0,2,0,3,3,11,11,6,
2,6,0,2,0,3,11,11,13,6,0,2,0,3,2,2,2,0,2,3,3,13,2,0,0,13,15}
```

This list contains the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 30, 51, 57, 58, 59, 62....$ It seems that the amount of data presented above is insufficient to obtain a conjecture.

Example 4.4. Here, let $k = 4$ and $s = 6$. In this example we study a chocolate game of which the coordinates $\{x, y, z\}$ satisfy the inequality $y \leq \lfloor (z + 6)/4 \rfloor$. We can obtain the same result by using the following CGSuite code.

```
k:=4;
s:=6;
x:=40;
z:=40;
for p1 from 0 to 1 do
for z1 from 0 to z do
for y1 from 0 to ((z+s)/k).Floor do
for x1 from 0 to x do
if examples.GrundyWithAPass(x1,y1,z1,p1,k,s).CanonicalForm==0 then
Worksheet.Print(*(x1+s)+*y1+*(z1+s)+*p1);
end
end
end
end
end
```

Example 4.5. Moreover, we can almost obtain the same result by using the following C++ code.

```
#include <iostream>
#include <cmath>
#include <bitset>
using namespace std;

// prototype
int grundyCalcSub(int xx,int yy,int zz,int pp);
int G[1000][1000][1000][2];
int sortresult[1000];
// max coordinate value
int z=80;
int x=80;
int p=1;
// y<=floor((z+s)/divk)
int divk=4;
int s=6;

// main
int main(int argc, const char * argv[]) {
// initialize
for(int l=0;l<=p;l++){
for(int k=0;k<=z;k++){
for(int j=0;j<=(int)floor(double((k+s)/divk));j++){
for(int i=0;i<=x;i++){
G[i][j][k][l]=99999;
}}}}
```

```
for(int i=0;i<=1000;i++) sortresult[i] = 99999;
G[0][0][0][0]=0;
G[0][0][0][1]=0;

// calculation
for(int l=0;l<=p;l++){
for(int k=0;k<=z;k++){
for(int j=0;j<=(int)floor(double((k+s)/divk));j++){
for(int i=0;i<=x;i++){
if(grundyCalcSub(i,j,k,l)==0){
cout<<((i+s)^j^(k+s)^p)<< ",";
if(sortresult[((i+s)^j^(k+s)^p)]==99999) sortresult[((i+s)^j^(k+s)^p)]=1;
}}}}}

// sort output
for(int i=0;i<=1000;i++){
if(sortresult[i]==1){
cout<<i<<",";
}}
return 0;
}

// Grundy Number Calculation
int grundyCalcSub(int xx,int yy,int zz,int pp){
int gFlag[xx+yy+zz+pp+1];
// initialize
for(int i=0;i<xx+yy+zz+pp+1;i++) gFlag[i] = 0;
if(xx==0&&yy==0&&zz==0) return 0;

// x
for(int i=0;i<xx;i++){
if(G[i][yy][zz][pp]==99999){
gFlag[grundyCalcSub(i,yy,zz,pp)]=1;
}else if(G[i][yy][zz][pp]!=99999){
gFlag[G[i][yy][zz][pp]]=1;
}}

// y
if(yy<=(int)floor(double((zz+s)/divk))){
for(int j=0;j<yy;j++){
if(G[xx][j][zz][pp]==99999){
gFlag[grundyCalcSub(xx,j,zz,pp)]=1;
}else if(G[xx][j][zz][pp]!=99999){
gFlag[G[xx][j][zz][pp]]=1;
}}}else{
for(int j=0;j<(int)floor(double((zz+s)/divk));j++){
if(G[xx][j][zz][pp]==99999){
gFlag[grundyCalcSub(xx,j,zz,pp)]=1;
```

```
}else if(G[xx][j][zz][pp]!=99999){
gFlag[G[xx][j][zz][pp]]=1;
}}}

// z
for(int k=0;k<zz;k++){
if(yy<=(int)floor(double((k+s)/divk))){
if(G[xx][yy][k][pp]==99999){
gFlag[grundyCalcSub(xx,yy,k,pp)]=1;
}else if(G[xx][yy][k][pp]!=99999){
gFlag[G[xx][yy][k][pp]]=1;
}}else{
if(G[xx][(int)floor(double((k+s)/divk))][k][pp]==99999){
gFlag[grundyCalcSub(xx,(int)floor(double((k+s)/divk)),k,pp)]=1;
}else if(G[xx][(int)floor(double((k+s)/divk))][k][pp]!=99999){
gFlag[G[xx][(int)floor(double((k+s)/divk))][k][pp]]=1;
}}}

// pass
if((xx!=0||yy!=0||zz!=0)&&pp==1){
if(G[xx][yy][zz][0]==99999){
gFlag[grundyCalcSub(xx,yy,zz,0)]=1;
}else if(G[xx][yy][zz][0]!=99999){
gFlag[G[xx][yy][zz][0]]=1;
}}

// mex
for(int l=0;l<=xx+yy+zz+pp+1;l++){
if(gFlag[l]==0){
G[xx][yy][zz][pp]=l;
break;
}}

return G[xx][yy][zz][pp];
}
```

This produces the following list.

```
{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
```

```
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,14,14,0,14,14,2,2,0,12,2,0,2,6,0,2,6,0,2,0,2,0,2,0,2,0,2,0,30,0,2,0,30,
0,2,0,6,2,0,2,0,0,2,0,2,0,2,6,0,2,0,6,6,0,2,0,0,2,2,0,2,0,2,0,2,0,2,0,11,
11,14,0,2,0,11,9,14,0,2,0,11,7,3,2,0,2,0,11,11,3,2,0,2,0,3,9,11,6,0,2,0,3,
7,11,6,0,2,0,3,7,3,58,2,0,2,0,3,57,59,3,2,0,2,0,59,59,2,58,62,0,2,0,59,7,
58,2,62,0,2,0,59,57,4,2,12,2,0,2,0,59,59,2,2,14,2,0,2,0,3,3,59,59,12,6,0,2,
0,3,0,59,59,2,6,0,2,0,3,8,3,3,51,51,2,0,2,0,3,3,10,10,51,51,2,0,2,0,11,11,
11,3,14,5,14,0,2,0,11,0,11,11,10,14,14,0,2,0,11,8,10,5,2,12,14,2,0,2,0,11,
11,2,2,14,2,6,2,0,2,0,3,3,11,11,12,6,2,6,0,2,0,3,0,11,11,18,13,17,6,0,2,0,
3,24,2,2,20,11,21,21,2,0,2,0,3,3,26,26,13,23,21,19,2,0,2,0,27,27,27,27,13,
15,17,17,30,0,2,0,27,0,27,5,20,13,13,19,30,0,2,0,27,24,26,2,18,11,17,19,10,
2,0,2,0,27,27,2,2,20,20,12,17,21,2,0,2,0,3,3,27,27,22,12,21,15,21,6,0,2,0,
3,0,27,27,12,11,0,21,7,6,0,2,0,3,8,2,2,10,5,17,17,21,2,2,0,2,0,3,3,10,10,
6,12,17,17,8,21,2,0,2,0,11,11,11,2,12,5,13,0,21,21,14,0,2,0,11,0,11,11,10,
3,2,13,21,21,14,0,2,0,11,8,10,5,13,13,12,0,11,4,119,2,0,2,0,11,11,2,2,5,5,
114,13,6,10,6,2,0,2,0,3,3,11,11,13,7,112,15,119,4,119,6,0,2,0,3,0,11,11,10,
13,5,113,119,117,6,6,0,2,0,3,120,2,2,10,3,117,119,114,116,118,116,2,0,2,0,
3,3,122,122,13,13,117,117,116,116,116,116,2,0,2,0,123,123,123,123,12,12,2,
12,118,116,117,120,126,0,2,0,123,0,123,5,116,115,13,15,10,116,120,116,126,
0,2,0,123,120,122,2,10,13,113,115,118,116,8,7,27,2,0,2,0,123,123,2,2,117,
15,113,12,118,113,118,5,25,2,0,2,0,3,3,123,123,13,13,117,115,114,6,4,116,
24,6,0,2,0,3,0,123,123,10,3,117,15,114,116,6,6,106,6,0,2,0,3,8,2,2,10,115,
6,115,3,118,112,118,25,25,2,0,2,0,3,3,10,10,13,13,4,4,119,116,118,112,106,
25,2,0,2,0,11,11,11,2,4,15,0,15,7,10,116,118,108,106,14,0,2,0,11,0,11,11,
10,13,0,0,4,10,118,116,106,108,14,0,2,0,11,8,10,5,13,3,13,13,7,9,120,7,100,
102,11,2,0,2,0,11,11,2,2,5,15,6,19,11,6,7,120,9,100,100,2,0,2,0,3,3,11,11,
13,13,4,4,23,23,6,22,100,27,100,6,0,2,0,3,0,11,11,10,19,17,4,20,9,23,6,27,
100,28,6,0,2,0,3,24,2,2,10,21,0,20,7,19,23,6,30,9,96,31,2,0,2,0,3,3,26,26,
13,23,0,15,7,21,23,16,24,30,31,96,2,0,2,0,27,27,27,27,13,15,13,16,8,23,23,
26,30,30,0,2,0,27,0,27,5,20,13,16,12,10,21,20,30,30,0,2,0,27,24,26,2,10,19,
14,16,20,8,2,0,2,0,27,27,2,2,21,21,17,13,20,2,0,2,0,3,3,27,27,13,23,20,20,
22,6,0,2,0,3,27,27,10,13,19,20,16,6,0,2,0,3,2,2,16,16,20,22,2,0,2,3,3,13,
14,22,20,23,9,2,0,0,2,13,13,10,22,21,9}
```

The above list contains the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,$ $16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 51, 57, 58, 59, 62, 96, 100, 102,$

$106, 108, 112, 113, 114, 115, 116,117, 118, 119, 120, 122, 123, 126.$

Examples 4.3 and 4.5 seem to show that the set $\{(x+s) \oplus y \oplus (z+s) \oplus p : \{x, y, z, p\}$ is a $\mathcal{P}$-position$\}$ contains all the non-negative integers. Therefore, we obtain the following conjecture.

Conjecture 4.2. Let $s = k2^t + m2^{t+1}$ for non-negative integers t, m such that $m = 0, 1, 2, ..., \frac{k}{2}-1$. Then the set $\{(x+s) \oplus y \oplus (z+s) \oplus p : \{x, y, z, p\}$ is a $\mathcal{P}$-position$\}$ contains all the natural numbers.

Acknowledgements. The authors are indebted to Shunsuke Nakamura and Naoki Hirabayashi. Although they were not the primary authors, their contributions were significant. The authors would like to express their thanks to the anonymous referee whose time and patience improved the quality of this work. Finally, many thanks to the Integers staff for the time which they put into this paper.

References

[1] S. Nakamura and R. Miyadera, Impartial Chocolate Bar Games, *Integers* **15**, (2015), Article G4.

[2] A.C. Robin, A poisoned chocolate problem, Problem corner, *The Mathematical Gazette* **73**, (1989), 341-343. An Answer for the above problem can be found in **74**, (1990), 171-173.

[3] D.Zeilberger, Three-Rowed CHOMP, *Adv. in Appl. Math.* **26** (2001), 168-179.

[4] R.E. Morrison, E.J. Friedman, and A.S. Landsberg, Combinatorial games with a pass: A dynamical systems approach, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, **21** (2011), 43-108.

[5] E.J. Friedman and A.S. Landsberg, On the geometry of combinatorial games: A renormalization approach, *Games of No Chance 3 MSRI Publications* **56** (2009).

[6] R. Berlekamp, J.H. Conway, and R.K. Guy, *Winning Ways for Your Mathematical Plays*, Academic Press London, Chap. IX, 1979.

[7] G. Horrocks and R.J. Nowakowski, Regularity in the G-Sequences of Octal Games with a Pass, *Integers* **3**, (2003), Article G1.

[8] M.H. Albert, R.J. Nowakowski, and D. Wolfe, *Lessons In Play*, A K Peters/CRC Press, 2007, 139.

[9] A.N. Siegel, *Combinatorial Game Theory (Graduate Studies in Mathematics)*, American Mathematical Society, 2013.